THESIS

LENGTHS OF CYCLE TIMES IN RANDOM NEURAL NETWORKS

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LENGTHS OF CYCLE TIMES IN RANDOM NEURAL NETWORKS

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BIOGRAPHICAL SKETCH

Mr. Sloane was born on October 10, 1939 in Beaumaris, Wales. He graduated B. Elec. Eng. in 1959 (taking First Class Honours at the final examination) and B. A. (Honours) in Mathematics in 1960, from Melbourne University, Australia; and received the Master of Science degree from Cornell University in June, 1964. He has worked for the Post Master General's Department of the Commonwealth of Australia, for Bell Telephone Laboratories, Incorporated, and for the Cognitive Systems Research Program, Cornell University; and is a member of M.A.A., A.M.S., I.E.E., I.E.E.E., I.R.E. (Aust.), the Tensor Society of Great Britain, and the American Cryptogram Association.

To Ann

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ABSTRACT

A feasibility study of timing circuits for a model of the human brain. It is shown that certain random neural networks have cycle times which increase exponentially with the size of the network, and that these networks may be used as timing devices even for a period of time equal to a human lifetime. The principal neural networks considered are characterized by either property (1) or property (2):

(1) Exactly one branch originates at each node, and exactly one branch terminates at each node. These are the graphs of the n! permutations of n objects.

(2) Exactly one branch originates at each node, and the terminating node of each branch is selected at random from all the nodes. These are the graphs of the n^n mappings of a set of n objects into itself.

The methods used in this report are those of probability, combinatorial analysis, and number theory.

ORGANIZATION

Chapter 1 contains a detailed description of the problem and its application to brain models, and is arranged as follows: In sections 1.1 and 1.2 we give a brief account of perceptron theory and indicate how our problem arose, and then in 1.3 the problem is restated and the basic quantities LC, LBF and LF are defined. It is shown that the main problem is to estimate these quantities for various classes of random networks. In 1.4 a special network is constructed with a very high LC and LF.

In 1.5 we define certain special classes of networks, of which Type 1A networks are the most important. With this background completed, in Chapter 2 we give a summary of the main results and state our conclusions.

Chapter 3 establishes some basic properties of the Type 1A networks and is arranged as follows: We begin in 3.1 with a discussion of the structure of a Type 1A network and the meaning of LBF, LC and LF. Then in 3.2 several probability spaces associated with T_n are defined, for convenience and explicitness in later calculations. Also in 3.2 and in the remaining two sections of this chapter we investigate the number and sizes of the components and of the loops in a Type 1A network.

Chapter 4 is the most important part of this work and contains the estimates for the expected values of LBF and LC. Chapter 4 is arranged as follows: In 4.1 a probabilistic argument is used to obtain a bound to E[LBF]. In 4.2 we use a graph theoretical approach to obtain both upper and lower bounds for E[LBF], although the lower bounds is redundant in view of the preceding section. (We state it nevertheless because of its simplicity of form and its similarity to the upper bound.) At the end of this section we show how the upper bound may be evaluated numerically and state a conjectured upper bound based on the computational results. Bounds for E[LC] and E[LCM of loop lengths] then follow in sections 4.4 to 4.9; the organization of these sections is described in section 4.3.

Chapter 5 contains further discussions of Type 1A networks, of which a detailed description is given in 5.1.

Chapter 6 deals with Type 4 networks. In 6.1 and 6.2 known properties of Type 4 networks are stated, and in Theorems (6.2.4) and (6.3.1) the LCM of the loop lengths of a typical Type 4 network is estimated. In (6.2.2) an asymptotic upper bound is given to LC and LF for both Type 1A and Type 4 networks.

Chapter 7 contains bounds for LBF and LC for Type 5A networks.

Finally in Chapter 8 we describe an approximate method for obtaining LF known as the "birthday model" method.

At the end of this report are: Appendix I, some lemmas frequently used; Appendix II, definitions and results from graph theory; Appendix III, a list of symbols used; and a bibliography.

CHAPTER 1

DETAILED DESCRIPTION OF THE PROBLEM AND ITS APPLICATION TO BRAIN MODELS

1.1 PERCEPTRONS

The problem with which we shall be dealing arose in Dr. F. Rosenblatt's work on perceptrons. We will therefore begin with a brief description of perceptron theory. A perceptron is a neural network which is a model for the neural activity in the brain. For our purposes a neuron will be considered as a threshold-logic element which responds to the sum of a set of input signals (excitatory and inhibitory) by generating an output signal, provided that the sum of the inputs is greater than a given threshold, Θ.

The simplest perceptron consists of 3 layers of neurons (see Figure 1.1). The first layer, the sensory units or S-units, act as transducers for physical signals originating outside the network; the second layer, the association units or A-units, represent nodal points at which the flow of impulses through the network is regulated; and the third layer, which in Figure 1.1 consists of only one neuron, contains the response units or R-units which represent output signal generators which can be observed from outside the network.

> "The connections of the network, c_{ij}, from unit u_i to unit u_j, each have a quantity associated with them called the value, v_{ij}. In most perceptrons, the signal transmitted by a connection is simply equal to the value, v_{ij}, or else equal to zero, if no signal is transmitted. The values may either be fixed (not time-dependent) or else they may be variable, with magnitudes depending on the past history of the network. All perceptrons are assumed to have at least some variable values."*

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*All quotations in sections 1.1 and 1.2 are from Dr. Rosenblatt's papers [49] or [50].

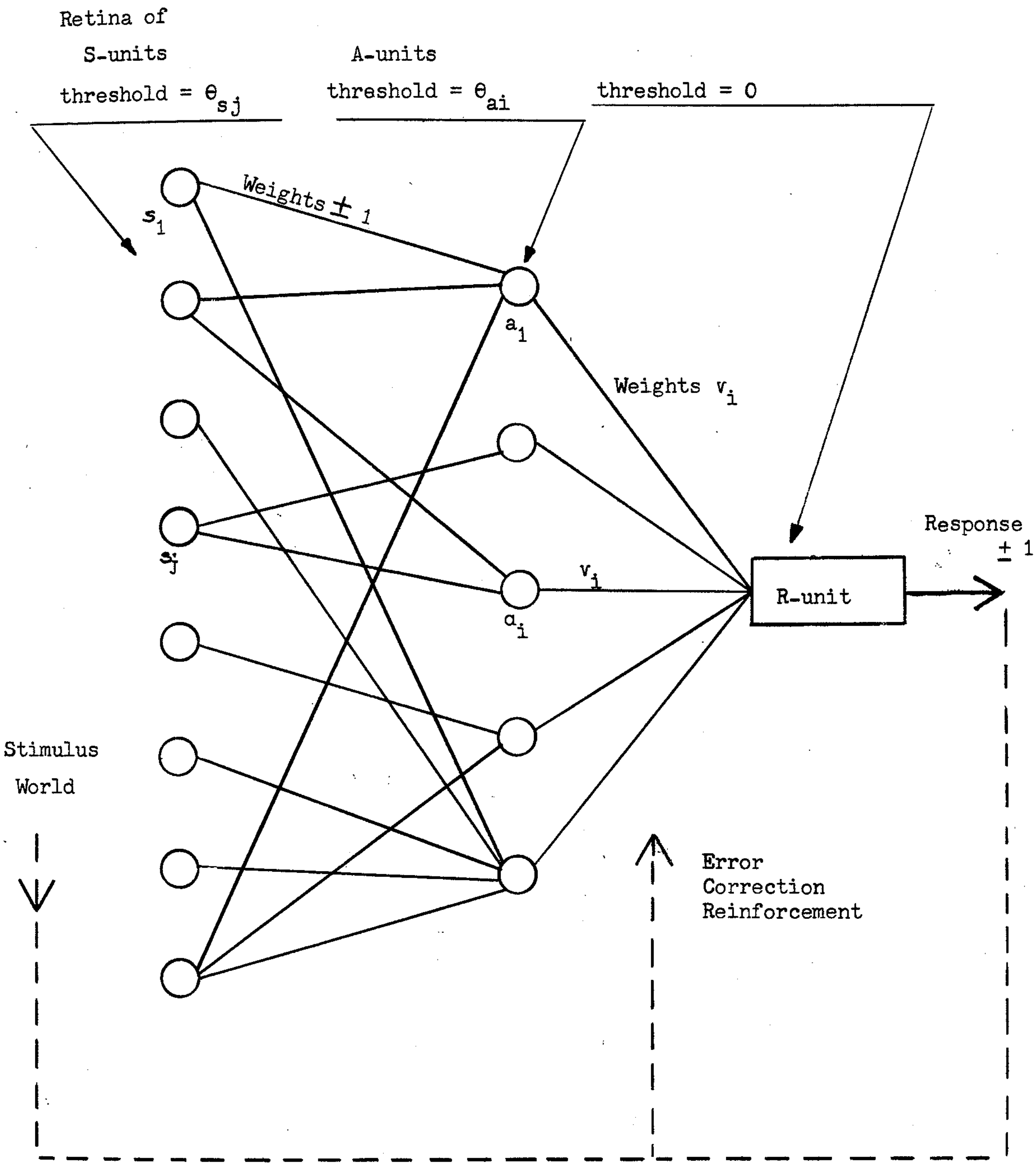


FIG 1.1 AN ELEMENTARY PERCEPTRON

In the elementary perceptron, the R-unit

> "emits an output of +1 if the total input signal is strictly positive, or -1 if the total input signal is negative. For a zero signal, the output is ambiguous, and is tentatively left undefined. Every A-unit has a variable-valued connection to the R-unit, and receives a set of fixed-value connections from the S-units. These S-A connections are generally assumed to have weights (or values of +1 or -1), and origin points on the retina may be assigned either systematically or at random, depending on the particular model.
>
> "The term stimulus is taken to mean any set of sensory points which are activated at a given time. The set of S-points activated at time t constitute the stimulus, S(t)."

A discrimination experiment is a learning experiment with perceptrons. This is the description of the general case, when the R-layer may have more than one unit:

> "The perceptron is first exposed to a sequence of stimuli from some well-defined environment (or admissible set of stimuli), with some modification procedure, or reinforcement rule, applied to the variable-valued connections of the network. The perceptron is then tested on a sample of stimuli to see whether it has learned a 'correct' output from the R-units for each stimulus in the test sample. In a discrimination experiment, the object is to teach the perceptron to assign an appropriate response to all stimuli, identifying the class of the stimulus in each case. For example, the experimenter might show the perceptron the four letters A, B, C, and D, in all possible positions on the retina, and require that the perceptron activate one R-unit for any A, a different one for any B, a third for any C, and a fourth for any D. An assignment of responses to all stimuli in the environment W is called a classification, C(W)."

A solution is said to exist for a given classification problem, C(W), and a given perceptron, if there is some assignment of weights to the variable connections, such that each stimuli produces the correct response.

Although a great variety of training procedures and reinforcement rules have been studied in perceptron theory, for the purposes of this introduction it is sufficient to describe just one method.

> "This is the so-called quantized α-system error correction procedure, which operates as follows: Some sequence of training stimuli is presented to the perceptron. For each of these stimuli, the evoked response is determined. If the response for a given stimulus is correct, no change is made in the values of the connections. If the response is wrong, an increment or decrement, of magnitude Δv, is added to the values of all connections which originate from active association units (whose input signals exceed the threshold, for the current stimulus). The sign of Δv is positive if the desired response is positive, and negative if the desired response is negative. With this system, if a set of values is obtained for which the response to every stimulus is correct, there will obviously be no further changes, and the solution which has been obtained will necessarily be stable."

The importance of simple perceptrons lies in the following theorem:

> "Given a simple perceptron, an environment, W, and some classification C(W) for which a solution exists, then the error correction procedure defined above will always converge to a solution after a finite number of stimuli, regardless of the initial values of the connections, and regardless of the sequence in which the stimuli occur, provided each stimulus ultimately reoccurs."

(For proof of this theorem, and extensions, see [7] and [48], and for further information about perceptrons see [6], [48], and [49].)

Remark: The S-units may be simple transducers, such as photoelectric cells, or more elaborate recoding devices,

> "which may detect such features as straight lines or edges in the stimulus pattern, transmitting only information about these important features to the A-units [49]. Short-time sequences, rather than momentary stimuli, may form the input patterns; these may be encoded in the association system as a nontemporal (spatial) pattern by means of a distribution of

> transmission delays in the S to A network, or by means of a closed-loop cross-coupled network (either in the A-system itself or prior to it), or else by means of a combination of 'On' and 'Off' neurons in the early layers which signal the onset and termination of the activity induced by a moving or changing stimulus[48]. There is increasing evidence that the last of these three mechanisms may be largely responsible for motion detection in the cat's visual system (Hubel and Wiesel[33])."

1.2 PERCEPTRONS WITH SEQUENTIAL MEMORY

After the above introduction we can now describe an important extension to perceptron theory, that is, a perceptron with sequential memory (for more details and mathematical analysis see [50]). It is in this context that our problem arose.

> "The sequential memory model will operate, basically, by reconstituting the succession of A-unit activity states which occurred when the original experience took place. This reconstitution is, generally, far from perfect, but it can be shown that it can be made close enough to the original activity states to permit the previously learned responses to occur, or alternatively, to learn new responses to stimuli in retrospect, which will then generalize satisfactorily to stimuli appearing in the environment. In order to do this, an auxiliary network is necessary, as shown in Fig. 1.2.
>
> "In this figure, as in the subsequent ones, broken arrows are used to represent adaptive connections (with variable weights), while solid arrows represent fixed connections. The circles represent sets of neurons. A normal arrowhead generally represents excitatory connections (or mixtures of excitatory and inhibitory connections), while a small circle in place of the arrowhead represents inhibitory connections.
>
> "The two main additions to the system shown in the previous figure are the threshold servomechanism for the association system, and the C-system, or clock network, which has variable connections to the A-units, and input connections from the R-units. The Θ-servo is simply a negative feedback system which tends to maintain a constant level of activity in the association system. It might consist, physiologically, of a set of cells whose input connections are drawn from the whole of the association network, and whose output connections

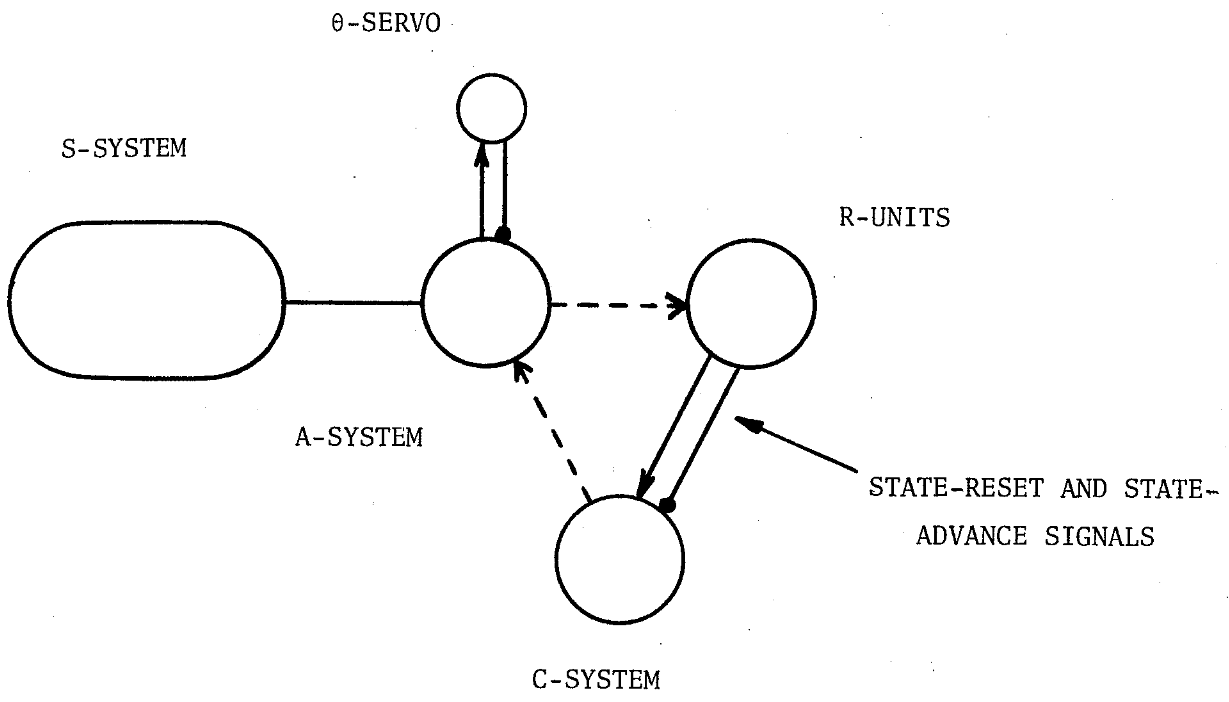


FIG 1.2 PERCEPTRON WITH SEQUENTIAL MEMORY

deliver an inhibitory signal to all A-units, which increases with the magnitude of the input signal. With such a control mechanism the association system will tend to find and maintain a constant level of activity despite changes in the distribution or intensity of input signals.

