Tournaments (n ≤ 6)

Tournaments are just what they look like. Team X beats team Y, indicated by an arrow directed from X to Y. All matches are one-on-one, every team plays every other team, and there are no draws.

The degree sequence for a tournament is an inventory of wins. For example the transitive 3-tournament has sequence 210, meaning the players have 2 wins, 1 win, and no wins. Degree sequences are far from unique.

The proper (or Condorcet) winner of a tournament beats all other players (with degree n - 1), but the existence of a proper winner is unlikely. By examining all possible states of the arrows, we can assign a probability to a tournament class as a random outcome (listed up to n = 5). From these we can find the probability of a proper winner for n players (see Table 8.1).

Converse tournaments (connected by the sign ~) are related by reversing all of their arrows. Those tournaments not paired with a converse are self-converse.

\[
\begin{array}{c}
\text{n = 2} \\
+ \quad \rightarrow \quad - \\
10
\end{array}
\]

\[
\begin{array}{c}
\text{n = 3} \\
\begin{array}{c}
+ \quad \rightarrow \\
\begin{array}{c}
\text{210 transitive } P = 3/4 \\
\text{111 cyclic } P = 1/4
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\text{n = 4} \\
\begin{array}{c}
3210 \\
P = 3/8 \\
\text{transitive}
\end{array} \\
\begin{array}{c}
3111 \\
P = 1/8 \\
\sim
\end{array} \\
\begin{array}{c}
2220 \\
P = 1/8
\end{array} \\
\begin{array}{c}
2211 \\
P = 3/8 \\
\text{4-cycle}
\end{array}
\end{array}
\]
\( n = 5 \)

\[
\begin{align*}
&+ \quad \frac{15}{128} \quad 43210 \\
&+ \quad \frac{5}{128} \quad 43111 \\
&- \quad \frac{5}{128} \quad 33310 \\
&+ \quad \frac{5}{128} \quad 42220 \\
&+ \quad \frac{15}{128} \quad 42211 \\
&- \quad \frac{15}{128} \quad 33220 \\
&+ \quad \frac{15}{128} \quad 33211 \\
&+ \quad \frac{15}{128} \quad 33211 \\
&- \quad \frac{15}{128} \quad 32221 \\
&+ \quad \frac{5}{128} \quad 32221 \\
&+ \quad \frac{15}{128} \quad 32221 \\
&- \quad \frac{3}{128} \quad 22222
\end{align*}
\]
\( n = 6 \)

1. \( 543210 \)
2. \( 542220 \)
3. \( 533310 \)
4. \( 533220 \)
5. \( 543111 \)
6. \( 542211 \)
7. \( 533211 \)
8. \( 533211 \)
9. \( 532221 \)
10. \( 532221 \)
11. \( 532221 \)
12. \( 522222 \)
Table 8.1 Numbers of Tournaments

<table>
<thead>
<tr>
<th>n</th>
<th>T(n)</th>
<th>*self-converse</th>
<th>#Hamiltonian</th>
<th>Condorcet probability n/2^{n-1}</th>
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<td>1</td>
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<td>1</td>
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<td>2</td>
<td>1</td>
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</tr>
<tr>
<td>4</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>1/2</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
<td>8</td>
<td>6</td>
<td>5/16</td>
</tr>
<tr>
<td>6</td>
<td>56</td>
<td>12</td>
<td>35</td>
<td>3/16</td>
</tr>
<tr>
<td>7</td>
<td>456</td>
<td>88</td>
<td>353</td>
<td>7/64</td>
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<tr>
<td>8</td>
<td>6880</td>
<td>176</td>
<td>6008</td>
<td>1/16</td>
</tr>
<tr>
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<td>2752</td>
<td>178133</td>
<td>9/256</td>
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<td>9733056</td>
<td>8784</td>
<td>9355949</td>
<td>5/256</td>
</tr>
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</table>

*Alistair Farrugia, Univ of Malta

*A Hamiltonian tournament has at least one n-cycle. Equivalently, every node is reachable from every node.

Table 8.2 Converse pairs for n = 6

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<th>sc</th>
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<td>22</td>
<td>sc</td>
<td>40</td>
<td>~ 41</td>
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<tr>
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<td>~ 2</td>
<td>23 ~ 25</td>
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<td>43</td>
<td>sc</td>
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<td>sc</td>
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<td>~ 13</td>
<td>26 ~ 29</td>
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<td>~ 49</td>
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<td>46</td>
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<td>~ 54</td>
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<td>sc</td>
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<td>sc</td>
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<tr>
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<td>~ 20</td>
<td>36 ~ 42</td>
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