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John Norris Tangan

Phone 1-617-729-3822

16 Winslow Road

Winchester, Mass. 01890

Dear N. J. A. Sloane,

...the reader perhaps a "bar" letter.

In regard to the opening pages of your 1973 book:

A Handbook of Integer Sequences

I think a serious terminological problem has arisen. First of all this is, of course, my own opinion. But also please note that Frank Harary and Edgar Palmer in their 1973 book Graphical Enumeration have (mainly) described digraphs not as "reflexive relations"

but (because no loops are allowed) as irreflexive relations. Surely they are correct. The concept of a loop is properly identified with xRx, the relation which is reflexive...
Thus we include all of the "connected" digraphs, plus all of the completely trivial digraphs, consisting only of \( n \) points. But there is some overlap. 0 0 is counted as "connected"

0 0 is not counted as connected but we want it in our set of 3 because it is trivial. Of the 16 digraphs for 3 points, 13 are connected, and one is trivial 5 0 0 3 = completely trivial

So we are counting 3 or 3

14 \times 16

14 \times 16 + 18 = 200 + 18

200 + 14 + 3 = 217, unfortunately the digraph 0 0 is counted twice, once as trivial, once as connected, thus we confuse,

1 + 3 + 14 + 200 = 218
I propose a revision of terminology which would make use of the concept of a loop. You describe symmetric relations as "graphs with loops of length 1 allowed". I think such relations, with loops of length 1 allowed, should be called symmetric reflexive relations.

Hence, by your terminology, "graphs", the relations usually described as graphical would hereafter be called symmetric graphs.

When we refer to "digraphs with loops of length 1 allowed", we should say "reflexive relations", in deference to the loops, \((xRx\), which are allowed.

More "digraphs" than are truly unrestricted, yet less symmetric...
In two-dimensional space we have 7 strip patterns for which the dimension is, strictly speaking, \( \frac{3}{2} \) or \( 1 \frac{1}{2} \) (per Coxeter), and we have 17 discrete groups of direct isometries for 2-d space. Thus each of the 7 strip patterns can multiply the 17 group (full) patterns to give us 119 imaginary unity elements, each of which has a conjugate (projectum), much as the space is 3-dimensional, with digraphs for 4 or fewer points. The numbers of (connected) digraphs for 1, 2, 3, 4 points are

\[ 1 + 2 + 13 + 199 = 215 \]

so that

\[ 1 + 1 + 1 + 2 + 1 + 13 + 1 + 199 = 219 \]

where 219 is Coxeter's number of "purely geometric" groups.
I am now in a position to set out the proposal new terminology.

**symmetric graphs**

- symmetric only (no loops)

**graphs**

- symmetric reflexive (loops allowed)

**looped digraphs**

- reflexive relations (loops allowed)

**digraphs**

- unrestricted relations (no loops)

Change some names
We now have \( 2 + 6 + 6 = 14 \)

There are 6 remaining 3 point digraphs, which are complicatedly connected but not maximally connected.

\[
\begin{align*}
\frac{1 + j - k + i(e)}{2} & \quad \frac{1 + j + k + i(e)}{2} & \quad \frac{1 - j + k - i(e)}{2} \\
\frac{1 - j + k - i(e)}{2} & \quad \frac{1 + j - k + i(e)}{2} & \quad \frac{1 + j + k - i(e)}{2} \\
\frac{1 - j - k + i(e)}{2} & \quad \frac{1 + j - k - i(e)}{2} & \quad \frac{1 - j + k + i(e)}{2}
\end{align*}
\]

These 6, plus the 218 digraphs for 4 points give us the 224, hence we have

\[ 2 + 6 + 6 + 224 = 238 \]

Since the 238 imaginary unity elements exist in 119 pairs, we note that

\[ 119 = 7(17) \]
These digraphs 1/3/16, 2/18 are extremely useful for work with algebras because if we assume an abstract entity \((+1)=1\) then \(1=+1\) and \(0=1+1=-2\) gives us the second unity element which is necessary for real algebra.

\[
1+3 = 4 = \left\{ +1, \pm \sqrt{-1} \right\}
\]

so we have the necessary units for complex (Gaussian) algebra.

\[
1+3+16 = 20
\]

so we add the units \((\pm j)\) and \((\pm k) = \pm (ij)\).

Thus the 16 digraphs for 3 points correspond to the 16 quaternion unity elements

\[
\frac{1}{2} \left( \pm 1 \pm \sqrt{-1} \pm j \pm k \right)
\]

Thus \(\pm j, \pm k, +1+3+16 = 24\)

Thus we have the 24 unity elements of quaternion algebra.
Some progress seems possible given $0$ and $\infty$ are trivial

$0 \to 0$ and $\infty \to \infty$ can be grouped with $0\cdot\infty$ and $\infty\cdot 0$ and $\infty\cdot\infty$.

![Diagram 1](image1)

By the 16 digraphs for 3 points, we find that 3 are not connected.

![Diagram 2](image2)

Whereas one is maximally connected.