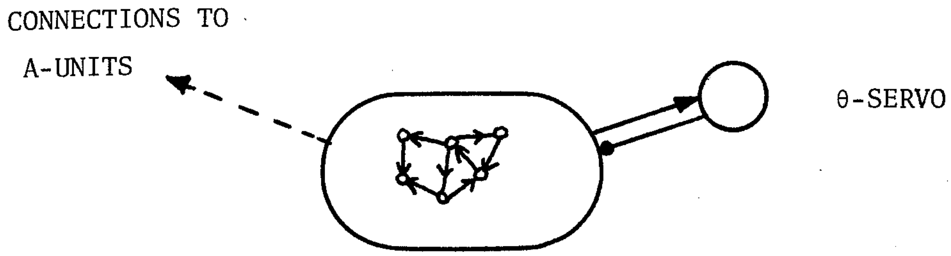
"The A-units may receive signals from two sources, apart from the servo system itself. Normally, their chief input source would be the sensory network, which is assumed to send strong signals to the A-units whenever sensory events occur. The second source is the set of adaptive connections from the C-network (the functioning of which will be elaborated shortly). These connections, however, are assumed to be limited to weights which are considerably smaller in magnitude than the weights of the S- to A- connections. Consequently, as long as sensory signals are arriving at the A-units, the state of the association system will be 'S-determined,' the signals which might be coming in simultaneously from the C-system constituting only a negligible perturbation in the total input signals. Under the action of the θ-servo, the A-units will act essentially like high-threshold units in a simple perceptron, and the C-system will have little or no influence on the operation of the primary information channel, from S to A to R. In this state (as long as sensory inputs continue) the perceptron can be trained or interrogated in the usual fashion, and all previous analyses of such performances remain applicable. On the other hand, when sensory signals cease (either due to lack of environmental stimulation or due to an active cutoff mechanism in the perceptron itself, which might be controlled by one of the R-units) the θ-servo will immediately act to lower the thresholds of the A-units until previous activity levels are restored. Under these conditions, the relatively weak signal component coming from the C-system becomes the primary determinant of the state of the system, and the A-units will respond to the C-network as if it were alternate sensory field.

"We must now consider the C-system itself in greater detail. Two alternative organizations are illustrated in Fig. 1.3. The C-network, as its name suggests, operates as a 'clock' for the memory of sequences. This clock may either be synchronous (progressing through a sequence of states at a rate which is independent of external events) or asynchronous, in which case it advances from one state to the next only when a suitable trigger-event occurs to make it do so. A synchronous clock is exemplified by a simple cross-coupled network (Fig. 1.3a) which will advance through a succession of states, each

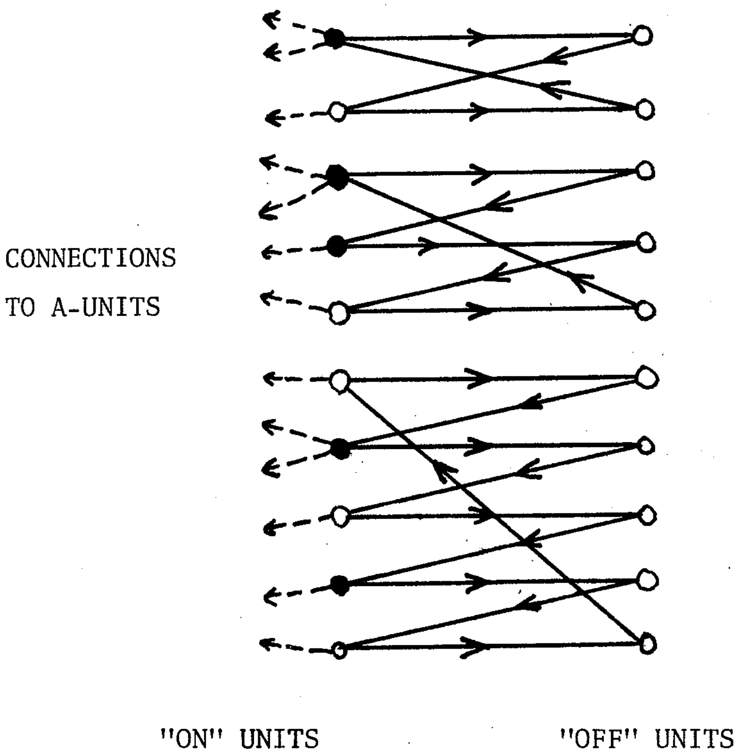
> determined by the preceding state, with a speed which depends only on the transmission time of the connections and synaptic delays. The Θ-servo acts to prevent 'blowups' or extinction of activity. While it would be quite possible for our model to operate with such a simple mechanism, the asynchronous clock, which permits the events constituting the recorded sequence to occur at one rate and to be recalled later at a different rate, is inherently of much greater interest."

An example of an asynchronous clock is shown in Fig. 1.3b. In this case,

> "the C-network is subdivided into two sets (or layers) of neurons. One layer consists of 'On' neurons, which deliver a sustained burst of impulses in response to an excitatory input signal; the second layer consists of 'Off' neurons, which are effectively inhibited during an input signal, but deliver a brief burst of high-frequency inpulses when the input ceases.
>
> "The manner of operation of the asynchronous clock network can best be understood from Fig. 1.3b. Assume that those 'On' units which are filled in solidly in the diagram are active at the present time. They will continue to emit impulses until some inhibitory signal arrives to cut them off. This inhibitory signal is provided by an 'On' burst or 'Off' burst of short duration, from any of the R-units, signaling some change in the response of the perceptron, and thus the beginning or end of a distinguishable event. If we assume that the coupling from the R-units to the C-network is dense enough and powerful enough, then any change in response will momentarily quench the activity of the On units in the C-system. During all of the time that these On units have been firing, however, they have not only been transmitting signals back to the A-units (by way of the variable connections, which will soon be discussed in detail); they have also been sending 'priming signals' to the Off units, which thus begin to fire as soon as the On units are cut off. This Off burst occurs only in the subset of cells which were connected to the active On units. These cells will immediately transmit excitatory signals back to the On layer, activating a new subset of On units, which will then continue to fire until it is finally quenched by



(a) FREE-RUNNING (SYNCHRONOUS)



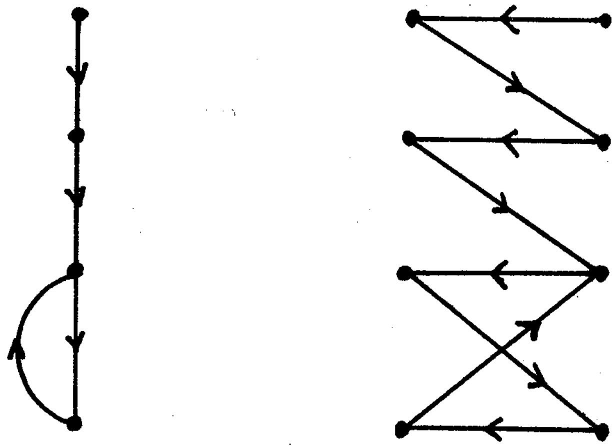
(b) ASYNCHRONOUS 1 - 1 SYSTEM

FIG 1.3 TYPES OF C-SYSTEMS

> the next change in the R-units. Thus the C-system will advance through a deterministic succession of states, changing abruptly to a new state whenever the response of the perceptron is altered in a significant fashion.
>
> "Ultimately, since the number of C-units must be finite, the network must return to its initial state, and the cycle will repeat."

For a "good" C-network, then, this event will not happen for an extremely long time. A second requirement is that the C-network be biologically plausible, for instance a network (like Fig. 1.3b) made up of loops whose lengths are different primes would be considered quite implausible.

Our problem is to analyze some possible C-networks. We shall only consider synchronous C-networks, but since to every synchronous network corresponds an equivalent asynchronous network our results also apply to several families of asynchronous networks. We show this correspondence in the case that the synchronous network G contains only excitatory connections of value 1. The equivalent asynchronous network H is obtained as follows. Suppose G has n nodes labelled 1,2,...,n. Let H have 2n nodes, consisting of n "on" nodes 1,2,...,n and n "off" nodes n+1,..., 2n; and for i=1,...,n let node n+i be connected to node i by a connection of value 1. Finally for every connection in G, from a node α to a node β, say, let there be a connection in H of value 1 from node α to node $n+\beta$. It is clear that H will be the asynchronous equivalent of G. Fig. 1.4 shows an example of this procedure. The discussion of the design of the C-network is resumed in the next section. We have still to show here how the C-network is used in the perceptron with sequential memory.



SYNCHRONOUS NETWORK G

"ON" UNITS "OFF" UNITS

EQUIVALENT ASYNCHRONOUS NETWORK H

FIG 1.4

"In all that follows, it will be assumed that the C-system is of sufficient size that the likelihood of a state repeating itself, without the network having been deliberately reset, is entirely negligible.

"Although the successive states of the C-system form a deterministic sequence, each state being a predictable consequence of the preceding one, their interrelationships (particularly the measure of the intersections of active sets at different times) are, generally, indistinguishable from those that would pertain to a collection of randomly chosen states. This property is of great importance in the subsequent analysis of the memory system.

"It now remains to see how the states of the C-system can be made to induce a succession of states in the A-system corresponding to a recorded sequence of stimuli. For this we must specify more precisely the modification mechanism of the C-unit to A-unit connections.

"We assume that each A-unit receives connections from a fraction M of the 'On' units in the C-network. The connection system from C to A is a many-to-many system, the only important constraint being that the choice of connections to particular A-units should be statistically independent of the particular sets of C-units which are likely to be active in different 'clock states.' To be explicit, it will be assumed that the connections to each A-unit originate from a set of MN_c points chosen at random with a uniform probability distribution (where N_c is the total number of units in the 'On' layer of the C-system). Thus, if a fraction Q_c of the C-units are active in any given state, it is expected that a fraction MQ_c are actually transmitting signals to any particular A-unit."

The modification of connections take place according to a rule similar to α-system correction procedure described earlier.

"In the recording of a memory sequence the following succession of events occurs it is assumed that the C-system is set to some initial activity state. This could be achieved, in the simplest case, by activating one of the R-units which forces the on-units of the C-network to the desired starting condition. This R-unit, in turn, could be trained to respond to a starting command, such as the name of the recorded sequence. The initial weights of C-to-A connections are assumed to be zero.

"With the C-system in its initial state, the first stimulus pattern of the sequence appears in the sensory system, and induces a corresponding activity state in the association units. Say, for example, the first stimulus is a triangle. The set of A-units responding to this triangle will then have their connections from the active C-units augmented in value. As long as the triangle remains on the 'retina,' signals transmitted from the C-units to the A-units will tend to be ignored, due to the action of the Θ-servo, and the relatively high weights of connections from the sensory system. On the other hand, if the same C-state should recur, without the presence of a retinal input, the Θ-servo will lower the effective thresholds of the A-units, and the augmented connection weights to the previously active A-units will tend to reactivate the same set of units which responded to the triangle.

"There will be no systematic attempt to turn off the 'improper units' which did not respond to the triangle, but the Θ-servo will tend to find a level at which only the units receiving the strongest input signals will be reactivated, which has essentially the same effect.*

"As soon as the triangle is replaced by the next stimulus (say a square) which is sufficiently different so that the response of the perceptron changes, the C-system will advance to its second state, which we have seen to be statistically independent of the first, although it is a deterministic consequence of it. Due to this statistical independence and the use of the γ-system, it can be shown that the expected value of the signal now received by any A-unit from the C-system is equal to zero. Consequently, the modifications of the connections which now take place to the set of A-units responding to the square will have the same effect (except for a slight noise effect) as if no previous memory had been recorded.

---

*This is the so-called "asymmetric model".

"If the square, in turn, is replaced by another triangle the change in response (whether correct or not is immaterial) will cause the C-system to advance to its third state, from which the expected signal to the A-units will again be zero. A new change in weights then occurs as before. This process continues indefinitely until the C-system either recycles (an unlikely possibility) or is deliberately reset.

"To see how the system acts in recall, suppose the response which resets the C-system is evoked, followed by a 'silent period,' during which no sensory inputs occur. The θ-servo, striving to normalize the activity level in the A-system, now lowers the thresholds to the point where the A-units begin to respond to the C-unit signals. As we have seen, the first state of the C-system will tend to reactivate the set of A-units responding to the first triangle (without any interference, other than random-noise effects from any subsequently recorded memory). As soon as this state is, in fact reconstituted in the association system, however, the triangle response should occur, and this response will advance the C-system to its next state. This state induces the A-unit activity pattern corresponding to the entire sequence of sensory events tends to be reconstituted, in proper temporal order. If the states are reconstituted accurately enough they can be used for teaching the perceptron new discriminations, in retrospect, or for applying subsequently learned discriminations to events which were improperly recognized at the time they occurred. None of this interferes with the sequential memory system, which is independent of changes in the A- to R-unit network.

"Due to the fact that the expected interaction between recorded events is zero, the sequences which can be stored may be extremely long. Ultimately, noise effects, which show up as a gradually increasing variance in the transmitted signals, grow to such a degree that they effectively mask the residual traces of previous memory, and the system saturates. Before this happens, the accuracy with which the association states are reconstituted gradually diminishes, and consequently the discriminatory responses which occur to remembered stimuli become less and less accurate. In evaluating the performance of this model, the most important question is the probability that a discriminatory response to a remembered stimulus is correct, after a long history of experience has been recorded."

It is then shown in [50] that this probability may be very close to unity, even after as many as 10^{11} different stimuli have been seen and recorded.

This concludes the description of the background for our problem. In the next section we restate the problem in terms of graph theory.

1.3 STATEMENT OF THE PROBLEM

From what we have said in the previous section, it follows that the problem is to find a simple family of neural networks, each with n neurons, with the following property. First one of these networks is chosen at random. Representing a neuron by a 1 if it fires and by an 0 if it does not fire, the state of this network is a binary vector of length n, and there are 2^n possible states. One of these is chosen at random (as the initial state) by a procedure to be described later. Since there are only a finite number of possible states, eventually either a state must recur or the activity must die out. We shall always consider the latter case as if the zero state were recurring with period 1. The problem is to find a single family of networks such that with high probability the time, m, at which this first recurrence occurs, is exponentially large with respect to n.

