

✓ 801

Sept 29, 1970

Dear Neil:

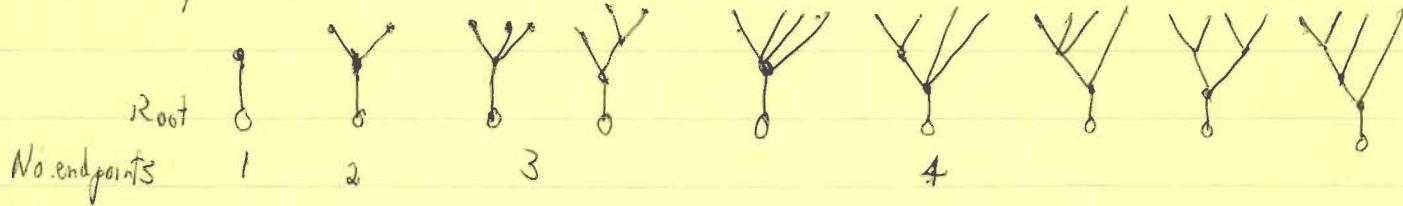
I have finally penetrated the meaning of the second tree enumeration in Cayley No. 203 (Collected Papers Vol. 3, 242-243) which has been bothering me for some time. It is

the enumeration of planted trees (by number of endpoints)

without points of degree 2 and with the root not counted,

The tree points are unlabeled.

The first few trees are



As you see the root & the stem (line at the root) may be removed, so the enumeration is that of rooted trees without points of degree two, with the exception where the root is of degree two, by no of endpoints

I also call such trees series-reduced (a branch with n lines in Series is reduced to a single line)

In your catalog you identify these numbers by "Trees by Number of Free Branches" which I find inadequate. I hope you agree

I have extended Cayley's few results as follows

n	1	2	3	4	5	6	7	8	9	10	11	12	13
B_n	1	1	2	5	12	33	90	261	766	2312	7068	21965	68954

n	14	15	16	17	18	19
B_n	218751	699534	2253676	7305788	23816743	78623602

and $B_{20} = 256738751$

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Of course this prodigious calculation occupied my whole weekend and is without benefit of any machine whatever. If you feel like using a machine it may be helpful to read the following.

Cayley gives

$$\begin{aligned} 1-x+2B(x) &= 1+x+2B_2x^2+2B_3x^3+\dots \\ &= (1-x)^{-1}(1-x^2)^{-B_2}\dots(1-x^n)^{-B_n}\dots \end{aligned}$$

with of course

$$B(x) = B_1x + B_2x^2 + \dots + B_nx^n + \dots, \quad B_1 = 1$$

And this is equivalent to (as for ^{unrooted} trees with like pts)

$$\begin{aligned} 1-x+2B(x) &= \exp [B(x) + \frac{B(x^2)}{2} + \dots + \frac{B(x^n)}{n} + \dots] \\ &= \exp \sum B_n^* \frac{x^n}{n}, \quad B_n^* = \sum_{d|n} dB_d \end{aligned}$$

$$(B_1^* = B_1, B_2^* = B_1 + 2B_2, B_3^* = B_1 + 3B_3, B_4^* = B_1 + 2B_2 + 4B_4, \dots)$$

Differentiating

$$-x+2xB'(x) = [1-x+2B(x)] \sum B_n^* x^n$$

$$-S_n + 2nB_n = B_n^* - B_{n-1}^* + 2 \sum_1^{n-1} B_k B_{n-k}^*$$

$$\text{E.g. } -1 + 2B_1 = B_1^* = B_1, \quad B_1^* = 1 \text{ and}$$

$$4B_2 = B_2^* - B_1^* + 2B_1 B_1^* = 2B_2 + 2B_1^2 \quad B_2 = 1, \quad B_2^* = 3$$

$$6B_3 = B_3^* - B_2^* + 2(B_1 B_2^* + B_2 B_1^*) = 3B_3 - 2B_2 + 8 \quad B_3 = 2, \quad B_3^* = 7$$

It is a little neater to write

$$\beta(x) = x(1-x+2B(x)) \sum B_n^* x^n, \quad \beta_1 = 1 = \beta_2^* \quad \beta_n = 2B_{n-1}, \quad n=3,4,5,\dots$$

so that

$$x\beta'(x) = \beta(x) + \beta(x) \sum B_n^* x^n$$

$$(n-1)\beta_n = \sum_1^{n-1} \beta_k B_{n-k}^* \quad (\beta_0 = B_0^* = 0)$$

Otherwise, using

$$(1-x^k)^{-1} = (1+x^k)(1+x^{2k})(1+x^{4k}) \dots (1+x^{2^{n_k}}) \dots$$

$$1-x+2B(x) = 1+x+2B_2x^2+\dots$$

$$= \prod (1-x^k)^{-B_k} = \prod (1+x^k)^{b_k}$$

$$\text{with } b_{2k} = B_{2k} + b_k$$

$$b_{2k+1} = B_{2k+1}$$

$$\text{Thus } b_1 = B_1 = 1. \text{ Then}$$

$$(1+x)(1+x^2)^{b_2} = 1+x+b_2x^2(1+x)+\dots$$

$$\text{and } 2B_2 = b_2 = B_1 + B_2 \quad B_2 = B_1 = 1$$

Next

$$(1+x)(1+x^2)^2(1+x^3)^{b_3} = (1+x+2x^2+2x^3+x^4+x^5)(1+b_3x^3+\dots)$$

$$2B_3 = 2 + b_3 = 2 + B_3 \quad B_3 = 2$$

In general if

$$B(x, N) = \underline{B_N(x)} = \prod_1^N (1+x^k)^{b_k} = \sum B_n(N)x^n$$

$$2B_n = B_{n-1} + b_n$$

so that

$$B_{2n} = B_{2n-1} + b_n$$

$$\underline{B_{2n+1} = B_{2n+1}(2n)}$$

I talked by telephone to Ed Beschler (he called me!) yesterday
and your book is going smoothly in the publication stream.

Remember me to Ann

As ever

John