![Diagram 3](image3)

So 12 remain, in 6 pairs, to complete one set of 6 we select

![Diagram 4](image4)

the latter is cyclic (clockwise)
When we come to Octave algebra it is clear that

\[ 1 + 3 + 16 + 218 = 238 \]

We can draw 238 digraphs for 4 or fewer points, so we only need add the abstract real units \( \pm 1 \), in order to have

the 240 unity elements necessary for the Octave algebra. It has always been argued that Octave algebra is non-distributive, but because the Norm of Product is equal to

the Product of Norms

but now we have a form of Octave algebra which is non-distributive, since we can draw each unity element in "graphical" form as digraphs with no loops allowed, and points unlabelled.
Thus an imaginary unity element not only

corresponds to a digraph, it is a digraph,

for 4 or fewer points.

\[
1 = \sqrt{-1} = i
\]

\[
3 = \begin{pmatrix}
\hat{1} \\
\hat{k} \\
\hat{e}
\end{pmatrix}
\]

\[
16 = -i, -j, -k, -e,
\]

\[
\pm i(e), \pm j(e), \pm k(e)
\]

\[
\begin{array}{c}
4 \\
+6 \\
+10 \\
+6 \\
16
\end{array}
\]

plus 6 q the 224

described by Coxeter in his classical (1946) paper on
octaves. The other 218 of the 224, require
4 points for digraphical representation.

Ideally we would wish for a sum

\[
2 + 6 + 6 + 224 = 238
\]

instead of

\[
1 + 3 + 16 + 218
\]

\[
2 = \pm \sqrt{-1}; \quad 6 = \begin{pmatrix}
\pm i \\
\pm e \\
\pm k
\end{pmatrix}; \quad 6 = \begin{pmatrix}
\pm i(e) \\
\pm j(e) \\
\pm k(e)
\end{pmatrix}
\]
By restricting the number of points to powers of 2, 1, 2, and 4, we get 

\[ 1 + 3 + 218 = 222 \]

which is the number of crystal classes or molecular "point groups" in 4-d space corresponding to the 32 such known for 3-d space.

I would predict that in 4-d space the number of atomic space groups would be the number of digraphs (no loops) which are possible for 5 (or perhaps 8) points:

For 5 points we have 9608 digraphs.

For 8 points we have 1,793,359,142 digraphs.

The analogous number for 3-d space is 218, which is close enough to the 219 sum computed by Coxeter.
\[ 0 = \sqrt{-1} \]
\[ 0 \cdot 0 = -\sqrt{-1} \]
\[ 0 \rightarrow 0 = \frac{1 + \sqrt{-1}}{\sqrt{2}} = \frac{1 + 0}{\sqrt{2}} \]
\[ 0 \leftarrow 0 = \frac{1 - \sqrt{-1}}{\sqrt{2}} = \frac{1 + 0}{\sqrt{2}} = \frac{1 - 0}{\sqrt{2}} \]

It is only in the case of the quaternion algebra that we need 4 abstract units, plus the 20 digraphs for 3 or fewer points.

In the case of the octonion algebra we need only the usual two abstract units, 1 and the 238 digraphs for 4 or fewer points. Thus each of the 238 digraphs is an imaginary unity element of octonion algebra.
It is customary to say that we have
65 + 165 = 230 atomic space groups
in 3-D space. But, as Coxeter has pointed out,
only 22 of the 65 exist in H enantiomorph pairs
the proper sum from the stereomeric gene geometry
is 54 + 165 = 219.

Thus if we have digraphs for any (n^2)
number of points, n = 1, 2, then
we have 1 + 218 = 219, as required
We should have at the 218 digraphs
(They are displayed in the 1969 Library book on
Graph Theory)
and see if we can select out 53.

1 + 53 + 165 = 1 + 218 = 219

Thus the "digraph" for a single point
corresponds to a rotation or a translation
of a "reflection" but not to a "reflective reflection"
Since 6 circles (2-D sphere) can be lattice packed about a central equal circle in 2-D space ("6 pennies on a table top")
I have always argued that

\[
\frac{1 + \sqrt{-1}}{\sqrt{2}} \quad \text{and} \quad \frac{1 - \sqrt{-1}}{\sqrt{2}}
\]

are "units" of Gaussian (complex) Algebra

\[
\frac{-1 + \sqrt{-1}}{\sqrt{2}} \quad \frac{-1 - \sqrt{-1}}{\sqrt{2}}
\]

\[
\frac{+1 + \sqrt{-1}}{\sqrt{2}} \quad \frac{+1 - \sqrt{-1}}{\sqrt{2}}
\]

along with \( \pm 1 \) and \( \pm \sqrt{-1} \)

This would be \( \pm 1 \) plus 4 digraphs

\[
\begin{cases}
0 
& \rightarrow \frac{\sqrt{-1}}{\sqrt{2}} \\
0 
& \rightarrow \frac{-\sqrt{-1}}{\sqrt{2}} \\
\circ 
& \rightarrow \frac{1 + \sqrt{-1}}{\sqrt{2}} \\
\circ 
& \rightarrow \frac{1 - \sqrt{-1}}{\sqrt{2}}
\end{cases}
\]
No doubt clarification is needed in regard to not only “di-graphs” but also in regard to reflections as operators and its reflexivity as an (abstract) relation. Despite numerous books on reflections, including high school texts on the geometry of reflections, the concept of a reflection in relation to the concept of reflexitivity remains less clear than it should be.

"x R x" implies a loop or a rotation,

x R x implies algebraically:

\[ x (x^{-1} x) = x = x (x x^{-1}) = (x x^{-1}) x = (x^{-1} x) x \]

\[ x + (x + (-x)) = x + 0 = x \]

\[ x \times (x \times x^{-1}) = x \times 1 = x \]

\[ x + y \in \{ x (y + z) - (x y + x z) \}^3 + x = x \]

\[ x + \{ x (y + z) - (x y + x z) \}^3 = x \]
\[ x + \left\{(x + yz) - (x + y)(x + z)\right\} = x \]

\[ y = 1 \text{ and } y = \pm \sqrt{-1}, \quad z = \pm \sqrt{-1} \]

\[ X \times \left\{ \frac{x + yz}{(x + y)(x + z)} \right\} = x \]

\[ X \times \left\{ \frac{x(y + z)}{xy + xz} \right\} = x \]

\[(1 + \sqrt{-1})^2 = +\sqrt{-1} \]

\[(1 - \sqrt{-1})^2 = -\sqrt{-1} \]

\[\left(\frac{1 + \sqrt{-1}}{\sqrt{2}}\right)(\frac{1 - \sqrt{-1}}{\sqrt{2}}) = 1 \]

\[\left(\frac{1 + \sqrt{-1}}{\sqrt{-2}}\right)(\frac{1 - \sqrt{-1}}{\sqrt{-2}}) = -1 \]
The counting $1 + 3 + 14 + 200$

includes all trivial digraphs $(0, 0, 0, 0, 0, 0)$

plus all connected digraphs. Probably the list of connected digraphs should not include $503$.