Each network is a deterministic system, and its state diagram is a graph* with 2^n nodes and 2^n directed branches, one branch originating at each node. Each connected component of the graph contains exactly one loop, and the complete state diagram will therefore look like Fig. 1.5.

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*Appendix 2 contains definitions of terms borrowed from graph theory.

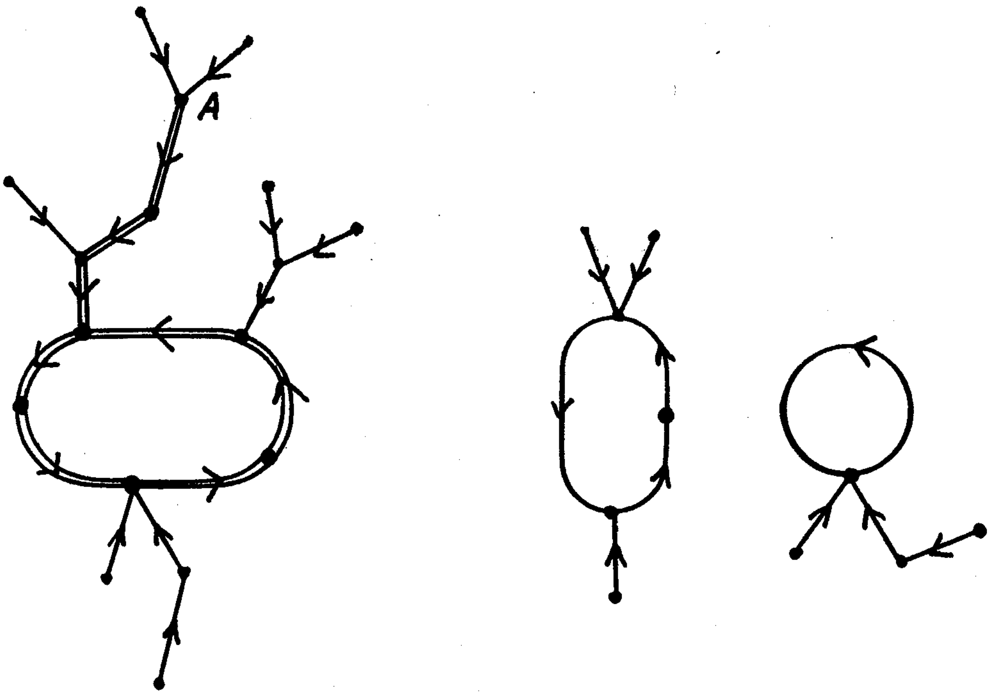


FIG 1.5 GENERAL FORM OF STATE DIAGRAM

The procedure described above is equivalent to choosing an initial node A from the state diagram and following it until it traverses a loop (see the heavy lines in Fig. 1.5).

Definitions of LBF, LC, LF. Throughout this work, LBF is the distance from the initial state A to where the path enters the loop, LC is the loop length, and LF = LBF + LC (see Fig. 1.6). (LC is Length of Cycle, LBF is Length Before cycle, and LF is Length to First repetition.)

Definition of Activity. The activity in a network is the number of active nodes.

Note: There are two graphs associated with each network. One is the graph of the network itself, showing the connections and weights; there are n nodes, representing neurons; and the branches represent axons. The other graph is the state diagram of the network, where the nodes represent the states of the network, and a branch goes from node i to node j if the network goes to state j immediately after state i. There are 2^n nodes and 2^n branches.

Briefly, the problem is to find a simple class of networks with n nodes such that with high probability LF is exponentially large compared to n.

1.4 DESIGN OF A SPECIAL NETWORK FOR WHICH LC AND LF ARE $\sim 2^{n/4}$

It is natural to ask if there exist special networks for which LF approaches its maximum, 2^n. In this section we construct a highly artificial network for which

$$LC \sim LF \sim 2^{n/4}$$

We design a bistable or flip-flop circuit using 4 nodes, and put a number, m (say), of these together to form a counting or clock network with n = 4m nodes.

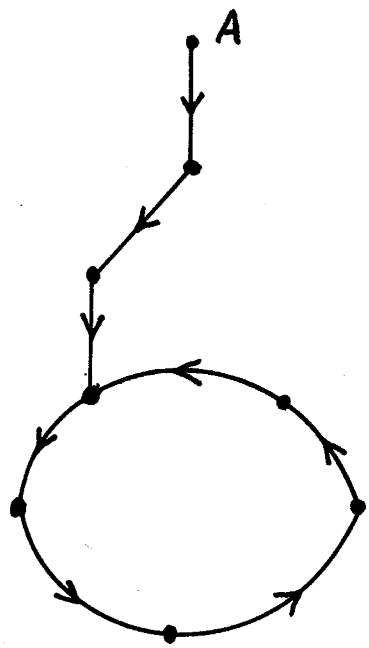


FIG 1.6 DETAIL OF FIG 1.5 LBF = 3, LC = 5, LF = 8

For this network:

$$LC = 2^{m+1} \sim 2^{n/4}$$

$$LBF = m$$

$$LF = 2^{m+1} + m \sim 2^{n/4}$$

Part of the state diagram of the complete network is shown in Table 1.2.

1.4.1 Design of Flip-Flop

The state of the flip-flop is the state of node A. We wish to realize the state transition table shown in Table 1.1. In other words, if no input pulse X is received, the flip-flop remains in the same state. If an input pulse is received, the state changes; and if an input pulse is received while in the "on" state, an output pulse Y is produced.

The network shown in Figure 1.7 has these characteristics. We give the equations describing its behavior and then verify that it works as a flip-flop by constructing a counting network out of several such networks.

A(t), B(t), C(t), X(t), Y(t) will be Boolean variables representing the states of nodes A-Y at time t. A bar, e.g., $\bar{A}(t)$, will denote the complement.

From the network we see that

$$C(t) = A(t-1)\ X(t-1) \tag{1}$$

$$B(t) = \bar{A}(t-1)\ X(t-1) \tag{2}$$

$$A(t) = \bar{A}(t-1)\ B(t-1)\ \bar{C}(t-1) + A(t-1)\ \bar{B}(t-1)\ \bar{C}(t-1) + A(t-1)\ B(t-1) \tag{3}$$

$$Y(t) = A(t-1)\ X(t-1) \tag{4}$$

Substituting (1), (2) in (3) gives

$$A(t) = A(t-1)\ \bar{X}(t-2) + \bar{A}(t-2)\ X(t-2) \tag{5}$$

TABLE 1.1

STATE TRANSITION TABLE FOR FLIP FLOP

Input X	Present State A	New State A'	Output
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

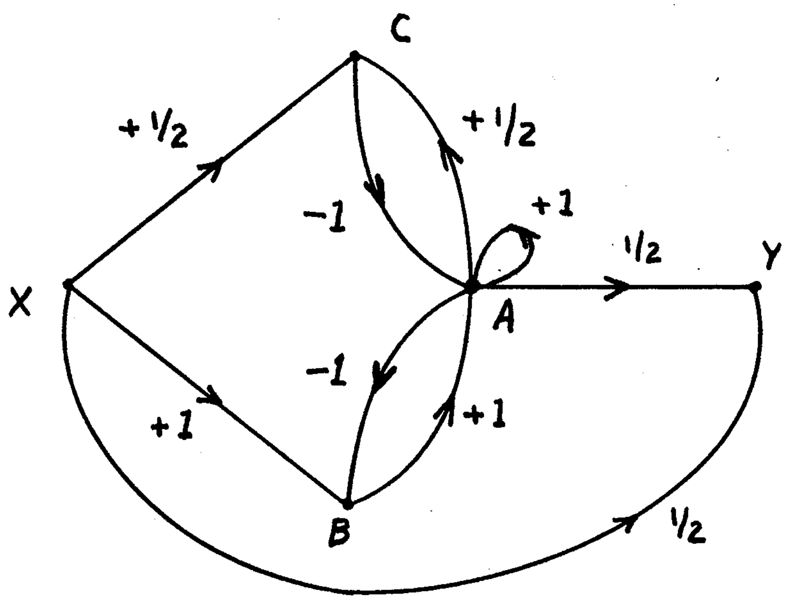


FIG 1.7 FLIP-FLOP NETWORK

In our application, A(t), X(t) will always satisfy either

<u>Condition 1</u>

$$\text{If } t = 2m,\ A(2m) = A(2m-1),\ X(2m) = X(2m-1) \tag{6}$$

or

<u>Condition 2</u>

$$\text{If } t = 2m,\ A(2m) = A(2m+1),\ X(2m) = X(2m+1) \tag{7}$$

If condition 1 is satisfied, and t is odd, (5) becomes

$$A(t) = A(t-2)\ \bar{X}(t-2) + \bar{A}(t-2)\ X(t-2) \tag{8}$$

and

$$\bar{A}(t) = A(t-2)\ X(t-2) + \bar{A}(t-2)\ \bar{X}(t-2) \tag{9}$$

If t is even,

$$A(t) = A(t-1) \quad \text{by (6)}$$

$$= \bar{A}(t-3)\ X(t-3) + A(t-3)\ \bar{X}(t-3) \qquad \text{by (8)}$$

$$= \bar{A}(t-2)\ X(t-2) + A(t-2)\ \bar{X}(t-2) \qquad \text{by (6)}$$

thus, (8), (9) hold for all t.

If condition 2 is satisfied, again we get (8) and (9) immediately from (5).

1.4.2 Design of Clock Circuit

With n = 4m nodes, m flip-flops are connected together as shown in Figure 1.8. The initial state is with node X_1 firing alone.

<u>Analysis of First Stage</u>

Because of the self-excitatory branch at X_1,

$$X_1(t) = 1,\ t \geq 0.$$

Also $A_1(0) = B_1(0) = C_1(0) = Y_1(0) = A_2(0) = \ldots = 0$

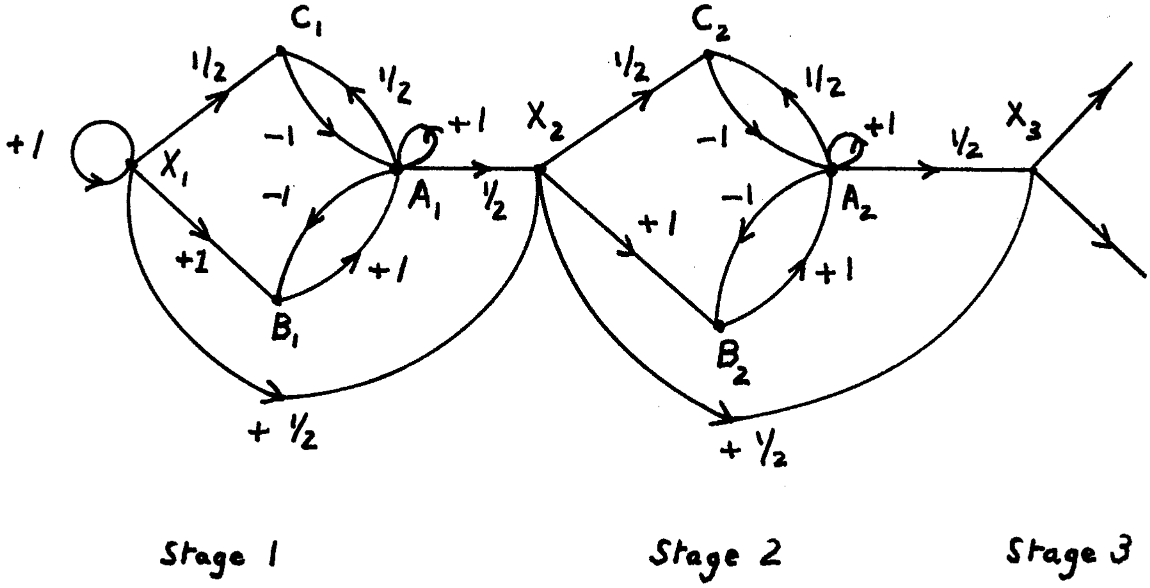


FIG. 1.8 CLOCK CIRCUIT

From equations (1) - (4),

$$B_1(1) = 1,\ C_1(1) = A_1(1) = Y_1(1) = 0$$

$$B_1(t) = \bar{A}_1(t-1),\ t \geq 1$$

$$C_1(t) = A_1(t-1),\ t \geq 1$$

$$A_1(t) = \bar{A}_1(t-2),\ t \geq 2$$

and so

$$B_1(t) = \bar{B}_1(t-2),\ t \geq 2$$

$$C_1(t) = \bar{C}_1(t-2),\ t \geq 3,\ C_1(2) = 0$$

$$Y_1(t) = \bar{Y}_1(t-2),\ t \geq 3,\ Y_1(2) = 0 \qquad (10)$$

These variables are the first 5 columns of the state diagram given in Table 1.2.

Notes: (1) Activity begins in Stage 1 at $t = 0$

(2) Condition (2) is satisfied here

(3) If there were only one stage, we would have LBF = 1, LC = 4, LF = 5 (a cycle is completed for the first time at $t = 5$).

Analysis of Second Stage

There is no activity in the second stage until $X_2 = Y_1$ fires, i.e., at $t = 3$.

Initial conditions:

$$X_2(0) = X_2(1) = X_2(2) = 0,\ X_2(3) = 1$$

$$A_2(0) = B_2(0) = C_2(0) = Y_2(0) = 0$$

$$= A_2(1) = B_2(1) = C_2(1) = Y_2(1)$$

$$X_2(t) = \bar{X}_2(t-2),\ t \geq 3, \qquad \text{by equation (10)}$$

Calculation of the first nonzero values of $A(t)$ indicates that condition (1) applies here-- this is verified later.

So we can apply (8), getting

$$A_2(t-4) = \bar{A}_2(t+2)\ X_2(t+2) + A_2(t+2)\ \bar{X}_2(t+2)$$

$$= \bar{A}_2(t)\ \bar{X}_2(t) + \bar{A}_2(t)\ X_2(t) \qquad \text{by (8) and (9)}$$

so

$$A_2(t+8) = A_2(t),\ t\geq 2$$

Similarly,

$$C_2(t+4) = X_2(t+3)\ A_2(t+3)$$

$$= \bar{X}_2(t+1)\ \bar{A}_2(t-1)$$

$$= X_2(t-1)\ \bar{A}_2(t-5)$$

$$= \bar{X}_2(t-3)\ A_2(t-5)$$

$$= X_2(t-5)\ A_2(t-5)$$

$$= C_2(t-4)$$

and likewise $Y_2(t)$, $B_2(t)$ have periods of 8.

These variables are shown in Table 1.2.

Notes: (1) We verify from the table that condition (1) applies here

(2) If there were only two stages, we would have

LBF = 2, LC = 8, LF = 10.

Analysis of Third Stage

Activity begins in the third stage when t = 8, and if there were only three stages, we would have

LBF = 3, LC = 16, LF = 19

Analysis of Fourth Stage

Activity begins when t = 17, and we find

LBF = 4, LC = 32, LF = 36

Analysis of Mth Stage

Activity begins when $t = 2^m + m - 3$, and

LBF = m, LC = 2^{m+1}, LF = $2^{m+1}+m$

which is what we were trying to prove.

TABLE 1.2

PARTIAL STATE TABLE OF CIRCUIT OF FIGURE 1.8

t	X_1	B_1	C_1	A_1	X_2	B_2	C_2	A_2
0	1	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0
2	1	1	0	1	0	0	0	0
3	1	0	1	1	1	0	0	0
4	1	0	1	0	1	1	0	0
5	1	1	0	0	0	1	0	1
6	1	1	0	1	0	0	0	1
7	1	0	1	1	1	0	0	1
8	1	0	1	0	1	0	1	1
9	1	1	0	0	0	0	1	0
10	1	1	0	1	0	0	0	0

Note that we have not shown that this is the best LF that can be obtained. An upper bound to that is of course 2^n; we have here obtained $(2^n)^{1/4}$.

