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THE NO-THREE-IN-LINE PROBLEM

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The problem considered here was given, in the case $n = 8$, by Dudeney [3] and Rouse Ball [2]. Given an n by n array of n^2 points of the unit lattice, select $2n$ of them so that no three are in a straight line.

It is not known whether such configurations exist for all n. For $n \leq 9$, solutions have been found by hand [1,4] and by computer [6]. The numbers of these are shown in Table I. Columns K, L, M, N, O, P, Q, R show the numbers of solutions having the symmetry of a kite (K, symmetry about one diagonal, D), a lozenge (L, symmetry about both diagonals), an isosceles trapezium (M, symmetry about one bisector, B, of a pair of opposite sides), no symmetry (N), that of an oblong (O, about both bisectors of pairs of opposite sides), a parallelogram (P, symmetry of rotation through 180°), a square (Q) and rotation (R) through a right angle; S and T are totals, the latter counting solutions as distinct even when obtain-

n	K	L	M	N	P	Q	R	S	T	t_n
2						1		1	1	0
3		1						1	2	8
4		1	1		1	1		4	11	44
5	2			3				5	32	152
6		2		4	2		3	11	50	372
7	1			11	10			22	132	824
8	5		1	40	7		4	57	380	1544
9	3			41	7			51	368	2712
10	3	1		132	13	1	6	156	1135	4448

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Table I

able one from another by a symmetry operation. The final column shows t_n (see later) calculated exactly from expression (8). The lattice of subgroups of symmetries of a square is illustrated in Figure I, the generators being given in parentheses.

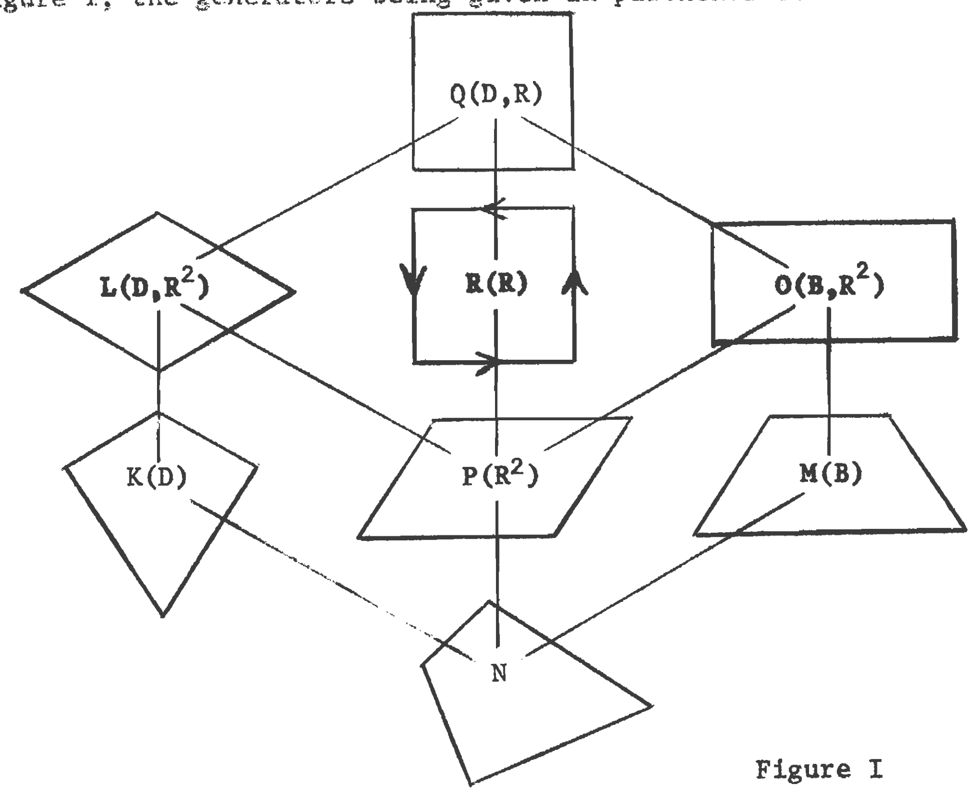

Figure I

Theorem 1. If n is odd, there is no solution with symmetry O, that of an oblong; a fortiori none with symmetry Q, the full symmetry of the square.