So $0 + 2 + 13 + 199 = 214$

$1 + 1 + 1 + 1 = \frac{9}{218}$

In that case we can match

which 218 digraphs with are trivial vs connected

with 218 digraphs (connected or not) for 4 points

Note that if we have an even number of points, the connected digraphs are $2 + 199 = 31 + 90 + 65 + 15$

\[
\frac{4}{6} \sum_{b=1}^{4} \left( \frac{b+1}{-1} \sum_{m=0}^{b} \frac{(b+1-m)^6}{(b+1-m)!} \right)
\]

\[
= 5(6.2) + 5(6.3) + 5(6.4) + 5(6.5)
\]
Clearly we have $2(2 + 10 + 10^4) = 232$.

The other eight are abstract, such as

$$+1, +\sqrt{-1}, +j, +k, +e, +\text{i}(e), -\text{j}(e) +\text{k}(e)$$

The eight permutations two things ($+$ and $-$) taken three at a time.

Thus, with 3 moments or points minimally required to establish oscillation or curvature, we have 8 forms (abstract) in 4 parts.

$$
\begin{array}{c}
\text{+++} \\
\text{---} \\
\end{array}
\quad
\begin{array}{c}
\text{+++} \\
\text{---} \\
\end{array}
\quad
\begin{array}{c}
\text{+++} \\
\text{---} \\
\end{array}
\quad
\begin{array}{c}
\text{+++} \\
\text{---} \\
\end{array}

\text{Axis}

\text{+++} = +1
\text{---} = -e
\text{+++} = +j
\text{---} = -j
\text{+++} = +k
\text{---} = -k
\text{+++} = +e
\text{---} = -e
$

\text{Cheers,}
\text{John}
$
These are the non-trivial, maximum disjoint subsets for \( 6 = 3! \) labelled elements, distributed into unlabelled (symmetric, transitive, reflexive) subsets.

Digraphs are not symmetric, not transitive, not reflexive.

So we see that, subject to conditions specified above, we can remove two labels from all 6 of a set of \( 3! = 6 \) points and remove 2 of the 4 points and then permit digraphical (connected) RELATIONS, which turn out to be UNRESTRICTED.

Neither symmetric nor transitive nor reflexive—(no loops allowed) — for 4 or fewer (even) numbers of points, if the number is odd, we get the 13 lattice paired spheres, in digraphical form all 13 digraphs are connected.
Each such looped digraph represents a "pair" (a conjugate pair) of algebraic elements, because every left loop (counter-clockwise) is similar to a right loop (clockwise).

For 2 points the 10 relations have

10 automorphisms

\[
\begin{aligned}
0 \rightarrow 0 & \quad \cong 0 \\
0 \rightarrow 0 & \quad 0 \\
0 \rightarrow 0 & \quad 0 \\
0 \rightarrow 0 & \quad 0 \\
0 \rightarrow 0 & \quad 0 \\
0 \rightarrow 0 & \quad 0 \\
0 \rightarrow 0 & \quad 0 \\
0 \rightarrow 0 & \quad 0 \\
0 \rightarrow 0 & \quad 0 \\
0 \rightarrow 0 & \quad 0
\end{aligned}
\]
\[ 2 + 13 = 15 \text{ equals } 3-1 \text{ simplex (tetrahedron)} \]

including the center.

Now if we think of just 2,32 of the 238 digraphs or unity elements, and if we think of them as existing in 116 pairs, then each such pair is represented by two reflexive relations (loops of length one allowed) for 1, 2, 3 unlabelled points

\[ 2 + 10 + 104 = 116 \]

The 8 extra unity \[ 8 + 232 = 240 = 120 \text{ pairs} \]

exist in 4 pairs, as shown above

\[
\begin{align*}
\pm 1 & \quad \pm \sqrt{-1} \\
\pm \frac{1 \pm \sqrt{-1}}{\sqrt{2}} & \quad \frac{-1 \pm \sqrt{-1}}{2}
\end{align*}
\]
So we have seen how a theory of digraphs, with or without loops, can be added to a theory of abstract algebraic "elements" such as +1,

to give us the 240 unity elements of the (Deveci-Grawes-Congley) "Octave" Algebra,

whilst the number of unlabelled points decreases from 4 to 3,

\[ 1 + 2 + 10 = 13 \]

\[ 2 + 4 + 20 = 26 \]

\[ 2 + 2 + 10 + 2(1 + 2 + 10 + 104) = 248 \]

\[ (10 - 2) + 2(2 + 10 + 104) = 240 \]

The idea that a unity element in an algebra should be a digraph (looped or not) in a few points adds great geometric clarity to algebra.
April 13, 1975

4-13-75

Dr. N.J.A. Sloane,

The terminology used in your *Handbook of Integer Sequences* (1973) is not only in conflict with much texts on graph theory as that of Harary but also is in conflict with common sense. For example, Harary suggests that digraphs are irreflexive rather than reflexive, whereas we refer to digraphs which do not include loops, i.e., to your series 1, 3, 16, 218, 9608, ... .