1.5 TYPES OF NETWORKS CONSIDERED

In this report we will restrict ourselves to quite simple classes of networks. The principal classes we consider are Type 1A and Type 4 networks, but we shall on occasion refer to the other types. The complete list of definitions of the various types follows.

Type 1A Networks

T_n, the family of all type 1A networks of n nodes, is defined to be the family of the directed graphs of the n^n mappings of a set of n objects into itself. Equivalently, T_n consists of the n^n graphs with n nodes (each of threshold 1) and one branch (of value 1) originating at each node, the terminating node of the branch being chosen at random (independently and with replacement) from all the n nodes.

Type 1 networks are a generalization of type 1A networks.

Type 1 Networks. Random Terminating Nodes.

(In the notation of [48], p. 132, this is a Poisson, constrained origin, network.)

X excitory branches (of weight + 1) and Y inhibitory branches (of weight - 1) originate at every node. The terminating nodes of these branches are not specified, and are to be chosen uniformly and independently from all the nodes, so that each node has the probability of $\frac{1}{n}$ of being the terminating node of any branch. There are $n^{n(X+Y)}$ possible networks. Clearly any number of excitory and inhibitory branches, up to nX and nY respectively, may terminate on any node.

The special case where X = 1, Y = 0, and threshold $\Theta = 1$ is then a type 1A network, as defined above.

Type 2. Random Originating Nodes.

(In the notation of [48], p. 129, this is a Binomial type network.)

Similar to type 1 except that X excitory branches and Y inhibitory branches terminate on each node, and the originating nodes are randomly selected.

The special case X = 1, Y = 0, Θ = 1 will be called a type 2B network.

Type 3. Both Ends Random.

(In the notation of [48], p. 132, this is a Poisson, random origin, network. See also [14].)

X excitory branches and Y inhibitory branches are chosen with both ends selected randomly and independently, i.e., any node has the probability $\frac{1}{n}$ of being the originating node of any branch, and any node has the probability $\frac{1}{n}$ of being the terminating node of any branch. There are $n^{(X+Y)}$ different networks.

Type 4 Network

This is just the graph of a permutation of n objects. It corresponds to a random network with exactly one branch, of weight + 1, originating and terminating at every node. More precisely, a permutation

$$\begin{matrix} 1 & 2 & 3 & \dots & n \\ a_1 & a_2 & a_3 & \dots & a_n \end{matrix}$$

is chosen at random, each of the n! different permutations being equally likely. The network is formed by connecting node 1 to node $a_1, \dots,$ node n to node a_n; and has threshold of 1.

Type 5A Network with n Nodes

Each node is the originating node of exactly one directed branch, of weight + 1, the terminating node being chosen uniformly (and independently) from the other n - 1 nodes. The threshold is 1.

Thus, a type 5A network is a type 1A network with self-loops, or loops of length 1, excluded. Clearly there are $(n - 1)^n$ different type 5A networks with n nodes, compared with n^n different type 1A networks.

It turns out that types 1A and 5A networks can usually be analyzed by almost identical techniques (see for example [47]). Mathematical sociologists have used type 5A networks as models of communication relations in groups of people. (See for example [53], p. 73, and [44].) There are certain situations which are more naturally represented by type 5A networks than by type 1A networks, for example the network formed by asking each person to name his best friend.

CHAPTER 2

SUMMARY OF RESULTS AND CONCLUSIONS

With the above background completed, we can now summarize the main results and state our conclusions.

RESULTS CONCERNING LBF, LC AND LF, FOR TYPE IA NETWORKS.

2.1 THEOREM (4.1.9)

If the initial state contains one active node, then

$$E[LBF] = \frac{\pi n}{8} + 0(1) \tag{1}$$

$$\sigma^2[LBF] = n\left(\frac{2}{3} - \frac{\Pi}{8}\right) + 0(\sqrt{n})$$

2.2 (Equation (4.1.12))

If the initial state contains p active nodes, $p<<n$, then $\frac{LBF}{\sqrt{n}}$ (approximately) has the density function

$$f_p(y) = pF(y)^{p-1}\, F'(y)$$

where

$$F(y) = \begin{cases} 1 - e^{-y^2/2} + y\displaystyle\int_y^\infty e^{-x^2/2}\,dx & y \geq 0 \\ 0 & y<0 \end{cases}$$

2.3 THEOREM (4.2.10)

Let $L_w = \dfrac{n!\ w}{(n-w)!\ n^{w+1}}$ and let $U_{m,n}$, $V_{m,n}$ be defined as in (4.2.5) and (4.2.6). Then

$$\sum_{m=1}^{n} L_m V_{n-m,m} \leq E[LBF] \leq \sum_{m=1}^{n} L_m U_{n-m,m}$$

and for low initial activity E[LBF] approaches the left-hand side, and for high initial activity the right-hand side. Further a method for computing $U_{m,n}$ is given at the end of section 4.2.

2.4 A CONJECTURE (Equation (4.2.22))

For some constant a, and $n \geq 1$,

$$E[LBF] < a\, n^{0.564}$$

2.5 THEOREM (4.6.44)

There exist constants α, β such that for $n \geq \alpha$, and if the initial state is formed by choosing

$$\sqrt{n}(\log n)^{3/4}$$

active nodes (with replacement), then

$$E[LC] > \exp\left[\frac{1}{2}(\log n)^{9/4} - \beta\,(\log n)^{5/4}\log\log n\right]$$

(The proof of this theorem depends on four approximations, (4.5.9), (4.6.18), (4.6.33), and (4.6.48).)

2.5 THEOREM (4.7.12)

There exists a constant α such that for $n \geq 32$,

$$E[\text{LCM of loop lengths}] > \exp\left[0.8\, n^{1/5} - \frac{1}{2}\log n - \alpha\right]$$

(The proof of this theorem depends on the approximations (4.5.9) and (4.6.48).)

2.7 THEOREM (4.8.9)

For $n \geq 100$ and any A such that

$$n/(2 \log n) > A > 1.6$$

the expected value of the cycle time LC of all networks with n nodes and fewer than A log n components is less than

$$\exp\left[\frac{A}{2} \log^2 n + A \log c_1 \log n\right]$$

where $c_1 = 1.33 \ldots$

2.8 THEOREM (4.8.13)

There exists a constant A such that for $n \geq 1$,

$$E[LC] < Ae^{2\sqrt{2n}}/\sqrt{n}$$

2.9 Two other methods for obtaining an upper bound to E[LC] are given in Theorem (4.8.11) and section 4.9.

CONCLUSIONS ABOUT TYPE 1A NETWORKS

From (2.1) - (2.9) we conclude that type 1A networks with n nodes form satisfactory clock networks provided either of the following sets of conditions is satisfied:

(a) The initial activity is chosen randomly, and in fact the initial state is formed by selecting $\sqrt{n}\,(\log n)^{3/4}$ nodes at random (independently and with replacement) from the n nodes; and $n \geq \alpha$, where α is some constant (it is only known at the moment that $\alpha \geq e^{e^{33}}$);

(b) The initial state is formed by selecting exactly one active node at random in each component, and $n \geq 32$.

(Conditions (a) and (b) are of course sufficient conditions--the necessary conditions may be much weaker.)

Comparison of Bounds

The dominating terms of the bounds on E[LC] are

(2.5*) $E[LC] > \exp\left[\frac{1}{2} \log^{9/4} n\right]$ under condition (a).

(2.6*) $E[LC] > \exp(0.8n^{1/5})$ under condition (b).

(2.7*) $E[LC] < \exp\left[\frac{A}{2} \log^2 n\right]$ for all networks with fewer than A log n components.

(2.8*) $E[LC] < \exp(2\sqrt{2n})$.

These functions are sketched in Fig. 2.1.1.

For a numerical comparison we calculate how large n must be in order to be able to record 10^3 stimuli per second for 100 years (i.e., for a human lifetime), using the bounds (2.5*), (2.6*) and (2.8*). (See Fig. 2.1.2).

Lower bound (2.5*)

n must satisfy

$$\exp\left[\frac{1}{2} \log^{9/4} n\right] = 10^3.60.60.24.365.100$$
$$= 3.15\ 10^{12}$$

i.e., $n = 1.1\ 10^6$ approximately

Lower bound (2.6*)

n must satisfy

$$\exp(0.8\ n^{1/5}) = 3.15\ 10^{12}$$

i.e., $n = 6.1\ 10^7$

Upper bound (2.8*)

n must satisfy

$$\exp(2\sqrt{2n}) = 3.15\ 10^{12}$$

i.e., n = 103

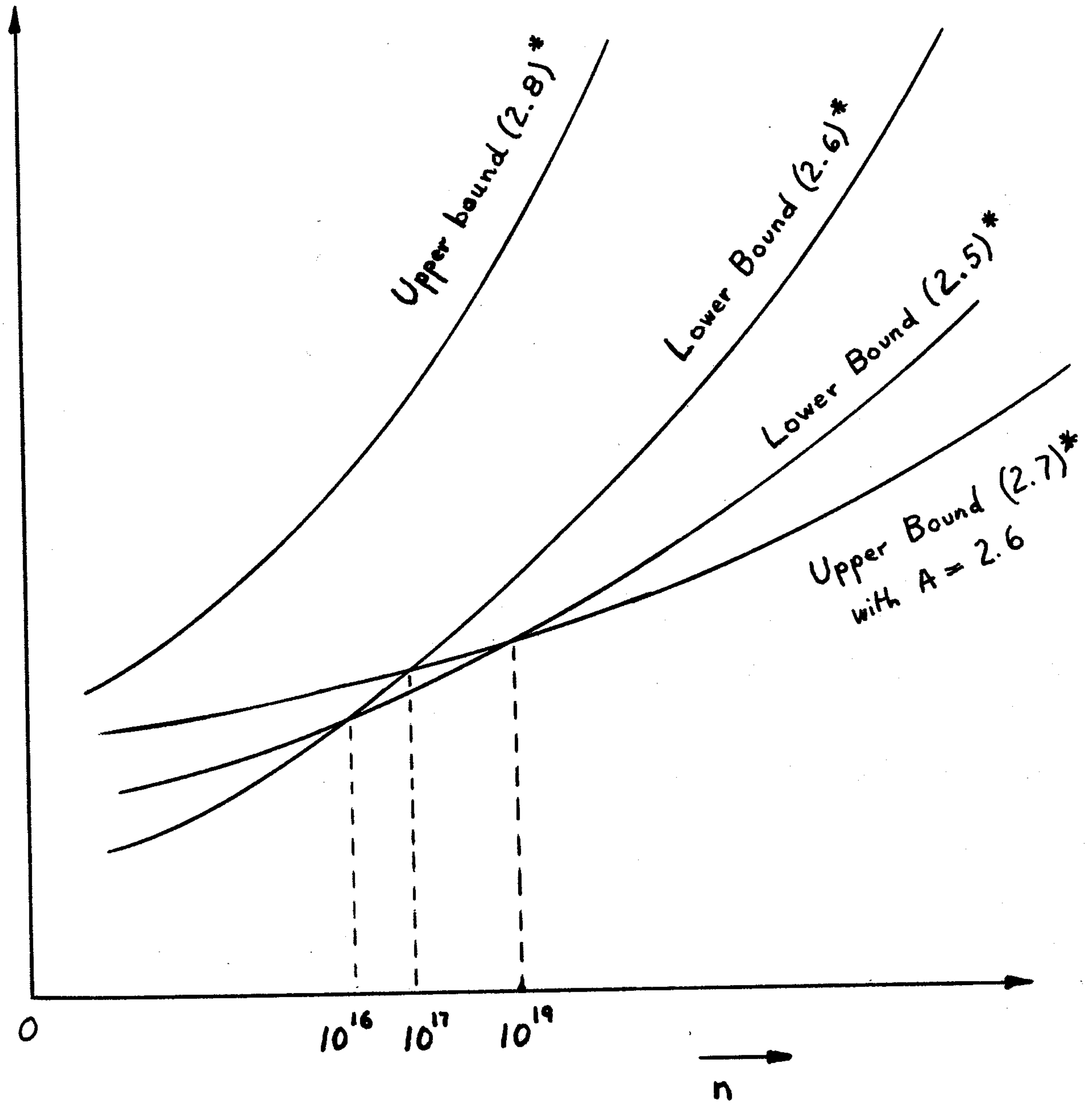


FIG. 2.1.1 COMPARISON OF BOUNDS ON E [LC]

We conclude that there is some n_0 in the range

$$103 \leq n_0 \leq 6.1\ 10^7$$

such that the average cycle time LC of all type 1A networks with n_0 nodes is

$$10^3.\ 60.60.\ 24.\ 365.\ 100 = 3.15\ 10^{12}$$

i.e., that these networks behave satisfactorily as clock networks for a period equal to a human lifetime.

REMARK

The results of sections 4.6-4.8 seem to lead to the following conclusion. "Most" type 1A networks with n nodes have about log n or fewer components, and for these networks the average value of the cycle time LC is of the order of

$$e^{\text{constant } (\log n)^2} \qquad (2.11.1)$$

However there are a "small" number of exceptional networks (exceptionally good) when considered as clock networks, whose number of components greatly exceeds log n, and for these it seems likely (in view of Theorems (4.7.12), (4.8.13)) that the average LC is the order of

$$e^{n^{\text{constant}}}$$

which for large n is much bigger than (2.11.1). Therefore, we are led to the following rule: A simple constraint on the maximum size of a component of a type 1A network with n nodes, say

$$\text{max component size} < \frac{n}{(\log n)^3},$$

(implying

$$\text{number of components} > (\log n)^3\ )$$

will greatly increase the cycle time LC, and therefore the performance as a clock network.

