Evaluations of several series in the base e^{π}

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In this note we provide proofs for several values of power series evaluated at $e^{-\pi}$, which appeared in a recent paper by Plouffe [3].

Theorem 1. *We have*

$$\sum_{n=0}^{\infty} \underline{\text{A081362}}(n)e^{-\pi n} = e^{-\pi/24}2^{1/8}. \tag{1}$$

Proof. Let $A(x) = \sum_{n=0}^{\infty} \underline{\text{A081362}}(n)x^n$ be the generating function of the sequence A081362. It is well-known that

$$A(x) = \prod_{n=1}^{\infty}(1 - x^{2n-1}).$$

Substituting $x = e^{-\pi}$ in $A(x)$, we see that (1) is equivalent to

$$\prod_{n=1}^{\infty}(1 - e^{-(2n-1)\pi}) = e^{-\pi/24}2^{1/8}.$$

Recall (e.g., [2]), that for every complex number τ, with $\mathrm{Im}(\tau) > 0$, the Dedekind eta function $\eta(\tau)$ is given by

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n),$$

where $q = e^{2\pi i\tau}$. We have $q^2 = e^{2\pi i(2\tau)}$. Thus,

$$\prod_{n=1}^{\infty}(1 - q^{2n}) = (q^2)^{-1/24}(q^2)^{1/24}\prod_{n=1}^{\infty}(1 - (q^2)^n) = q^{-1/12}\eta(2\tau).$$

Hence

$$\prod_{n=1}^{\infty}(1 - q^{2n-1}) = \frac{\prod_{n=1}^{\infty}(1 - q^n)}{\prod_{n=1}^{\infty}(1 - q^{2n})} = \frac{q^{-1/24}\eta(\tau)}{q^{-1/12}\eta(2\tau)} = q^{1/24}\frac{\eta(\tau)}{\eta(2\tau)}. \tag{2}$$

Now, let $\tau = i/2$. Then, $q = e^{-\pi}$. It follows from (2) that

$$\prod_{n=1}^{\infty}(1 - e^{-(2n-1)\pi}) = e^{-\pi/24}\frac{\eta(i/2)}{\eta(i)}. \tag{3}$$

It is well-known (e.g., A091343 and A248190) that

$$\eta(i) = \frac{\Gamma(1/4)}{2\pi^{3/4}}, \qquad \eta(i/2) = \frac{\Gamma(1/4)}{2^{7/8}\pi^{3/4}}, \tag{4}$$

where Γ is the gamma function. Thus, $\eta(i/2)/\eta(i) = 2^{1/8}$. Substituting this into (3), we obtain

$$\prod_{n=1}^{\infty}(1 - e^{-(2n-1)\pi}) = e^{-\pi/24}2^{1/8},$$

completing the proof. □

Theorem 2. *We have*

$$\sum_{n=0}^{\infty} \text{A000712}(n)e^{-\pi n} = e^{-\pi/12}2^{3/4}\frac{\Gamma(3/4)^2}{\sqrt{\pi}}. \tag{5}$$

Proof. Let $A(q) = \sum_{n=0}^{\infty} \text{A000712}(n)q^n$ be the generating function of the sequence A000712. It is well-known that

$$A(q) = \prod_{n=1}^{\infty}\frac{1}{(1-q^n)^2} = q^{1/12}\eta(\tau)^{-2}, \tag{6}$$

where τ is a complex number such that $\text{Im}(\tau) > 0$ and $q = e^{2\pi i\tau}$. Set $\tau = i/2$. Then $q = e^{-\pi}$.

$$\sum_{n=0}^{\infty} \text{A000712}(n)e^{-\pi n} = A(e^{-\pi}) = e^{-\pi/12}\eta(i/2)^{-2}. \tag{7}$$

Using (4), $\eta(i/2)^{-2} = 2^{7/4}\pi^{3/2}/\Gamma(1/4)^2$, we obtain

$$\sum_{n=0}^{\infty} \text{A000712}(n)e^{-\pi n} = e^{-\pi/12}\frac{2^{7/4}\pi^{3/2}}{\Gamma(1/4)^2}. \tag{8}$$

Now, by Euler's reflection formula (e.g., [1, 6.1.17]),

$$\Gamma(1/4)\Gamma(3/4) = \frac{\pi}{\sin(\pi/4)} = \pi\sqrt{2}.$$

Solving for $\Gamma(1/4)$ and substituting in (8) finishes the proof. □

Theorem 3. *We have*

$$\sum_{n=0}^{\infty} \underline{\text{A004018}}(n) e^{-\pi n} = \frac{\sqrt{\pi}}{\Gamma(3/4)^2}.$$

Proof. Let $A(q) = \sum_{n=0}^{\infty} \underline{\text{A004018}}(n) q^n$ be the generating function of the sequence A004018. It is well-known that $A(q) = \vartheta_3(0,q)^2$, where $\vartheta_3(0,q)$ is the Jacobi theta function (e.g., [1, 16.27.3]). Substituting $q = e^{-\pi}$, we obtain

$$\sum_{n=0}^{\infty} \underline{\text{A004018}}(n) e^{-\pi n} = \vartheta_3(0, e^{-\pi})^2 = \left(\sum_{n \in \mathbb{Z}} e^{-\pi n^2} \right)^2. \tag{9}$$

By a formula in A175574, the right-hand side of (9) is exactly $\sqrt{\pi}/\Gamma(3/4)^2$, and the proof is complete. □

References

[1] M. Abramowitz and I. A. Stegun (eds.), *Handbook of Mathematical Functions*, 1970.

[2] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Springer, New York, 1976.

[3] S. Plouffe, Numbers in the base e^{π}, arXiv:2509.15609 [math.NT] (2025). Available at: `https://arxiv.org/abs/2509.15609`

[4] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., `https://oeis.org`.