After all, since a reflexive relation relates \( x \) to \( x \): i.e., \( xRx \), it is a loop which most directly relates \( x \) to \( x \).

The figure \( \xrightarrow{0} \) allows \( x \) to indirectly relate to \( x \), but only via \( y \).
The series:

\[ 1 \ 3 \ 9 \ 33 \ 139 \ 718 \ 4535 \]
\[ 1 \ 3 \ 7 \ 18 \ 52 \ 208 \ 1252 \]
\[ 2 \ 15 \ 87 \ 510 \ 328 \ 3 \]

Suggest that the graphs for 5 or fewer points form a subset \( \{1, 5, 23, 3\} \) of the possible topologies for 5 points.

\[ 1 \ 4 \ 13 \ 46 \ 185 \ 903 \ 5438 \]
\[ 1 \ 3 \ 7 \ 18 \ 52 \ 208 \ 1252 \]
\[ 1 \ 6 \ 28 \ 133 \ 695 \ 418 \ 8 \]
\[ 7 \ 35 \ 168 \ 863 \ 504 \ 9 \]

\[ \frac{1}{3!} \ 1 \ 5 \ 4 \ 8 \ 4 \ 7 \ + \ 7 + 2 \]
\[ \frac{5}{5!} \ 2 \ 1 \ 6 \ 1 \ 5 \ \frac{6}{6!} \ 1 \ 4 \ 2 \ 4 \ 3 \ + \ 7 \]

\[ 2 \ 8 + 133 + 695 + 4 \ 86 = 2 \ 1 + 7 \ = 5042 \]

\[ 101 \ 456 \ 5042 \]
Thus the proper word and concept to describe
$0 \rightarrow 0 \iff xRy \Rightarrow yRx$ which is, asymmetry,
and as all seem to agree, the symmetric relation
I have no quarrel with the word and concept of
transitivity to describe
$xRy \text{ and } yRz \Rightarrow xRz$

but all this simpifies to me, graphically,
is that if there is a line from $x$ to $y$
\[ x \rightarrow y \]
\[ 0 \rightarrow 0 = 0 \rightarrow 0 \]
\[ x \rightarrow y \]
\[ \text{and a line from } y \to z \]

that this implies a line from $x$ to $z$
After much work I have concluded that we can identify series with RELATIONS implied in a rigorous and unambiguous manner, for unlabeled (undistinguished) points.

<table>
<thead>
<tr>
<th>2, 10, 104, 3044, 291, 968</th>
<th>symmetric reflexive (looped digraphs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 3, 9, 33, 139</td>
<td>symmetric transitive</td>
</tr>
<tr>
<td>540</td>
<td>transitive reflexive</td>
</tr>
<tr>
<td>273</td>
<td></td>
</tr>
<tr>
<td>1, 3, 16, 218, 9605</td>
<td>symmetric, digraph</td>
</tr>
<tr>
<td>88</td>
<td></td>
</tr>
<tr>
<td>1, 2, 4, 11, 34</td>
<td>unrestricted</td>
</tr>
<tr>
<td>1, 4, 13, 46, 185, 903, 5438</td>
<td>unrestricted</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\{ & 1, 3, 9, 33, 139, 718, 4535 \\
\{ & 1, 2, 4, 11, 34, 156, 1044 \\
& 1, 5, 22, 105, 562, 3491 \\
& 1, 6, 28, 133, 695, 4186 \\
-1 = 2 & 9-6, 33-28, 139-133, 718-695, 4535-1044 = 349 \\
& \frac{143+733}{2} = 349 < \frac{745}{349} < \frac{645}{2}
\end{align*}
\]
3.

\[ x 
\rightarrow \quad y 
\quad + \sqrt{-1} 
\rightarrow 
-\sqrt{-1} 
\]

The operation which relates \( x \) and \( y \) because \( \pm \sqrt{-1} \) are not only reverse but also inverse in respect to each other.

When this transitivity exists, it is exemplified by associativity (metric transitivity), and

\[
x y + z = (x + z)(y + z) = \frac{(xy + yz)}{x} \\
(x + y)z = (xz) + (yz) = \frac{xy - yz}{x}
\]

Suppose \( x = 1 \); \( y = \sqrt{-1} \); \( z = \sqrt{0} \), such that \( z^2 = 0 \).

Then \( xy = yx \) \( ; y^2 = 2x \) \( ; xz = zx \)

\[
x(y + z) = xy + xz = y + z
\]

\[
x(xy) = (xy)z = y \quad x(yz) = y = (yz)x
\]

\[
(xy)z = x(yz) = (yz)x
\]
\[ 4 + 6 + 21 + 165 = 196 = 14^2 \]

\[ 8 \cdot 12 + 42 + 165 = 227 \]

\[ 5 (2) \{3 + 8 + 12 \} \]

Consider 5 operations, 3 or fewer points

\[ 5 (4 + 42) = 230 \]

\[ 4 + 42 + 165 \]

\[ \frac{46}{11} = \frac{211}{211} \]

Translations, rotations, reflections

Plus 2 types of screw displacement

5 (zero points + 2 points)
\[ e = \sqrt{0} \]
\[ i = \sqrt{-1} \]
\[ 1 = \sqrt{1} \]

\[(3 + i)(3 + i) = 0 + 2i(3) - 1\]
\[= 2(2) - 1\]
\[= -1 + i(2) = (i+3)(i+3)\]

\[(1 + 1)(1 + 3) = 0 + 2 + 1 = 3\]
\[(1 + i)(1 + i) = 2i\]
\[(i + 1)(i + 1)\]

\[(i + 3)(i + 3) = -1 + 2i + 0\]

\[2 + i(4) = 4i + 2 = 0 2i + 2i + 2i \]
So that only the case of two points, dimensions \((\pi)\) is excluded, whereas dimensions \((\pi^1)\), \(0\) and \(\pi^2\) are included.