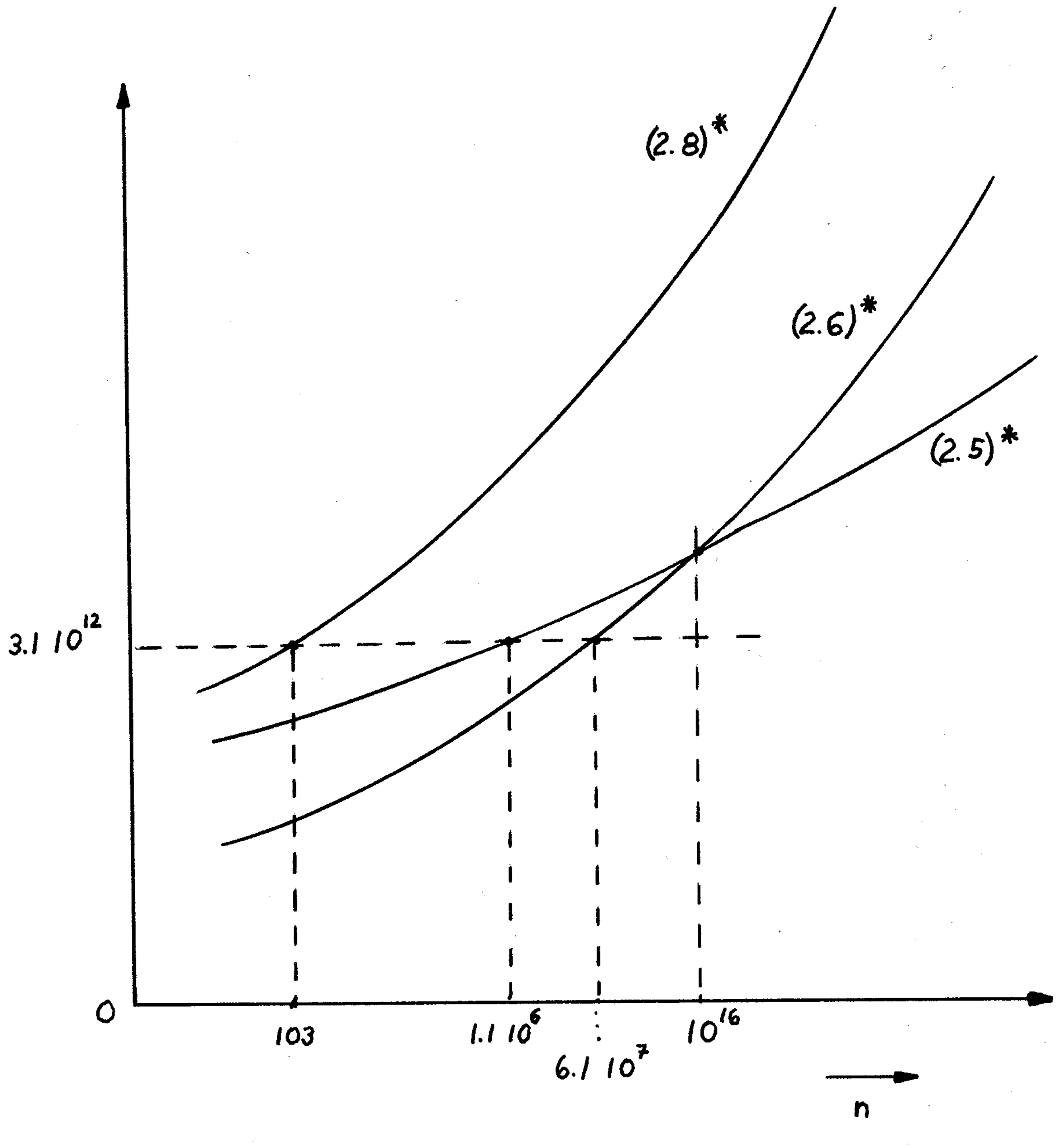


FIG 2.1.2 NUMERICAL COMPARISON OF BOUNDS TO E [LC]

The choice of the best such constraint is an interesting problem for future research.

RESULTS FOR TYPE 4 NETWORKS

2.10 THEOREM (6.2.4) (Erdos and Turan)

Given any positive ε, δ, there exists $n_o(\varepsilon,\delta)$ such that for $n > n_o(\varepsilon,\delta)$,

$$e^{(\frac{1}{2} - \varepsilon)\log^2 n} \leq \text{L.C.M. of loop lengths} \leq e^{(\frac{1}{2} + \varepsilon)\log^2 n}$$

holds, except for at most $n!$ exceptional networks.

2.11 THEOREM (6.3.1)

$$E(\text{LCM of loop lengths}) \leq \frac{e^{2\sqrt{n}}}{2\sqrt{e\Pi}\ n^{3/4}}\left[1 + 0\frac{1}{\left|\sqrt{n}\right|}\right]$$

2.12 (Equation (6.2.2))

Let $G(n)$ be the maximum LC over all type 4 networks of n nodes. Then (Landau)

$$\log G(n) \sim \sqrt{n \log n}$$

This bound also applies to type 1A networks.

CONCLUSION

From (2.10) we conclude that type 6 networks with n nodes form satisfactory clock networks provided the initial state is formed by selecting exactly one active node in each component, and provided n is sufficiently large.

Results for Type 5A Networks

In Chapter 7 it is shown that most of the above results for type 1A networks (namely 2-1, 2-5, 2-6, 2-7, 2-8, and 2-12) hold also for type 5A networks, and therefore the conclusion also holds, that under certain conditions type 5A networks are satisfactory clock networks.

RESULTS ABOUT THE STRUCTURE OF TYPE 1A NETWORKS

2.13 (Equation (3.2.9)) SIZE OF COMPONENTS

$P[m_1$ components of size $n_1,\ldots,m_{k_1}$ components of size $n_{k_1}]$

$$= \frac{n!}{n^n} \prod_{i=1}^{k_1} \frac{C_{n_i}^{m_i}}{n_i!^{m_i} m_i!}$$

where $C_i = \sum_{r=0}^{i-1} \frac{(i-1)!i^r}{r!}$

and $\sum_{i=1}^{k_1} m_i n_i = n$

2.14 (Equation (3.2.14)) SIZE OF LOOPS

P[a_1 loops of length 1,...,a_m loops of length m]

$$= \frac{n!w}{(n-w)!n^{w+1}} \sum_{i=1}^{m} \frac{1}{a_i! \, i^{a_i}}$$

where $w = \sum_{i=1}^{m} ia_i$

2.15 (Section 3.3) NUMBER OF COMPONENTS

$$P[k \text{ components}] = \sum_{j=0}^{n-1} \binom{n-1}{j} n^{-1-j} c(j+1,k)$$

where c(j+1,k) is a signless stirling number of the first kind.

$$E[\text{Number of components}] = \sum_{j=1}^{n} \frac{n!}{(n-j)!n^j j}$$

$$= \frac{1}{2} (\log 2n+\gamma) + 0(1)$$

where γ is Euler's constant, and further for $n \geq 1$

$$0 \leq E[\text{Number of components}] - \frac{1}{2} (\log 2n + \gamma) \leq e^{-1}$$

$$\sigma^2 [\text{Number of components}] < \frac{1}{4} \log^2 n + c_4 \log n,$$

$c_4 \doteqdot 3.471.$

2.16 (Section 3.4) NUMBER OF LOOP NODES

$$P[\text{w loop nodes}] = \frac{n!\ w}{(n-w)!\ n^{w+1}}$$

$$E[\text{Number of loop nodes}] = \sqrt{\frac{\Pi n}{2}} + 0(1)$$

$$\sigma^2[\text{Number of loop nodes}] = n(2 - \frac{\Pi}{2}) + 0(\sqrt{n})$$

2.17 (Section 4.4) ANALYSIS OF A COMPONENT WITH N NODES

P[component has k loop nodes|component has n nodes]

$$\sim \sqrt{\frac{2}{\Pi n}}\ e^{-k(k-1)|2n} \qquad \text{for } k = o(n^{2/3})$$

Let $E[k|n] \triangleq$ E[number of loop nodes|component has n nodes] then

$$E[k|n] \sim \sqrt{\frac{2n}{\pi}}$$

$$\sqrt{\frac{2n}{\Pi}}\ (1 - \frac{1}{12n}) < E[k|n] < \sqrt{\frac{2n}{\Pi}}\ (1 - \frac{1}{\sqrt{2\Pi}})^{-1}$$

$$\sigma^2[k|n] \sim n(1 - \frac{2}{\Pi})$$

2.18 AN IMPORTANT REMARK (4.4.18)

Suppose a typical type 1A network has components of sizes $n_1, n_2, \ldots, n_k$ containing loops of lengths $\ell_1, \ell_2, \ldots, \ell_k$ respectively, then

$$P[\ell_1, \ell_2, \ldots, \ell_k | n_1, \ldots, n_k] = \prod_{i=1}^{k} P[\ell_i | n_i]$$

2.19 (Section 5.3)

The average height of a node in a tree is $\sim\sqrt{\frac{\Pi n}{2}}$

2.20 THEOREM (5.4.2)

The expected number of trees in a type 1A network

$$\sim \sqrt{\frac{\Pi n}{2}}\ (1 - \frac{1}{e})$$

2.21 (Section (5.5)

The expected number of nodes with k incoming branches

$\sim \frac{n}{ek!}$ for fixed k, as $n \to \infty$.

CHAPTER 3

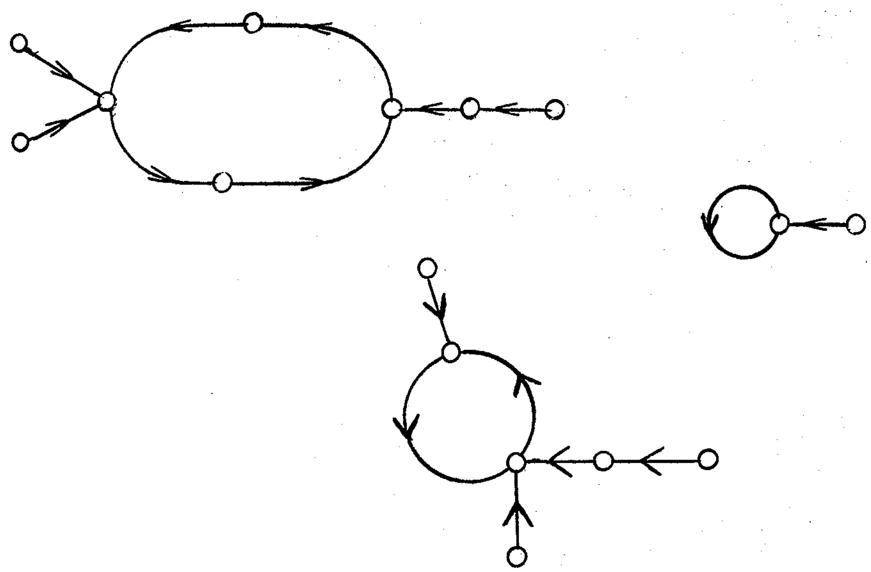
BASIC PROPERTIES OF TYPE 1A NETWORKS

3.1 TYPE 1A NETWORKS - INTRODUCTION

Structure of Type 1A Networks

As defined in section 1.2, T_n, the family of all type 1A networks of n nodes, consists of the n^n graphs* with n nodes and one (excitatory) branch originating at each node, the terminating node of the branch being chosen at random (independently and with replacement) from all the n nodes.

FIG. 3.1.1 TYPICAL TYPE 1A NETWORK



*For definitions borrowed from graph theory, see Appendix 2.

No node has two branches originating at it, so it is clear that a type 1A network consists of a number of components, each component consisting of exactly one directed loop, with trees attached to the nodes of the loop. Figure 3.1.1 shows a typical type 1A network with 16 nodes.

Behavior of Type 1A Networks as Neural Networks

Suppose the initial state contains a active nodes $X_1, X_2, \ldots, X_a$. Let dist (X_i) be the distance† of node X_i from the nearest loop (in the obvious sense). The activity in the trees moves in toward the loops, and after a time

$$LBF = \max_{i=1,\ldots,a} [\text{dist}\ (X_i)] \tag{3.1.1}$$

has elapsed all the activity is in the loops. Up to this time the total activity* has been decreasing or constant, and from now on it remains constant, and the transient behavior of the network has ceased. The network is now in a cyclic state, of period LC (say). The individual loops have periods Π_i equal to some divisor of their total length, as shown for example in Figure 3.1.2.

The period of the whole network is the least common multiple (abbreviated LCM) of the individual periods Π_i.

Then after a total elapsed time of

$$LF = LC + LBF,$$

some state will be repeated for the first time. Our goal is to estimate LBF, LC and LF.

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†Defined formally in section 4.2.

*Defined in section 1.3.

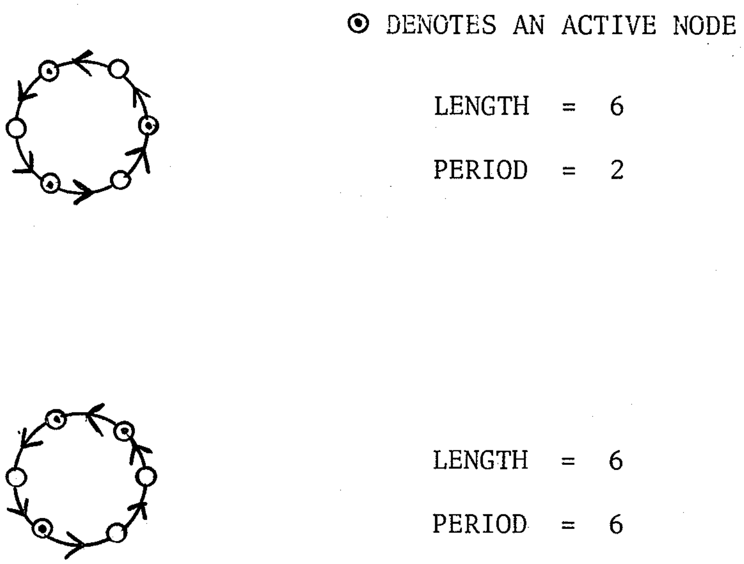


FIG 3.1.2 ACTIVITY IN A LOOP

3.2 DEFINITIONS OF SAMPLE SPACES

Let n be fixed, and let T_n be the family of all type 1A networks with n nodes. It is convenient to define some probability spaces which classify these networks in various ways. The motivation for this is notational convenience, as the following shows. Often we wish to find the average value over T_n of a quantity $X = X(c_1\ c_2\ \ldots\ c_k)$ which depends only on a gross property of type 1A networks, say their component sizes $c_1\ c_2\ \ldots\ c_k$. In this case it is most natural to evaluate $E[X]$ over the probability space Ω_2 (see below), and then we have simply

$$E_{T_n}[X] = E_{T_n}[X]$$

$$= \sum_{\sigma \varepsilon \Omega_2} P(\sigma)\ X(c_1\ c_2\ \dots\ c_k)$$

where σ is the subset of all type 1A networks containing k components of sizes $c_1\ c_2\ \dots\ c_k$.

Also, in this section we derive some basic properties of type 1A networks that will be frequently used later.

3.2.1 <u>Sample Space Ω_1(Same as T_n)</u>

$\Omega_1 = T_n$

There is a 1-1 correspondence between elements of Ω_1 and elements of T_n. Each element of Ω_1 has probability $\frac{1}{n^n}$

<u>Sample Space Ω_2(Classification of T_n According to Sizes of Components)</u>

Define an equivalence relation R_2 among the elements of T_n by $s\ R_2\ t$ if s and t are type 1A networks with the same sized components.

Then let

$$\Omega_2 = T_n/R_2 \tag{3.2.2}$$

i.e., a typical element σ of Ω_2 is the subset of T_n consisting of all type 1A networks with m_1 components of size n_1, m_2 of size n_2, ..., m_{k_1} of size n_{k_1} ; where

$$1 \leq n_1 < n_2 < \dots < n_{k_1} \leq n \tag{3.2.3}$$

and

$$\sum_{\gamma=1}^{k_1} m_\gamma n_\gamma = n \tag{3.2.4}$$

To find the probability of σ we proceed as follows. n elements may be divided into m_1 subsets of size $n_1, \dots, m_{k_1}$ subsets of size n_{k_1}, in

$$\frac{n!}{n_1!^{m_1} \cdots n_{k_1}!^{m_{k_1}} m_1! \cdots m_{k_1}!} \qquad (3.2.5)$$

ways. A subset of size n_γ can be formed into a connected type 1A network in C_{n_γ} ways, where

$$C_j = \sum_{r=o}^{j-1} \frac{(j-1)!j^r}{r!}, \qquad (3.2.6)$$

by equation (4.4.3) below. Therefore from lemmas A4, A5 and A9 of Appendix 1,

$$\sqrt{\frac{\Pi}{2}}\, n^{n-\frac{1}{2}} - \frac{n^{n-1}}{2} \leq C_n \leq \sqrt{\frac{\Pi}{2}}\, n^{n-\frac{1}{2}} e^{\frac{1}{12n}} - \frac{1}{3} n^{n-1} \qquad (3.2.7)$$

and

$$C_n \sim \sqrt{\frac{\Pi}{2}}\, n^{n-\frac{1}{2}} \qquad (3.2.8)$$

Therefore

$$P[\sigma] = \frac{n!}{n^n} \prod_{\gamma=1}^{k_1} \frac{C_{n_\gamma}^{m_\gamma}}{n_\gamma!^{m_\gamma} m_\gamma!} \qquad (3.2.9)$$

(also given by Harris [31])

$$\sim \frac{\sqrt{2\Pi n}}{e^n} \prod_{\gamma=1}^{k_1} \left\{ \frac{\frac{\Pi}{2} n_\gamma^{n_\gamma - \frac{1}{2}}}{\sqrt{2\Pi}\, n_\gamma^{n_\gamma + \frac{1}{2}} e^{-n_\gamma}} \right\}^{m_\gamma} \frac{1}{m_\gamma!} \qquad \text{(using (3.2.8))}$$

$$= \frac{\sqrt{2\Pi n}}{2^k} \prod_{\gamma=1}^{k_1} \frac{1}{n_\gamma^{m_\gamma} m_\gamma!} \qquad (3.2.10)$$

(by (3.2.4) and (3.2.12) below)

$= \frac{\sqrt{2\Pi n}}{2^k}$ P[a random permutation in S_n has m_1 cycles of length $n_1, \ldots, m_{k_1}$ cycles of length n_{k_1}] (3.2.11)

(by [45] p. 67)

where

$$k = \sum_{\gamma=1}^{k_1} m_\gamma \qquad (3.2.12)$$

is the number of components in every member of σ.