Proof: By considering the bisecting lines parallel to the sides. [See 6.]

Conjecture 1. There is no solution with symmetry O except those which also have the full symmetry, Q.

Conjecture 2. If $n > 10$, there is no solution with symmetry Q.

The solution shown in Figure II was discovered by Acland-Hood [1]. An unsuccessful search [6] was made for solutions of this type with $12 \leq n \leq 32$.

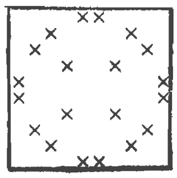

Figure II.

In support of a third conjecture we next prove the following theorem.

Theorem 2. The number, t_n, of sets of three points in line that can be chosen from an n by n portion of the unit lattice is given by

$$t_n = \frac{3}{\pi^2} n^4 \log n + O(n^4). \tag{1}$$

Proof: Take the n^2 points to be those with integer coordinates (x,y), $1 \leq x, y \leq n$. The number of sets of three points in lines parallel to either side of the square is

$$2n\binom{n}{3} = \frac{1}{3} n^2 (n-1)(n-2). \tag{2}$$

The number of sets in lines parallel to either diagonal is

$$2\binom{n}{3} + 4\left\{\binom{n-1}{3} + \binom{n-2}{3} + \ldots + \binom{3}{3}\right\} = \frac{1}{6} n(n-1)^2(n-2). \tag{3}$$

Thus the numbers (2) and (3) may be absorbed in the error term in (1). The main term will arise from triples chosen from lines of points (a,b), $(a+p, b+q)$, $(a+2p, b+2q)$, ... where

$$1 \leq q < p \leq [\tfrac{1}{2}(n-1)], \tag{4}$$

square brackets denote integer part, and $(p,q) = 1$, i.e. p is prime to q. Figure III is drawn to illustrate the case $n = 60$, $p = 7$, $q = 5$. Define $r = [(n-1)/p]$, so that $r = 8$ in this case. Points in regions marked 1 in Figure III are in lines originating