The 4 setting rules are:

I. Graphical set \(\rightarrow\) line \(\rightarrow\) broken line
II. Dipraphical set \(\rightarrow\) directed line = dual \(\leftrightarrow\)
III. Looped graph \(\circ\) circle (neutral rotation)
IV. Looped dipraph \(\circ\) directed circumference (directed rotation)

It is possible to get 154 relations, thus:

for zero, one and three points

and 165 relations for two and three points.

\[165 + 6 + 4 = 175 = \{\rho(15) - 1\}^2 = \{\rho(10+3+2)\}^2 - 1\]

\[154 + (6+3+2) = 165 = 4 + 6 + 14 + 2 + 3 + \theta\]

The 10 maximal relations on 2 points can be explained...
\[(2 + i4) = \sqrt{(2+i4)(2+i4)}\]
\[= \sqrt{4 + 16i - 16}\]
\[= \sqrt{-12 + 16i}\]

\[(R + iR + \varepsilon R)(S+iS+\varepsilon S)\]
\[RS + iBS + \varepsilon RS\]
\[-RS + iRS]\]
\[i\varepsilon RS + 0 \varepsilon iRS + \varepsilon RS\]
\[i(4RS) + RS = RS + i(4RS)\]

\[(A + iB + \varepsilon C)(x + iy + \varepsilon z) = \]
\[AX + iAY + \varepsilon AZ\]
\[-BY + BX\]
\[i\varepsilon BZ + \varepsilon CX\]
\[+ 0 \varepsilon iCY\]
\[i(AY + BX + BZ + CY) + \varepsilon (AZ + CX) = \sqrt{(-\varepsilon)}(X) + \sqrt{0}\]
The arrow indicates whether the limit (being zero) is being approached from a positive (clockwise) or negative (counterclockwise) direction.

Thus the theory of discrete groups of isometries in 3-d space is considerably simplified when we draw precise zeroes or limit points for each group.

In the simplest case, \( 11(14) = 154 \),

we see that if we have relations defined (abstractly or concretely) for zero points and 4 sets of rules, then we get

\[ 4 + 6 + 144 = 154 = 11(14) \]
Clearly
\[
(A + iB + Ec)(x + iy + c^2) \Rightarrow \{- (Ay + B(y+z) + cy)^2
\]
\[= - (Ay + B(y+z) + cy)^2
\]
\[= - [\{ y(A+c) + B(y+z) \}^2]
\]

We don't really have transitivity unless we have 3 units which behave in 3 distinct trajectories. A unit \( u \) is the square root of any negative number of the form
\[- N = - \{ B^2 \}
\]

and a unit \( e \) is the square root of any form of zero, such as
\[ o(N) = o(B^2) \]
where $384$ is the maximum number of linear spaces in reference to a single 
(flower) figure in the set given on p 16
of your book. Coxeter's ideas can
thus be refined still further, we can argue
that we have just $27 + 165 = 192$ pure groups
(Geometrically) when we speak of discrete groups of
isometries in 3-d space

\[
\begin{align*}
27 &= 3 \times 9 \\
165 &= 3 \times 55 \\
192 &= 6 \times 186 = 3 \times 27(7) = 6 + 6(31)
\end{align*}
\]

These can be regarded, also, as $384$ discrete groups
in $192$ enantiomorphie pairs. When a point is
represented by an arrow, it is a limit point.
A dual unit \( A + iB \)

is a square root of \( (A + iB)(A + iB) = A^2 - B^2 + i(2AB) \)

\[ = C + i(2AB) \]

\[ C = A^2 - B^2 \]

\[ (A + B)(A - B) \]

\[ = (A + B)(A + B) + i(2AB) \]

Thus, \( A + iB \) can be a square root of any real number \( \sqrt{C = (A + B)(A - B)} \)

Plus \( i \sqrt{2AB} \)

\[ (5 + i10)^2 = -75 + i(100) \]

\[ (10 + i5)^2 = +75 + i(100) \]

The real part of \((A + iB)^2\) is negative or positive depending on whether \( B > A \) or \( A > B \)
In the case of the 10 pairs of hooped digraph relations, I did not draw all the details, but I am sure it is clear as to how the symbolism and structure is reversed, giving

\[
\frac{(2^2)^2 + (2^2)}{(2^2)}
\]

Draw 54 groups in just 2 points as 1 point attained in this manner.

When we consider the 165 groups in 2 or 3 points, the 21 referring to 2 points are written with an extra (isolated) point on each background.

If we paired the 165 we would get

\[
165 + 165 + 54 = 384 = 2(4)6(8) = 3(27)
\]
\[(4 + \epsilon B)^2 = (4 + \epsilon B)(A + \epsilon B) = A^2 + 2AB + \epsilon^2 \{AB\}^2\]

Finally:

\[(A + iB + \epsilon C)\text{ is the square root of} \]

\[\sqrt{A^2 + iAB + AC + B^2 - B^2 + iBC + \epsilon CB + \epsilon^2 C^2}\]

\[(A + iB + \epsilon C)(A + iB + \epsilon C) = A^2 + iAB + AC + B^2 - B^2 + iBC + \epsilon CB + \epsilon^2 C^2\]

\[A + iB + \epsilon C = \sqrt{A^2 - B^2 + 2AC + i\{2AB + 2BC\}}\]

This is complicated but can be any real number plus \[i\{2AB + 2BC\}\]
I will draw the 54 relations in 27 pairs

- Translation
- Rotation
- Screw displacement
- Glide reflection

There are six pairs of one point relations.