Sample Space Ω_3. (Classification of T_n According to Sizes of Loops)

Define an equivalence relation R_3 among the elements of T_n by $s\ R_3\ t$ if s and t are type 1A networks with the same sized loops.

Then let

$$\Omega_3 = T_n/R_3 \qquad (3.2.13)$$

i.e., a typical element σ of Ω_3 is the subset of T_n consisting of all type 1A networks with a_1 loops of length 1, a_2 loops of length 2, ..., a_m loops of length m. Then

$$P[\sigma] = \frac{n!\ w}{n^{w+1}\ a_1!\ldots a_m!\ 1^{a_1}\ldots m^{a_m}\ (n-w)!} \qquad (3.2.14)$$

where $w=a_1+2a_2+\ldots+ma_m$ is the number of loop nodes.

Proof. The a_1 loops of length 1 can be chosen in $\binom{n}{a_1}$ ways. The a_2 loops of length can be chosen from the remaining $(n-a_1)$ nodes in

$$\frac{(n-a_1)!}{(n-a_1-2a_2)!}\ \frac{1}{a_2!\ 2^{a_2}}$$

ways; because we can choose an ordered sample of $2a_2$ nodes in $\frac{(n-a_1)!}{(n-a_1-2a_2)!}$ ways; and then we don't wish to take into account the order among the a_2 pairs, giving the factor $\frac{1}{(a_2)!}$, nor do we care which of the two nodes in any pair is first, giving the factor $1/2^{a_2}$.

Similarly the a_3 loops of length 3 can be chosen in

$$\frac{(n-a_1-2a_2)!}{(n-a_1-2a_2-3a_3)!\; a_3!\; 3^{a_3}}$$

ways from the remaining $(n-a_1-2a_2)$ nodes, and so on.

When the loops have been constructed, using w nodes, the remaining n-w nodes must be arranged to form a rooted w-tree with the w loop nodes as roots. It is known (see lemma A1 of Appendix 1) that this can be done in wn^{n-w-1} ways. Collecting these results,

$$P[\sigma] = \binom{n}{a_1} \frac{(n-a_1)!}{(n-a_1-2a_2)!\; 2^{a_2} a_2!} \cdots$$

$$\cdots \frac{(n-a_1-2a_2-\ldots-(m-1)a_{m-1})!}{(n-w)!\; m^{a_m} a_m!} \cdot \frac{wn^{n-w-1}}{n^n}$$

QED.

Sample Space Ω_4. (Classification of T_n according to Number of Components)

Define an equivalence relation R_4 among the elements of T_n by s R_4 t if s and t contain the same number of components.

Then let

$$\Omega_4 = T_n/R_4 \qquad (3.2.15)$$

i.e., a typical element σ of Ω_4 is the subset of T_n consisting of all type 1A networks with k components. In section 3.3 below we show that

$$P[\sigma] = \sum_{j=o}^{n-1} \binom{n-1}{j} n^{-1-j}\; c(j+1, k),$$

where c(j+1,k) is a signless Stirling number of the first kind.

Sample Space Ω_5. (Classification of T_n according to number of Loop Nodes)

Define R_5 by: $s\ R_5\ t$ if s and t have the same number of loop nodes. Then let

$$\Omega_5 = T_n/R_5 \tag{3.2.16}$$

i.e., a typical element σ of Ω_5 is the subset of T_n consisting of all type 1A networks containing w loop nodes. In section 3.4 we show that

$$P[\sigma] = \frac{n!\ w}{(n-w)!\ n^{w+1}} \tag{3.2.17}$$

Sample Space Ω_6. Initial States (Chosen Without Replacement)

Let $\Omega_6 = \Omega_6(\alpha)$ be the probability space consisting of all initial states with α active nodes (picked at random without replacement from the n nodes). Thus Ω_6 contains $\binom{n}{\alpha}$ elements, each of probability $\binom{n}{\alpha}^{-1}$.

Sample Space Ω_7. Initial States (Chosen With Replacement)

Let $\Omega_7 = \Omega_7(A)$ be the probability space consisting of all initial states formed by picking A nodes out of n (independently and with replacement) and making them active. Thus Ω_7 contains n^A elements, each of probability n^{-A}.

Notation

It will often be helpful to distinguish the probability space over which a particular expectation is taken. Our notation for this is that E_1 is an expectation taken over Ω_1, E_2 over Ω_2, etc.

3.3 CLASSIFICATION OF TYPE 1A NETWORKS BY NUMBER OF COMPONENTS

Let n be a large fixed integer, and let a member g of T_n be chosen at random. Let g have K components, so that K is a random variable. Then

$$P[K=k] = \frac{\text{Number of networks with n nodes and k components}}{n^n}$$

$$= \frac{Q_{n,k}}{n^n} \quad \text{(say)}$$

It is known (see [47])that

$$Q_{n,k} = \sum_{j=o}^{n-1} \binom{n-1}{j} n^{n-1-j} \; c(j+1,k)$$

where c(j+1,k) is a signless Stirling number of the first kind.

$$\text{Prob}\,[K=k] = \sum_{j=o}^{n-1} \binom{n-1}{j} n^{-1-j} \; c(j+1,k) \qquad (3.3.1)$$

From (3.3.1) it can be shown (see[47] page 181) that the j^{th} binomial moment B_{nj} of K is given by

$$B_{nj} = \sum_{k=o}^{n-1} \binom{n-1}{k} n^{-1-k} \; c(k+2,j+1) \qquad (3.3.2)$$

Mean of K

$$E[K] = B_{n_1} = \sum_{k=o}^{n-1} \binom{n-1}{k} n^{-1-k} \; c(k+2,2) \qquad (3.3.3)$$

But([35] p. 159)

$$c(m,2) = (m-1)\,!\,\left(1+\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m-1}\right), \; m \geq 2$$
$$c(1,2) = o \qquad (3.3.4)$$

Substituting (3.3.4) into (3.3.3) Riordan [47] shows

$$E[K] = \sum_{j=1}^{n} \frac{n!}{(n-j)\,!\; n^j \, j} \qquad (3.3.5)$$

Also Kruskal [37] has shown

$$E[K] = \frac{1}{2}\,(\log 2n + \gamma) + o(1) \qquad (3.3.6)$$

where γ is Euler's constant. Here are his main steps:

Let $E_n \overset{\Delta}{=} E[K]$, and define A_n by

$$E_n = \int_o^{A_n} (1-e^{-x}) \frac{dx}{x} \tag{3.3.7}$$

Kruskal then shows

$$\log A_n = \frac{1}{2} (\log 2n-\gamma) + o(1) \tag{3.3.8}$$

whose γ is Euler's constant, and from this that

$$E_n = \frac{1}{2} (\log 2n + \gamma) + \int_{A_n}^{\infty} e^{-x} \frac{dx}{x} \tag{3.3.9}$$

which using (3.3.8) becomes

$$E_n = \frac{1}{2} (\log 2n + \gamma) + o(1) \tag{3.3.10}$$

$$= \frac{1}{2} \log n + 0(1) \tag{3.3.11}$$

Kruskal also shows that E_n approaches infinity monotonically with n.

We will now use these results to get bounds for E_n. Since the integrand of (3.3.7) is positive for $x \geq 0$, it follows that A_n approaches infinity monotonically with n. Now for $n \geq 1$ certainly $E_n \geq 1$, by definition. Also

$$-e^{-x} \leq -1+x, \quad x \geq o$$

Therefore from (3.3.7)

$$E_n \leq \int_o^{A_n} dx = A_n,$$

$$A_n \geq 1 \text{ for } n \geq 1,$$

$$\int_{A_n}^{\infty} e^{-x} \frac{dx}{x} \leq \int_1^{\infty} e^{-x} \frac{dx}{x} \leq \int_1^{\infty} e^{-x} dx = e^{-1}$$

Therefore from (3.3.9),

$$0 \le E_n - \frac{1}{2}(\log 2n + \gamma) \le e^{-1} \quad \text{for } n \ge 1 \tag{3.3.12}$$

Variance of K

In terms of the binomial moments we have

$$\sigma^2[K] = 2B_{n,2} + B_{n,1} - B_{n,1}^2 \tag{3.3.13}$$

Now $c(m+1,\ 3) = \frac{m!}{2}\left[\left(\sum_{r=1}^{m} \frac{1}{r}\right)^2 - \sum_{r=1}^{m} \frac{1}{r^2}\right]$ ([35] p. 159)

$$= m! \sum_{\substack{r,s=1 \\ s<r}}^{m} \frac{1}{rs} \tag{3.3.14}$$

From (3.3.2):

$$B_{n,2} = \sum_{k=1}^{n-1} \binom{n-1}{k} n^{-1-k} (k+1)! \sum_{\substack{r,s=1 \\ s<r}}^{k+1} \frac{1}{rs}$$

$$= \sum_{r=2}^{n} \sum_{s=1}^{r-1} \frac{1}{rs} \sum_{k=r-1}^{n-1} \binom{n-1}{k} n^{-1-k} (k+1)!$$

$$= \sum_{r=2}^{n} \sum_{s=1}^{r-1} \frac{1}{rs} A_{nr}$$

where $A_{nr} = \sum_{k=r-1}^{n-1} \binom{n-1}{k} n^{-1-k} (k+1)!$. Luckily it is known

that

$$A_{nr} = \frac{n!}{(n-r)!\ n^r}$$ ([47])

so

$$B_{n,2} = \sum_{r=2}^{n} \sum_{s=1}^{r-1} \frac{1}{rs} \frac{n!}{(n-r)!\ n^r}$$

$$= \frac{1}{2}\left[\sum_{r,s=1}^{n} \frac{1}{rs} \frac{n!}{(n-r)!\ n^r} - \sum_{r=1}^{n} \frac{1}{r^2} \frac{n!}{(n-r)!\ n^r} \right]$$

$$= \frac{1}{2}\left[\sum_{s=1}^{n} \frac{1}{s} \sum_{r=1}^{n} \frac{n!}{(n-r)!\ n^r r} - \sum_{r=1}^{n} \frac{n!}{r^2 (n-r)!\ n^r} \right] \tag{3.3.15}$$

Now $\sum_{s=1}^{n} \frac{1}{s} < 1 + \log n$

so (3.3.15) becomes, using (3.3.5) and (3.3.12),

$$B_{n,2} < \frac{1}{2}(1 + \log n)\left(\frac{1}{2} \log n + \frac{1}{2} \log 2 + \frac{\gamma}{2} + e^{-1}\right)$$

$$< \frac{1}{4} \log^2 n + c_1 \log n \quad \text{for } n \geq 3 \tag{3.3.16}$$

where $c_1 = \frac{1}{2} \log 2 + \frac{\gamma}{2} + e^{-1} + \frac{1}{2} = 1.503\ldots$

From (3.3.12), (3.3.13), (3.3.16) for $n \geq 3$

$$\sigma^2[K] < \frac{1}{4} \log^2 n + c_2 \log n + c_3$$

where $c_2 = 2c_1 + \frac{1}{2} - \frac{1}{2}(\log 2 + \gamma) \approx 2.871\ldots$

$$c_3 = \frac{1}{2}(\log 2 + \gamma) + e^{-1} - \frac{1}{4}(\log 2 + \gamma)^2 = 0.600\ldots$$

or more simply, for $n \geq 3$,

$$\sigma^2[K] < \frac{1}{4} \log^2 n + c_4 \log n \tag{3.3.17}$$

where $c_4 = |c_2| + |c_3| = 3.471\ldots \quad (3.3.18)$

3.4 CLASSIFICATION OF TYPE 1A NETWORKS BY NUMBER OF LOOP NODES

3.4.1 Theorem

The probability that a type 1A network contains w loop nodes is

$$L_w = \frac{n!\ w}{(n-w)!\ n^{w+1}} \tag{3.4.2}$$

Proof. The w loop nodes can be chosen in $\binom{n}{w}$ ways. For each such choice the w loop nodes can be arranged into loops in w! ways, since there is a 1-1 correspondence between such arrangements and permutations of w objects. The remaining n-w nodes do not lie in loops, and if we imagine that the w branches forming loops to be removed, the n node network remaining is a w-tree with the w loop nodes as roots (see Appendix 2 for definitions). By lemma A2, there are wn^{n-w-1} such w-trees. Therefore

$$L_w = \binom{n}{w}\ w!\ \frac{wn^{n-w-1}}{n^n}$$ QED

Notation. It is convenient to define

$$\phi_n(w) = \frac{n!}{(n-w)!\ n^w} \tag{3.4.3}$$

so that $L_w = \frac{w}{n}\,\phi_n(w)$

Lemma (3.4.4)

The mean and variance of w are given by

$$\text{Mean} = \sqrt{\frac{\pi n}{2}} + 0(1)$$

$$\text{Variance} = n(2 - \frac{\Pi}{2}) + 0(\sqrt{n})$$

Proof. $\text{Mean} = \sum_{w=1}^{n} \frac{n!}{(n-w)!}\ \frac{w^2}{n^{w+1}}$ (3.4.5)

$$E[w^2] = \sum_{w=1}^{n} \frac{n!}{(n-w)!} \frac{w^3}{n^{w+1}}$$

and the lemma follows by lemmas A7, A8 of Appendix 1. QED

CHAPTER 4

LBF AND LC FOR TYPE 1A NETWORKS

4.1 LOWER BOUND TO EXPECTED VALUE OF LBF

Case I Initial state contains exactly one active node β. (In this case $\Omega_6 = \Omega_7$)

Suppose that β is given. The event that LC = j and LF = k can only happen if node β is connected to a different node β_1, β_1 is connected to a different node β_2,...and β_{k-1} is connected to β_{k-j}. Therefore

$$P_{\Omega_1 x\Omega_6}[LC=j,\ LF=k] = \frac{(n-1)\ (n-2)\ldots(n-k+1)}{n^k}$$

$$= \frac{(n-1)!}{(n-k)!n^k} \tag{4.1.1}$$

$$P_{\Omega_1 x\Omega_6}[LBF=b] = \sum_{k=b+1}^{n} P[LC = k-b,\ LF = k] \tag{4.1.2}$$

Therefore

$$E_{\Omega_1 x\Omega_6}[LBF] = \sum_{b=o}^{n-1} b \sum_{k=b+1}^{n} \frac{(n-1)!}{(n-k)!n^k}$$

$$= \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!n^k} \sum_{b=0}^{k-1} b$$

$$= \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!n^k} \frac{k(k-1)}{2}$$

$$= \frac{(n-1)!}{2n^n} \sum_{r=o}^{n-1} (n-r)\ (n-r-1)\ \frac{n^r}{r!} \tag{4.1.3}$$

$$= \frac{n!}{2n^n} \sum_{s=o}^{n-1} \frac{n^s}{s!} - \frac{1}{2} \quad \text{by lemmas A6, A7.} \tag{4.1.4}$$

From lemmas A5, A9, and (4.1.4) we obtain

$$\frac{\sqrt{2\Pi n}}{4} - \frac{3}{4} < E[LBF] < \frac{\sqrt{2\Pi n}}{4} e^{\frac{1}{12n}} - \frac{2}{3} \tag{4.1.5}$$

and therefore

$$E[LBF] = \frac{\sqrt{2\Pi n}}{4} + 0(1) \tag{4.1.6}$$

In the same way

$$E[(LBF)^2] = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!n^k} \frac{k(k-1)(2k-1)}{6} \tag{4.1.7}$$

$$= \frac{(n-1)!}{6n^n} \sum_{r=o}^{n-1} \frac{n^r}{r!} [2(n-r)^3 - 3(n-r)^2 + (n-r)] \quad \text{where } r=n-k$$

$$= \frac{(n-1)!}{6n^n} \left(\frac{4\ n^{n+1}}{(n-1)!} - 5n \sum_{s=o}^{n-1} \frac{n^s}{s!} + \frac{n^n}{(n-1)!} \right) \text{by lemmas A6-A8}$$

$$\sim \frac{2n}{3}$$ by lemma A4. We conclude that:

(4.1.9) <u>Theorem</u> If the initial state contains one active node,

$$E[LBF] = \frac{\sqrt{2\Pi n}}{4} + 0\ (1) \tag{4.1.10}$$

$$\sigma^2[LBF] = n\left(\frac{2}{3} - \frac{\Pi}{8}\right) + 0(\sqrt{n}) \tag{4.1.11}$$

and for arbitrary initial activity (4.1.10) is a lower bound to E[LBF].