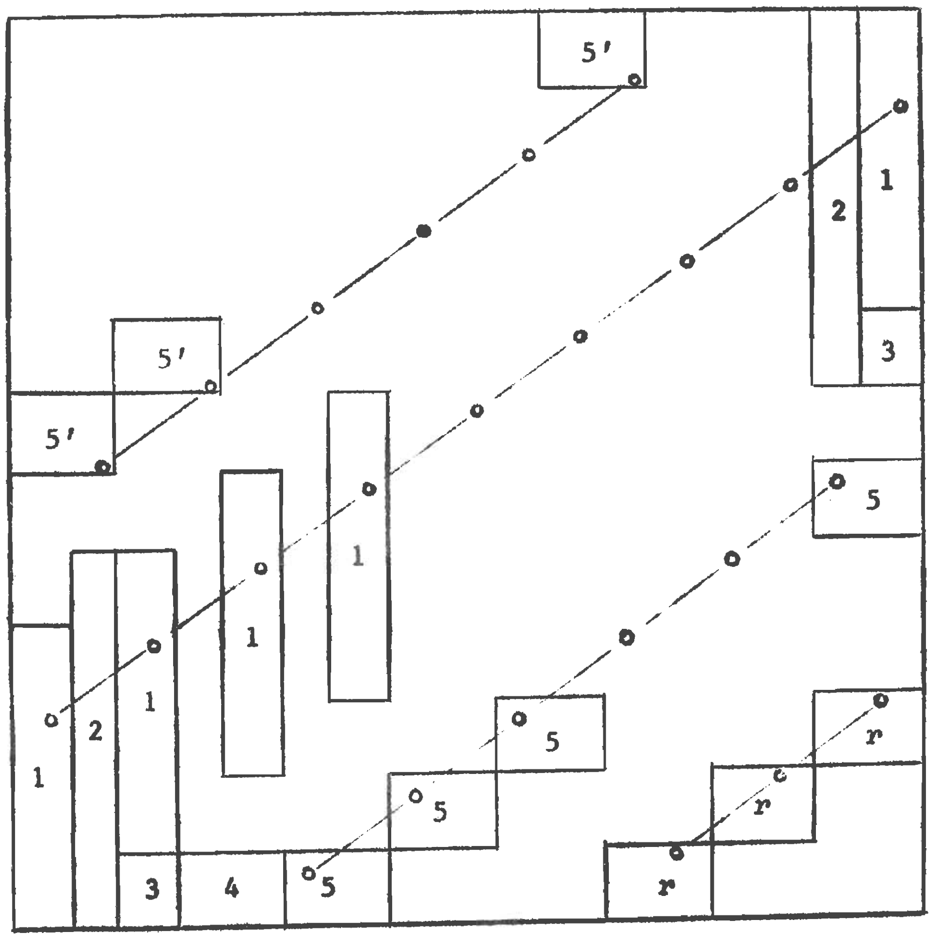

Figure III

in the rectangle $1 \leqslant a \leqslant n - rp$, $1 \leqslant b \leqslant n - rq$, each line containing $r+1$ points. Those in regions 2 have $n - rp + 1 \leqslant a \leqslant p$ and $1 \leqslant b \leqslant n - (r-1)q$, and r points in each line. Triples arising from regions 1 and 2 should be counted 4 times, to allow for the cases where one or both of p and q are negative. Triples arising from regions j $(3 \leqslant j \leqslant r)$ are counted 8 times, for the same reason, together with the fact that they are each repeated (see the regions 5' in Figure III). These regions have $n - (r+3-j)p + 1 \leqslant a \leqslant n - (r+2-j)p$ and $1 \leqslant b \leqslant q$, except for $j = 3$ where the range for a is $p + 1 \leqslant a \leqslant n - (r-1)p$. The lines in these cases contain $r+3-j$ points. The required number of triples is thus

$$4\{(n-rp)(n-rq)\binom{r+1}{3} + ((r+1)p - n)(n - (r-1)q)\binom{r}{3}\} + \\ + 8\{(n-rp)q\binom{r}{3} + pq\sum_{j=4}^{r}\binom{r+3-j}{3}\} =$$

$$= \frac{1}{3}r(r-1)\{6n^2 - 4n(p+q)(r+1) + pq(r+1)(3r+2)\},$$

summed for p and q in the range (4), and augmented by (2) and (3), so that

$$t_n = \frac{1}{6}n(n-1)(n-2)(3n-1) + \\ + \sum_{p=2}^{[\frac{1}{2}(n-1)]} \sum_{\substack{q=1 \\ (p,q)=1}}^{p-1} \frac{1}{3}r(r-1)\{6n^2 - 4n(p+q)(r+1) + pq(r+1)(3r+2)\}. \qquad (5)$$

The inner summation, over q, is facilitated by writing

$$\sum_{\substack{q=1 \\ (p,q)=1}}^{p-1} 1 = \phi(p), \tag{6}$$

Euler's totient function, and by the following lemma.

Lemma 1.

$$\sum_{\substack{q=1 \\ (p,q)=1}}^{p-1} q = \frac{1}{2} p\phi(p), \quad p \geq 2, \tag{7}$$

whose proof follows from the observation that $(p,q) = 1$, if and only if $(p,p-q) = 1$.

By using (6) and (7), we may rewrite (5) as

$$t_n = \frac{1}{6} n(n-1)(n-2)(3n-1) + $$

$$+ \sum_{p=2}^{[\frac{1}{2}(n-1)]} \frac{1}{6} r(r-1)\{12n^2 - 12np(r+1) + p^2(r+1)(3r+2)\}\phi(p). \tag{8}$$

$$t_n = \frac{1}{2} \sum_{p=2}^{[\frac{1}{2}(n-1)]} r(r-1)(2n - p(r+1))^2 \phi(p) -$$