The 24 pairs of two point relations are

2 graph + 2
3 digraph + 3
6 looped graph + 6
10 looped digraph + 10

Reverse all of the
symbolism & structure
That is \((A + iB + ceC)\) is the square root of a complex number, just as is \((A + iB)\), whereas \((A + iB)\) is the square root of the real number \(A^2 + 2(AB)\).

Note that \((A + iB + ceC)\) is the square root of a complex number for which the imaginary part is even, not odd!!!

Note that

\[ A^2 + B^2 + 2AC + i \left\{ 2B(A + c) \right\}^\frac{3}{2} \]

reaches a wider class of real numbers.
we have 22 which are known to exist in
11 enantiomorphous pairs, so that Coxeter
argues in favor of just 54 such groups.
It is thus easy for the rest of us to see
that these 54 groups exist in 27 pairs,
and we allow each of the 6 relations on
a single point to have an analogy which
is, however, a relation between 2 points.
This leaves 15 relations between 2 points
which have no analogues for relations on a single point
6 + 6 + 6 + 6 + 15 + 15 = 54.
That is, we simply allow each of 27 relations
to have an enantiomorphous relation,
The maximum number of relations amongst
3 unlabelled (undistinguished points) is

\[ 104 = 2^9 \times 3^3 = 2^3 \times 3^3 = 2^3 (1 + 2 + 10) \]

\[ = 2^3 \left( u(0) + u(1) + u(2) \right) \]

\[ = C(3) \left\{ u(0) + u(1) + u(2) \right\} \]

\[ u(0) = 1 \]

The maximum number of relations amongst
3 unlabelled points is 4

\[
\begin{array}{cccc}
4 & 16 & 20 & 104 \\
12 & 41 & 84 & 88 \\
11 & 218 & 90 & 3044 \\
11 & 90 & 218 & 3044 \\
79 & 128 & 2824 \\
\end{array}
\]
If we allow 2 or 3 points, but not none,
our total becomes \( 21 + 144 = 165 = 3 \times 5 \times 11 \)

If we allow 1 or 2 points but not 3 our total becomes 27.

It is customary in discussing the
theory of symmetry groups in 3-dimensional space to refer
to 230 discrete groups, of which 165
include not only direct isometries but also
opposite (reflection) isometries. Thus these 165
groups are called mixed symmetry groups, by which
we mean that they involve relations between
2 or 3 points. There are 65 discrete groups
of direct isometries which are generally
recognized, but amongst these 65
<table>
<thead>
<tr>
<th>Operation</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$34 \div 544 \div 9608 \div 291468$</td>
<td>$\frac{9608}{9608}$</td>
</tr>
<tr>
<td>$510 \div 9064 \div 282360$</td>
<td>$\frac{282360}{9064}$</td>
</tr>
<tr>
<td>$255 \div 4532 \div 141180$</td>
<td>$\frac{141180}{4532}$</td>
</tr>
<tr>
<td>$15(17) \div 2266 \div 47060$</td>
<td>$\frac{47060}{2266}$</td>
</tr>
<tr>
<td>$7(5)17 \div 1133 \div 11(103)$</td>
<td>$\frac{11(103)}{1133}$</td>
</tr>
<tr>
<td>$166 \div 171 \div 282360 \div 1660$</td>
<td>$\frac{1660}{282360}$</td>
</tr>
<tr>
<td>$272 \div 103 \div 171 \div 282360 \div 142$</td>
<td>$\frac{142}{282360}$</td>
</tr>
<tr>
<td>$141180 \div 70590$</td>
<td>$\frac{70590}{141180}$</td>
</tr>
<tr>
<td>$3135295 \div 11765$</td>
<td>$\frac{11765}{3135295}$</td>
</tr>
<tr>
<td>$13 \times 2353 \div 181$</td>
<td>$\frac{2353}{181}$</td>
</tr>
<tr>
<td>$13^3 + 13^2 - 13(3)2^3(5)$</td>
<td>$13^3 + 13^2 - 13(3)2^3(5)$</td>
</tr>
<tr>
<td>$23610 \div 26310$</td>
<td>$\frac{26310}{23610}$</td>
</tr>
</tbody>
</table>
consequently we run up a total of 3363 relations. For other polyhedra its analogous product would be

\[ 59(62) = 3487 \]

\[ 22(26) = 52 = 572 \]

\[ 23(26) = 598 \]

\[ 11(14) = 154 = 2 \phi(12) \]

Note 6 21 144
27 171
\[ 26 - 154 \]
\[ = 17 \]

A cube or icosahedron, with a center, is the sum of possible relations for 2r fewer points
With just one point and 4 sets of rules
we have 6 possible relations

With two points and 4 sets of rules we have
21 possible relations

With three points and 4 sets of rules we have
144 possible relations

With 4 points and 4 sets of rules we have
3363 possible relations

\[
6 + 21 + 144 + 3363 = 3539
\]

\[
213534 \div 3 = 2 (9) 14 (31)
\]

\[
1767 \div 3 = 589
\]

\[
19 \div 31 = 9 \text{ R } 20
\]

\[
57 \div 62 = 3 \text{ R } 1
\]
Clearly this set of all possible relations amongst 4 or fewer points can be described as follows. Consider a dodecahedron, which has:

\[20 \text{ vertices} + 30 \text{ edges} + 12 \text{ faces} = 62\]

i.e. 62 components. Since each face is a pentagon (forming 5 edges) we think of the face being preserved but its edges not counted (hence remaining invariant).