<u>Distribution of LBF</u>

Still for case I, let $Z = \frac{LBF}{\sqrt{n}}$. Then Harris shows ([31]) using Stirling's approximation (in a different context) that from (4.1.2) it follows that (as $n \to \infty$) Z has the density function

$$= \frac{n!}{2n^n} \sum_{s=o}^{n-1} \frac{n^s}{s!} - \frac{1}{2} \qquad \text{by lemmas A6, A7.} \tag{4.1.4}$$

From lemmas A5, A9, and (4.1.4) we obtain

$$\frac{\sqrt{2\Pi n}}{4} - \frac{3}{4} < E[LBF] < \frac{\sqrt{2\Pi n}}{4} e^{\frac{1}{12n}} - \frac{2}{3} \tag{4.1.5}$$

and therefore

$$E[LBF] = \frac{\sqrt{2\Pi n}}{4} + O(1) \tag{4.1.6}$$

In the same way

$$E[(LBF)^2] = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!n^k} \frac{k(k-1)(2k-1)}{6} \tag{4.1.7}$$

$$= \frac{(n-1)!}{6n^n} \sum_{r=o}^{n-1} \frac{n^r}{r!} [2(n-r)^3 - 3(n-r)^2 + (n-r)] \quad \text{where } r=n-k$$

$$= \frac{(n-1)!}{6n^n} \left(\frac{4\, n^{n+1}}{(n-1)!} - 5n \sum_{s=o}^{n-1} \frac{n^s}{s!} + \frac{n^n}{(n-1)!} \right) \text{by lemmas A6-A8}$$

$\sim \frac{2n}{3}$ by lemma A4. We conclude that:

(4.1.9) Theorem If the initial state contains one active node,

$$E[LBF] = \frac{\sqrt{2\Pi n}}{4} + O(1) \tag{4.1.10}$$

$$\sigma^2[LBF] = n\left(\frac{2}{3} - \frac{\Pi}{8}\right) + O(\sqrt{n}) \tag{4.1.11}$$

and for arbitrary initial activity (4.1.10) is a lower bound to E[LBF].

Distribution of LBF

Still for case I, let $Z = \frac{LBF}{\sqrt{n}}$. Then Harris shows ([31]) using Stirling's approximation (in a different context) that from (4.1.2) it follows that (as $n \to \infty$) Z has the density function

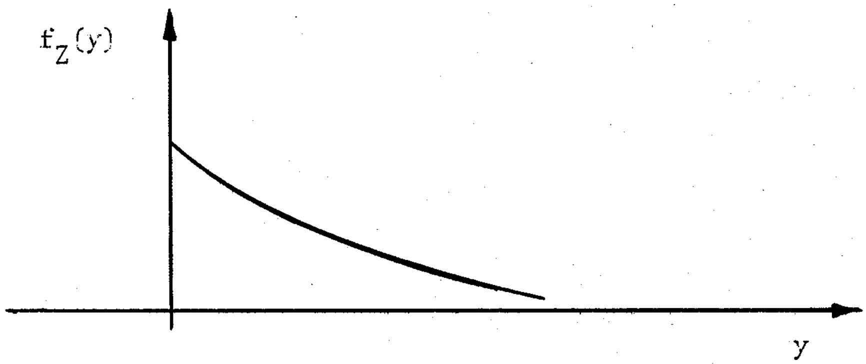
$$f_Z(y) \sim \begin{cases} \sqrt{2\Pi}\ (1 - \Phi(y)) & y \geq 0 \\ 0 & y < 0 \end{cases} \tag{4.1.12}$$

where

$$\Phi(y) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{y} e^{-x^2/2} dx \tag{4.1.13}$$

Fig. (4.1.1) shows a sketch of $f_Z(y)$

FIG. 4.1.1



Therefore Z has the distribution (as $n \to \infty$)

$$F_Z(y) \sim \begin{cases} 1-e^{-y^2/2} + y\sqrt{2\Pi}\ (1 - \Phi(y)) & y \geq 0 \\ 0 & y < 0 \end{cases} \tag{4.1.14}$$

Case II Initial state contains p active nodes $a_1, a_2, \ldots, a_p$, where p is small compared with n.

Then

$$LBF(a_i) = \max_{i = 1,\ldots,p} [\text{dist}\ (a_i)] \tag{4.1.15}$$

$$P[LBF(a_i) < x] = P[\max_i [dist(a_i)] < x] \quad (4.1.16)$$

$$= P[dist(a_i) < x, \forall i]$$

$$= \prod_{i=1}^{p} P[dist(a_i) < x]$$ (This is an approximation since the events are not really independent. Once it is known that $dist(a_1) < x$, a small piece of the network is fixed, and the distribution of $dist(a_2)$ is, strictly speaking, no longer based on (4.1.1). However, for p small compared with n, the approximation will be a good one.)

$$= P[\text{LBF of Case I} < x]^p \quad (4.1.17)$$

Let $Z_p = \frac{LBF}{\sqrt{n}}$, then Z_p approximately has the distribution function

$$F_p(y) = (F_Z(y))^p \quad (4.1.18)$$

where $F_Z(y)$ is defined by equation (4.1.14) above.

Z_p has the approximate density function

$$f_p(y) = pF_Z(y)^{p-1} f_Z(y) \quad (4.1.19)$$

where $f_Z(y)$ is defined by equation (4.1.12) above.

4.2 UPPER BOUND TO EXPECTED VALUE OF LBF

In this section we give a graph theoretical approach which leads to an interesting pair of bounds for E[LBF] (Theorem (4.2.10)). Because of the results of the previous section we do not do anything further with the lower bound. At the end of this section we show how the upper bound may be evaluated on a computer, and give a conjectured upper bound based on computational results, (equation (4.2.22)).

Our bounds are based on assuming an initial activity either of 1 node, or of n nodes, and so apply equally well to both

$E_{\Omega_1 x \Omega_6}$ [LBF] and $E_{\Omega_1 x \Omega_7}$ [LBF].

Let $G_{n,m}$ be the family of all rooted m-trees, containing n+m nodes, of which m are specified in advance as roots. (For definitions see appendix 2.) By lemma A1, $G_{n,m}$ has $m(n+m)^{n-1}$ elements.

If $g \varepsilon G_{n,m}$ and x is a node of g, define

dist(x)

to be the length of the (unique) path in g joining x to its root node. Let

$$\text{weight } (g) \triangleq \sum_{\text{all nodes } x \varepsilon g} \text{dist}(x) \quad (4.2.1)$$

$$\text{height } (g) \triangleq \max_{\text{all nodes } x \varepsilon g} [\text{dist}(x)] \quad (4.2.2)$$

$$W_{n,m} \triangleq \sum_{g \varepsilon G_{n,m}} \text{weight } (g) \quad (4.2.3)$$

$$Y_{n,m} \triangleq \sum_{g \varepsilon G_{n,m}} \text{height } (g) \quad (4.2.4)$$

$$V_{n,m} \triangleq \frac{W_{n,m}}{m(n+m)^n} \quad (4.2.5)$$

$$= \underset{g \varepsilon G_{n,m}}{\text{average}} \left[\underset{\text{nodes of } g}{\text{average}} \text{ (distance of a node from its root)} \right]$$

$$U_{n,m} \triangleq \frac{Y_{n,m}}{m(n+m)^{n-1}} \quad (4.2.6)$$

$$= \underset{g \varepsilon G_{n,m}}{\text{average}} \left[\underset{\text{nodes of } g}{\text{maximum}} \text{ (distance of a node from its root)} \right]$$

Bounds for E[LBF]

Case I One active node in initial state

Then E[LBF|given that there are m loop nodes] = $V_{n-m,m}$

$$E[LBF] = \sum_{m=1}^{n} V_{n-m,m} L_m \tag{4.2.7}$$

where L_m is the probability that a type 1A network has m loop nodes, and by (3.2.17)

$$L_w = \frac{n!\ w}{(n-w)!\ n^{w+1}}$$

It follows that for any number of active nodes in the initial state,

$$E[LBF] \geq \sum_{m=1}^{n} L_m V_{n-m,m} \tag{4.2.8}$$

Case II n active nodes in initial state, i.e., 100% initial activity

As in case I, we obtain

$$E[LBF|m \text{ loop nodes}] = U_{n-m,m}$$

$$E[LBF] = \sum_{m=1}^{n} L_m U_{n-m,m}$$

and for any number of active nodes in the initial state,

$$E[LBF] \leq \sum_{m=1}^{n} L_m U_{n-m,m} \tag{4.2.9}$$

Conclusion:

Theorem (4.2.10)

$$\sum_{m=1}^{n} L_m V_{n-m,m} \leq E[LBF] \leq \sum_{m=1}^{n} L_m U_{n-m,m}$$

and for low initial activity E[LBF] approaches the left hand bound, for high initial activity the right.

In view of the lower bound obtained in the previous section, we do not pursue the left hand side of this any further here.

Calculation of $Y_{n,m}$

Let $R^{(m)}_{n,h}$ be the number of rooted m-trees with n labelled* nodes and height h, and $S^{m}_{n,h}$ the number with height at most h.

Then

$$Y_{n,m} = \sum_{h=1}^{n} \frac{h\, R^{(m)}_{n+m,h}}{\binom{n+m}{n}} \tag{4.2.11}$$

(the $\binom{n+m}{n}$ appears because this is the number of ways of choosing the root nodes), and

$$R^{(m)}_{n,h} = S^{(m)}_{n,h} - S^{(m)}_{n,h-1} \tag{4.2.12}$$

Let us write $S_{n,h}$ for $S^{(1)}_{n,h}$. Then Riordan [46] has shown that $S_{n,h}$ is given by the recurrence relation

$$S_{1,k} = 1,\ k \geq 1;\ S_{k,1} = k,\ k \geq 1$$

$$(n-1)\, S_{n,h+1} = \sum_{k=1}^{n-1} k \binom{n}{k} S_{n-k,\,h+1}\, S_{k,h} \quad \text{for } n > 1,\ h \geq 1 \tag{4.2.13}$$

From Polya's theorem,

$$\sum_{n=m}^{\infty} S^{(m)}_{n,h} \frac{x^n}{n!} = \frac{1}{m!} \left(\sum_{n=1}^{\infty} S_{n,h} \frac{x^n}{n!} \right)^m \tag{4.2.14}$$

From (4.2.6), (4.2.11), (4.2.12),

$$U_{n,m} = \frac{Y_{n,m}}{m(n+m)^{n-1}} = \frac{1}{\binom{m+n}{n} m\ (n+m)^{n-1}} \sum_{h=1}^{n} h \left[S^{(m)}_{n+m,h} - S^{(m)}_{n+m,h-1} \right]$$

$$= \frac{1}{\binom{m+n}{n} m(n+m)^{n-1}} \left[n\ S^{(m)}_{n+m,n} - \sum_{i=o}^{n-1} S^{(m)}_{n+m,i} \right] \qquad (4.2.15)$$

Now $S^{(m)}_{n+m,n}$ is just the number of m-trees of n+m nodes

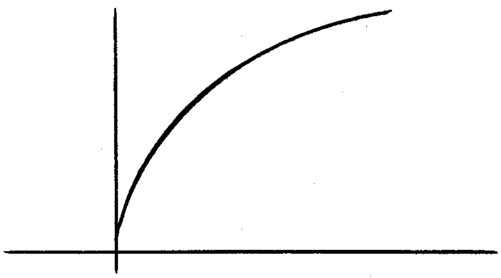
$$= \binom{n+m}{m}\ m\ (n+m)^{n-1} \quad \text{by lemma A1} \qquad (4.2.16)$$

From (4.2.15), (4.2.16),

$$U_{n,m} = n - \frac{1}{\binom{m+n}{n} m\ (n+m)^{n-1}} \sum_{s=1}^{n-1} S^{(m)}_{n+m,i} \qquad (4.2.17)$$

so we may compute $U_{n,m}$ from (4.2.13), (4.2.14), (4.2.17).

Remark It is much more feasible to calculate the transient quantities LBF, $U_{n,m}$, $V_{n,m}$ with a computer than it is to calculate the cycle time, because the former are roughly of the order of $n^{1/2}$ and rapidly show their true shape,



whereas the latter is of the order of $n^{\log n}$ and does not grow until n is very large , thus:

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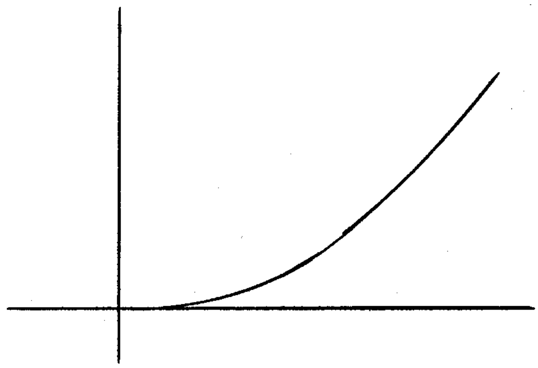
*For the explanation of why we use labelled graphs, see the remark at the end of Appendix 2.



We may obtain a simpler but cruder upper bound to E[LBF] than (4.2.9) as follows.