$$\frac{1}{6} \sum_{p=2}^{[\frac{1}{2}(n-1)]} r(r^2-1)\; p^2\phi(p) + O(n^4)$$

$$= \frac{1}{2} \sum_{p=2}^{[\frac{1}{2}(n-1)]} r(r-1)(n -(p-t))^2 \phi(p) + O(n^4)$$

where we write $n = pr+t$ $(1 \leq t \leq p)$ in the first term, and note that the second term is $O(n^4)$ since $r < n/p$ and $\phi(p) < p$. Hence

$$t_n = \frac{1}{2} \sum_{p=2}^{[\frac{1}{2}(n-1)]} (n-t)(n-t-p)(n+t-p)^2 \phi(p)/p^2 + O(n^4)$$

and the inequalities just given, together with $t \leq p < n$, show that

$$t_n = \frac{1}{2} n^4 \sum_{p=2}^{[\frac{1}{2}(n-1)]} \phi(p)/p^2 + O(n^4).$$

Theorem 2 will now follow from a second lemma.

Lemma 2.

$$\sum_{p=1}^{m} \phi(p)/p^2 = \frac{6}{\pi^2} \log m + O(1).$$

Proof:

$$\sum_{p=1}^{m} \phi(p)/p^2 = \sum_{p=1}^{m} \frac{1}{p^2} p \sum_{d|p} \frac{\mu(d)}{d},$$

where $\mu(d)$ is the Möbius function [5], and, on writing $p = dd'$,

$$\sum_{p=1}^{m} \phi(p)/p^2 = \sum_{\substack{d,d' \\ dd' \leq m}} \frac{\mu(d)}{d^2 d'} = \sum_{d=1}^{m} \frac{\mu(d)}{d^2} \sum_{d'=1}^{[m/d]} \frac{1}{d'}$$

$$= \sum_{d=1}^{m} \frac{\mu(d)}{d^2} (\log m - \log d) + O(1)$$

$$= \log m \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O(1)$$

$$= \frac{6}{\pi^2} \log m + O(1).$$

For large n, the probability that three points, chosen at random, should be in line is thus

$$\frac{3}{\pi^2} n^4 \log n \Big/ \binom{n^2}{3} \sim \frac{18 \log n}{\pi^2 n^2}$$

and the probability that three such points should not be in line is

$$1 - \frac{18 \log n}{\pi^2 n^2} + O(\frac{1}{n^2}).$$

If we assume that the events are independent, the probability that $2n$ points contain no three in line is

$$\left(1 - \frac{18 \log n}{\pi^2 n^2} + O(\frac{1}{n^2})\right)^{\binom{2n}{3}} = e^{-\frac{24}{\pi^2} n \log n + O(n)}.$$

Hence an estimate of the number of solutions to the no-three-in-line problem is given by

$$\binom{n^2}{2n} n^{-24n/\pi^2} e^{O(n)},$$

which the use of Stirling's formula shows to be

$$O(n^{-c_1 n} c_2^n), \quad \text{where} \tag{9}$$

c_1 and c_2 are constants, with $c_1 = -2 + 24/\pi^2$. The expression (9) supports our third conjecture.

Conjecture 3. There is only a finite number of solutions to the no-three-in-line problem.

It is clear, by the pigeon-hole principle, that it is not possible to select more than $2n$ points without there being three in line. In the other direction, even to choose a much smaller number with no three in line seems difficult. Erdős [1] has shown that n points may be so chosen, provided n is prime. Choose those with coordinates (x,x^2), reduced modulo n. Three such points are in line if and only if

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} \equiv 0, \text{ mod } n.$$

I.e., If and only if $(x-y)(y-z)(z-x) \equiv 0$, mod n, which is not possible if x, y, z are incongruent, mod n, and n is prime. If n is composite the construction fails, and it is not known if n points can always be chosen with no three in line.

If we reproduce the probability argument with kn points in place of $2n$, the corresponding value of c_1 in (9) is $-2 + 3k^3/\pi^2$, which tends to zero as $n \to \infty$, provided $k > (2\pi^2/3)^{1/3} \approx 1.873856$.

Conjecture 4. For large n, we expect to be able to select approximately $(2\pi^2/3)^{1/3}n$ points with no three in line, but no larger number.

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