Varying thereafter we have 20 vertices, +25 edges + 12 faces, hence 57 components varying. Each varying component is allowed to become a transform over an entire set of 62.
5049 = 3(1683) = 9(561) = 27(187) = 3³(11)17

5049 = \((0+0!)! + (0+1!)! + (1+2!) + (1+3!)!\)!

\((1+0)! + (1+1)! + (1+2)! + (1+(0+1+2)!))!

\((1+0)! + (1+(0+1)! + (1+(0+2)! + (1+(0+1+2)!))

Three things 0, 1, 2 and if we combine them 3 at a time we get only zero. If we combine them 2 at a time we get the combination 1+2 (1st n), because, we use only zero pairing 0, 1 and 0, 2.

When however, we combine 3 at a time, we get 0+1+2 = 3 which is the same result as 1+2 as a pairing, the only unlimited pairing.

The requirement is that we are to use only the 4 or 8 possible combinations of 0, 1, and 2 which include zero.
For 6 or fewer points, the number of connected graphs (143) is precisely equal to the number of geometries for 7 or fewer points (143) provided the number of points is larger than one.

\[ G(7) = C_7^2 = 143 = \sum_{i=1}^{7} G(i) = \sum_{i=1}^{6} C_6^i G(i) \]
Saying it another way, we omit \(1, 2, 1+2\), and the empty-set all of those combinations \(c \cup 4\) which don't explicitly display \(a \in 6 = 0\).

The sum must have 2 or more parts if we are to have the factorial of it.

\[
\begin{align*}
(1+5,0) & = (1+0)! = 1! \\
(1+5,1) & = (1+(0+1))! = (1+1)! = 2! \\
(1+5,2) & = (1+(0+2))! = (1+2)! = 3! \\
(1+5,3) & = (1+(0+4+2)) = (1+3)! = 7!
\end{align*}
\]

\[1+5079 = 5050 = \frac{100^2 + 100}{2} = \sum_{b=0}^{3} (1+b)!\]

\[100 = 5! - 4! + 2! + 1! + 0!\]

\[= 5! - 4! + 2! + 1! + 0! - 2! 3 - 5! 4! 2! 3\]

Some regular or semi-regular pattern exists.

In this computer, a series
It seems worthwhile to make a serious effort to match geometries for 5+1 points with graphs for 5 points; that is we should seek some compromise to make the total come out precisely. In the case of connected graphs:

\[1 + 1 + 2 + 6 + 21 + 112 + 853\]
\[1 \ 2 \ 4 \ 10 \ 31 \ 143 \ 996\]
\[2 \ 4 \ \underline{11} \ \underline{34} \ \underline{156} \ \underline{1044} \ \underline{48}\]
\[= 3 (3+13)\]

The total number of connected graphs for 5 or fewer points is only slightly less than this number of graphs for just 5 points.
\[
\begin{align*}
\lambda &= 1 + \sqrt{5} \\
\beta &= 1 - \sqrt{5} \\
\alpha - \beta &= \sqrt{5} \\
\alpha &= \frac{1 + \alpha - \beta}{1 + \alpha + \beta} \\
\beta &= \frac{1 - \alpha - \beta}{1 + \alpha + \beta} \\
\lambda &= \frac{\alpha + \beta + \alpha - \beta}{\alpha + \beta + \alpha + \beta} \\
\beta &= \frac{\alpha + \beta - \alpha - \beta}{\alpha + \beta + \alpha + \beta}
\end{align*}
\]
Also if $g = 5$ then the total number of graphs plus points for 5 or fewer points is

$T(5)$ i.e., the number of topologies for 5 points

1 1 2 4 11 34 156 1044 12346
1 1 2 4 5 26 101 950 11051
2 8

$55 = 11051

94 = F(11) + F(5)$

The number of graphs for 5 points is always

$g(5) < g(5+1)$, where $g(5+1)$ is the number of Geometries for $(5+1)$ points

1 1 2 4 19 34 156 1044 12346
1 1 2 4 9 26
1 1 2 6 21 112 853 11117
1 2 3 5 11 32

$123 = 55

123 = 86

11117 = 97

Geometries for $(5+1)$ points are similar to connected graphs for 5 points
For those who still think that the simplest graphs are symmetric reflexive, the pattern of comparison between transitive reflexive and symmetric reflexive will be considered as a pair rejection pattern comparison, but I regard the simplest graphs as neither symmetric nor reflexive. They are simply topologies which have become not also only non-reflexive but intransitive. In particular, they have become intransitive. I have some doubts as to topologies being reflexive (even) because they don’t have loops, whereas loops would empty $x \times x$ which is reflexivity.
Clearly, the total number of f(5) and graphs plus points is similar to the

Total number of possible geometries for \((n+2)\) points
if \(n \leq 5\). If \(n = 6\) then this sum is \(\frac{1}{2} G(6)\)

\[ G = 249 \ 26 \ 101 \ 950 \]
\[ S + AS = 249 \ 27 \ 97 \ 2(474) + 2 \]

8-d space 950 geometries 474 A + AS for 6 ≤ n ≤ 6
7-d space 101 " " 97 A + AS for n = 5
6-d space 26 " " 27 A + AS for n = 4
5-d space 9 " " 9 A = 3
4-d space 4 " " 4 A + AS n = 2
3-d space 2 " " n ≤ 1

450 = 2 + 2 3
101 = 4 + 3
26 = -1 + 3
\[ \{ q = 3 \} \\ \{ s = 3 \} \]
At most the topologies are transitive and a metric topology is possessed of metric transitivity, i.e., associativity — I cannot accept the suggestion that topologies are reflexive. If they are not very transitive then they are symmetric transitive.

Addition is, for example, symmetric transitive (because addition is not "distributive" i.e., reflexive with respect to multiplication — Topology is only non-metric in the sense of being based upon an addition, an accumulation, which is not distributive with respect to multiplication.
The one thing which is clear is that graphs must meet again soon, to agree upon definitions of the meanings of geometrical representations of such RELATIONS as symmetry, transitivity, and reflexivity.

\[\begin{array}{cccccccc}
1 & 2 & 4 & 11 & 34 & 156 & 1049 & 12346 \\
\hline
1 & 2 & 5 & 16 & 63 & 378 & 2045 \\
2 & 4 & 9 & 27 & 97 & 474 & 3089 \\
2 & 6 & 15 & 42 & (139) & 613 & 3702 \\
(2) & 3 & 9 & 33 & & & 4535 \\
1 & 4 & 13 & 46 & 185 & 903 & 5438 \\
\end{array}\]

\[1+3=4\]
\[4+9=13\]
\[2+4+27=33\]

\[0 \quad 0 \quad 1 \quad 5 \quad 29 \quad 162 \quad 1001 \quad =6-3-2\]
\[\begin{array}{cccccccc}
1 & 6 & 35 & 197 & 1198 \\
\hline
4 & -12 & -46 & -185 & -1252 & (-54) \\
34 & 156 & 1044 & -74 & 41 & 154 \\
\end{array}\]
143 is the number of 7th powers required to represent any number as a sum of 7th powers or to represent any number as a sum of

1 first power
4 squares
9 cubes
19 bi-quadrates
33 bi-cuboids
70 fifth powers
143 sixth powers

It is therefore fascinating that if we have more than 1 point but less than 8, the total number of possible geometries is 143, which is also the number of connected graphs if the number of points is 6 or fewer: 6, 5, 4, 3, 2, 1.

We have plenty of questions to raise, still, about representations by figures of logical relations.
of 16 proposed digraphs, only 1 has the property that \( xRy \Rightarrow yRx \).

Can logical relations exist even when they cannot be rigorously represented in graphical form?

The various figures are simply partitions which build up to the full figure with a number of lines of \( (p) = l \) such that \( l > p \), which represents all of the possibilities for logical relations on \( p \) points, according to 1 of 4, 5, or 6 sets of rules.
Mr. J. N. Tangen  
16 Winslow Road  
Winchester, Massachusetts 02138

Dear Mr. Tangen:

I am sorry it has taken me so long to reply to your two very interesting letters. However, for the last six months I have been working on different problems altogether, and have had very little time for the Handbook of Integer Sequences.

I greatly appreciate the time and effort you have put into those letters, and I shall try to reply to them as best I can.

Concerning the definitions of graphs and relations. I tried to follow standard terminology, which of course is not always logically correct, and you are right in pointing this out. But I do not altogether agree with your proposed changes. For example, you propose that sequence 646 be called graphs. This conflicts with the almost universal convention in graph theory, that a graph should contain no loops. When the time comes for a second printing (or edition) of the book, I shall re-examine these names more carefully in the light of your letter.

Your discussion of the relationship between the sequence 1,3,16,218,... and various algebras is extremely interesting, although I am not sure I completely understand what you are saying. Have you written to Coxeter about this?

(Later) I'm sorry, although I have tried once more to follow your argument, but I am afraid it is over my head.

It is clear, is it not, that the number of relations with any given property in which every point is reflexive
is the same as the member with the same property in which every node is irreflexive? For in the first case there is a loop at every point, while in the second case there is a loop at no point. So the numbers are equal. It is for this reason that I did not pay too much attention to the distinction between reflexive and irreflexive relations.

I thank you again for taking the trouble to write to me.

Yours sincerely,

N. J. A. Sloane

P.S. Perhaps you will find the enclosed amusing.

Enc.
As above
Dear [Name],

Due to the long delay it happens I am now back in Oxford again, having left your "Handbook" in Cambridge, Mass. "study-studio", my wife and I have now the necessity of maintaining not only this house in Oxford but also our studio apartment in Mass. I am very sure that I can correct these names & labelling for you, but it will take time away from my regular work and it will involve some expenses for me. This work is sufficiently important so that your company should be willing to authorize a modest outlay (a total of $100). When I shall hear from you or your company, I'll get a check for $100. I will feel fully obliged to drop what I am doing — my current work is underfunded — and get on with the correction of your graph names, I refer only to corrections of the graph names, other ideas which I have communicated to you are sufficiently speculative so that contract work is not justified. Whenever I wrote to you about #646 I may possibly have been wrong, but that was many months ago.

I have great admiration for Harold Scott MacDonald Coxeter, but he is age 69. I won't be writing him often on these matters, not would I expect him often to reply. The question is not one of authority but experience.
As O. Wilde once said about (law) —
the life of mathematics has not been logic but experience. In recent years, and in recent months, my work in the area where Coxeter is experienced has gone far beyond the work of Coxeter or anyone else. But I never would have ventured, this year, to do the graph names, if my wife had not decided to forward your letter to me.

I am not precisely looking for extra work — as I am overwhelmed with work — but my integrity requires me to follow up on this naming project for the graphs. 

Sincerely,

N. Tangen
Oct 1, 1976

Sender's name and address (Please show your postcode)

John Nornis Tangen
11 Tackley Place
Oxford England
Ox 2 GRR

An air letter should not contain any enclosure

By air mail  Air letter
Par avion  Aerogramme

N. J. A. Sloane
Bell Laboratories
600 Mountain Avenue
Murray Hill, New Jersey
07974 USA