It is clear that a type 1A network containing exactly two trees has on the average a lower LBF than a type 1A network with exactly one tree; etc. Thus E[LBF] is a maximum in the case that there is exactly one tree (i.e., in the case that there is one component, and this component contains a loop of length one-- from sections 3.3 and 4.4 it easily follows that this event has probability $\sim\sqrt{\frac{\Pi}{2n} \cdot \frac{2}{\Pi n}} = \frac{1}{n}$, which is relatively "large".) Thus a cruder bound than (4.2.9) is

$$E[LBF] \leq u_{n-1,\ 1} \qquad (4.2.18)$$

<u>A Conjecture</u>

Certainly

$$u_{n,1} < n \qquad (4.2.19)$$

and section 4.1 suggests

$$\lim_{n \to \infty} \quad u_{n,1} \sqrt{\frac{2}{\Pi n}} > 1$$

So we are led to conjecture that

$$u_{n,1} \sim a\ n^{b}\ ,\ n \to \infty,\ a,\ b \text{ constants,} \qquad (4.2.20)$$

Computer calculation of $u_{n,1}$ using (4.2.13), (4.2.1]), (4.2.17) suggests that this is true, and that

$$0.564 > b \geq 0.5 \tag{4.2.21}$$

From (4.2.18), (4.2.20), (4.2.21) we would then have

$$E[LBF] \leq an^{0.564} \tag{4.2.22}$$

for some constant a, which agrees closely with the lower bound $\sqrt{\frac{\Pi}{8}}\, n^{0.5}$ obtained in Theorem (4.1.9) above.

4.3 BOUNDS ON EXPECTED VALUE OF LC - INTRODUCTION

In the next six sections we give bounds on the expected value of the cycle time LC and on the expected value of the LCM of the loop lengths. We begin by considering various properties of a typical component of a type 1A network (in section 4.4), then give a lower bound to the expected value of the LCM of k integers picked at random (in section 4.5). Following this we derive these bounds:

(i) A lower bound to E[LC], Theorem (4.6.44) (depending on assumptions (4.5.9), (4.6.18), (4.6.33) and (4.6.48), provided that the initial activity is chosen according to (4.6.3).

(ii) A lower bound to E[LCM of loop lengths], Theorem (4.7.12), (depending on assumptions (4.5.9) and (4.6.48)).

(iii) An upper bound to E[LC] over all networks with fewer than A log n components, (4.8.9).

(iv) Two upper bounds to E[LC] over all type 1A networks, Theorems (4.8.11) and (4.8.13), by two different methods.

(v) A third method of obtaining an upper bound to E[LC], section 4.9. This requires for its successful completion an as-yet unknown upper bound to the LCM of k numbers, equation (4.9.1).

4.4 ANALYSIS OF A COMPONENT WITH N NODES

Here we analyze the number of loop nodes, k, in a particular connected component of a type 1A network, given that the component has n nodes. Then $k = 1, 2, \ldots, n$.

Let C_{nk} = number of possible different components with n nodes and k loop nodes. It follows from the proof of Theorem (3.4.1) and the fact that the number of ways of seating k people at a circular table is $(k-1)!$ that

$$C_{n,k} = \binom{n}{k} k\, n^{n-k-1} (k-1)! = \phi_n(k) n^{n-1} \qquad (4.4.1)$$

where we define

$$\phi_n(h) = \frac{n!}{(n-k)!\,n^h} \qquad (4.4.2)$$

Let C_n = total number of different components with n nodes $= \sum_{k=1}^{n} C_{n,k}$

$$= (n-1)! \sum_{k=o}^{n-1} \frac{n^k}{k!} \qquad (4.4.3)$$

$$\sim \sqrt{\frac{\pi}{2}}\, n^{n-1/2}, \quad n \to \infty, \qquad (4.4.4)$$

by lemma A4.

Let $p_{nk} = \frac{C_{nk}}{C_n}$ = probability that a component with n nodes has k loop nodes

$$= \frac{n^{n-k}}{(n-k)!} \cdot \frac{1}{\sum_{r=o}^{n-1} \frac{n^r}{r!}} \qquad (4.4.5)$$

$$\sim \sqrt{\frac{2}{\pi n}}\, \phi_n(k) \quad \text{for } n \to \infty, \text{ uniformly in } k, \qquad (4.4.6)$$

by lemma A4.

Note $\frac{C_{n,k+1}}{C_{n,k}} = \frac{n-k}{n} < 1$. So

$$C_{n,k} > C_{n,k+1}, \text{ for all } k \tag{4.4.7}$$

Expected value of k, given n,

$$E[k|n] = \sum_{k=1}^{n} kp_{nk} = \frac{1}{\sum_{k=o}^{n-1} \frac{n^k}{k!}} \sum_{k=1}^{n} \frac{kn^{n-h}}{(n-k)!}$$

From lemmas A5, A6, A9 this becomes

$$\sqrt{\frac{2n}{\Pi}} \left(1- \frac{1}{12n}\right) < E[k|n] < \sqrt{\frac{2n}{\Pi}} \left(1 + \frac{B}{\sqrt{n}}\right) \leq C\sqrt{\frac{2n}{\Pi}} \tag{4.4.8}$$

for $n \geq 1$, where $B = \frac{1}{\sqrt{2\Pi} - 1}$ and $C = \left(1 - \frac{1}{\sqrt{2\Pi}}\right)^{-1}$

Therefore

$$E[k|n] \sim \sqrt{\frac{2n}{\Pi}} \text{ as } n \to \infty \tag{4.4.9}$$

$$E[k^2|n] = \frac{1}{\sum_{k=o}^{n-1} \frac{n^k}{k!}} \sum_{k=1}^{n} \frac{k^2\ n^{n-k}}{(n-k)!}$$

$= n$ (by lemmas A4, A7)

$$\sigma^2\ [k|n] \sim n - \frac{2n}{\Pi} = n\left(1 - \frac{2}{\Pi}\right) \tag{4.4.10}$$

For $k = o(n^{2/}3)$, by lemma A2,

$$\phi_n(k) = e^{\left[\frac{-k(k-1)}{2n} + 0\left(\frac{k^3}{n^2}\right)\right]} \tag{4.4.11}$$

$$p_{nk} \sim \sqrt{\frac{2}{\Pi n}}\ e^{-k(k-1)/2n}, \quad k = o(n^{2/}3), \tag{4.4.12}$$

So, roughly, k has an exponential distribution.

For $\alpha,\beta = o(n^{2/3})$, from (4.4.12) we have

$$P[\alpha \le k \le \beta] \sim \sum_{k=\alpha}^{\beta} \sqrt{\frac{2}{\Pi n}}\; e^{-k(k-1)/2n} \tag{4.4.13}$$

$$\sim \sqrt{\frac{2}{\Pi n}} \int_{\alpha}^{\beta} e^{-x(x-1)/2n}\, dx \tag{4.4.14}$$

Roughly, $P[\alpha \le k \le \beta] \sim \sqrt{\frac{2}{\Pi n}} \int_{\alpha}^{\beta} e^{-x^2/2n}\, dx$

$$= \sqrt{\frac{2}{\Pi}} \int_{\alpha/\sqrt{n}}^{\beta/\sqrt{n}} e^{-y^2/2}\, dy \tag{4.4.15}$$

To summarize this section:

Theorem (4.4.16) For type 1A networks:

E [number of loop nodes in a component | component has n nodes] =

$E[k|n]$ satisfies, for $B = (\sqrt{2\Pi}-1)^{-1}$, $C = (1 - \frac{1}{\sqrt{2\Pi}})^{-1}$

$$\sqrt{\frac{2n}{\Pi}}\left(1 - \frac{1}{12n}\right) < E[k|n] < \sqrt{\frac{2n}{\Pi}}\left(1 + \frac{B}{\sqrt{n}}\right) \le C\sqrt{\frac{2n}{\Pi}}, \quad n \ge 1 \tag{4.4.8}$$

$$E[k|n] \sim \sqrt{\frac{2n}{\Pi}} \quad \text{as } n \to \infty, \tag{4.4.9}$$

and

$$E[k|1] = 1 \tag{4.4.17}$$

$$\sigma^2[k|n] \sim n\left(1 - \frac{2}{\Pi}\right) \tag{4.4.10}$$

Remark The results of this section are independent of the total number of nodes in the network. It follows that if a typical type 1A network has components of sizes $n_1, n_2, \ldots, n_k$ containing loops of lengths $l_1, l_2, \ldots, l_k$, resp., then

$$P[l_1, l_2, \ldots, l_k | n_1, n_2, \ldots, n_k] = \prod_{i=1}^{k} P[l_i | n_i] \quad (4.4.19)$$

4.5 ESTIMATES FOR GCD AND LCM OF K NUMBERS - INTRODUCTION

Let k integers $X_1, X_2, \ldots, X_k$ be chosen at random, independently and with replacement, from $[1, 2, \ldots, n]$. In this section we find a lower bound to

$$\frac{E\,[LCM\,(X_1,\ldots,X_k)]}{E\,[X_1, X_2, \ldots, X_k]}$$

This lower bound is based on an approximation, namely (4.5.9), however in (4.5.24) we give several arguments in support of this approximation, the main one being that when it is used to estimate

$$E\,[GCD(X_1,\ldots,X_k)]^{\lambda} \qquad, \lambda \leq k-1$$

it gives the correct result as obtained (by a different and rigorous method) by Gegenbauer [19].

Decomposition of X_i into Prime Factors

Let k integers $X_1, X_2, \ldots, X_k$ be chosen at random, independently and with replacement, from $[1, 2, \ldots, n]$.

Let $a_p(X_i)$ be the highest power of the prime p that divides X_i. Thus X_i and $a_p(X_i)$ are random variables.

Probability distribution of $a_p(X_i)$. Suppose $r \geq 1$, and let $n = Ap^r + B$, where $0 \leq B < p^r$. Then

$$P[a_p(X_i) \geq r] = P[p^r \text{ divides } X_i] = \frac{A}{n} = \frac{n-B}{np^r} \quad (4.5.1)$$

$$= \frac{1}{p^r} - \frac{B}{np^r} = \frac{1}{p^r} - \frac{\varepsilon}{n}, \text{ where } 0 \leq \varepsilon < 1$$

Of course

$$P[a_p(X_i) \geq 0\ ] = 1 \tag{4.5.2}$$

Let p_j, j = 1, 2, ..., be different primes. In the same way that (4.5.1) was proved we get, for any $r_j \geq 0$,

$$P[a_{p_j}(X_i) \geq r_j;\ j=1,2,\ldots,\mu] = \frac{1}{\prod_{j=1}^{\mu} p_j^{\,r_j}} - \frac{\varepsilon}{n},\ o \leq \varepsilon < 1 \tag{4.5.3}$$

$$\begin{aligned} P[a_p(X_i) = r] &= P[a_p(X_i) \geq r] - P[a_p(X_i) \geq r+1] \qquad (4.5.4)\\ &= \frac{1}{p^r} - \frac{\varepsilon}{n} - \frac{1}{p^{r+1}} + \frac{\varepsilon'}{n} \\ &= \frac{p-1}{p^{r+1}} + \frac{\delta}{n},\ |\delta| < 1 \end{aligned}$$

(4.5.4) is true for all $r \geq o$, but of course $a_p(X_i)$ cannot exceed $[\log_p n]$, and so

$$P[a_p(X_i) = r] = 0 \quad \text{for } r > [\log_p n] \tag{4.5.5}$$

Let Z_j, j=1,2,...,μ be any random variables taking only integer values. Then

$$P[Z_j = r_j,\ j=1,\ldots,\mu] = \sum_{\varepsilon_j = o \text{ or } 1} (-1)^{\sum_{j=1}^{\mu} \varepsilon_j} P[Z_j \geq r_j + \varepsilon_j,\ j=1,\ldots,\mu] \tag{4.5.6}$$

where the summation extends over the 2^μ possible choices for $\varepsilon_1,\ldots,\varepsilon_\mu$. Thus the generalization of (4.5.4) to several primes is

$$P[a_{p_j}(X_i) = r_j,\ j=1,\ldots,\mu] = \sum_{\epsilon_{j=o,1}} (-1)^{\sum_{j=1}^{\mu} \varepsilon_j} \left\{ \frac{1}{\prod_{j=1}^{\mu} p_j^{\,r_j+\varepsilon_j}} - \frac{\delta(p_j,r_j,\varepsilon_j)}{n} \right\} \quad \text{where } |\delta| < 1$$

(from (4.5.3) and (4.5.6))

$$= \sum_{\varepsilon_j=0,1} \frac{(-1)^{\sum_{j=1}^{\mu} \varepsilon_j}}{\prod_{j=1}^{\mu} p_j^{r_j+\varepsilon_j}} + \frac{2^\mu \delta}{n}, \quad |\delta| < 1,$$

$$= \prod_{j=1}^{\mu} \left(\frac{1}{p_j^{r_j}} - \frac{1}{p_j^{r_j+1}} \right) + \frac{2^\mu \delta}{n} \qquad |\delta| < 1 \tag{4.5.7}$$

(verify by multiplying out the latter product)

whereas by (4.5.4),

$$\prod_{j=1}^{\mu} P[a_{p_j}(X_i) = r_j] = \prod_{j=1}^{\mu} \left(\frac{1}{p_j^{r_j}} - \frac{1}{p_j^{r_j+1}} + \frac{\delta}{n} \right) \qquad |\delta| < 1$$

$$= \prod_{j=1}^{\mu} \left(\frac{1}{p_j^{r_j}} - \frac{1}{p_j^{r_j+1}} \right) + \text{remainder} \tag{4.5.8}$$

Comparison of (4.5.7) and (4.5.8) suggests (but certainly does not prove) the following

(4.5.9) Approximation For large n, small error is introduced by assuming the random variables $a_2(X_i)$, $a_3(X_i)$, $a_5(X_i)$, ..., $a_p(X_i)$, ... to be independent; and to have the distributions

$$P[a_p(X_i) = r] = \frac{p-1}{p^{r+1}}, \text{ all } r \geq 0 \tag{4.5.10}$$

For the remainder of this section, we assume that (4.5.9) is true. The following equations, therefore, are to be considered as conditional on the truth of (4.5.9). (See also the remarks (4.5.24).)

Definition of the Random Variables A_p, B_p, C_p

Let

$$A_p = \sum_{i=1}^{k} a_p(X_i), \quad B_p = \max_{i=1}^{k} a_p(X_i), \quad C_p = \min_{i=1}^{k} a_p(X_i) \tag{4.5.11}$$

Then clearly

$$X_1 . X_2 . \dots . X_k = \prod_{p \le n} p^{A_p} \tag{4.5.12}$$

$$LCM[X_1, X_2, \dots, X_k] = \prod_{p \le n} p^{B_p} \tag{4.5.13}$$

$$GCD[X_1, X_2, \dots, X_k] = \prod_{p \le n} p^{C_p} \tag{4.5.14}$$

Estimate for $GCD[X_1, X_2, \dots, X_k]$

From (4.5.14) and the Approximation (4.5.9) we have

$$P[GCD[X_1, X_2, \dots, X_k] = g]$$

$$= P[\prod_{p \le n} p^{C_p} = \prod_{p \le n} p^{g_p}] \quad \text{where } g = \prod_{p \le n} p^{g_p}, \text{ (say)}$$

$$= \prod_{p \le n} P[C_p = g_p] \tag{4.5.15}$$

From (4.5.10) we have

$$P[a_p(X_i) \ge r] = \frac{1}{p^r}, \ r \ge 0 \tag{4.5.16}$$

$$P[a_p(X_i) \le r] = 1 - \frac{1}{p^{r+1}}, \ r \ge 0 \tag{4.5.17}$$

So that

$$P[C_p \ge r] = \prod_{i=1}^{k} P[a_p(X_i) \ge r] = \frac{1}{p^{rk}} \tag{4.5.18}$$

from (4.5.16), and

$$P[C_p = r] = \frac{p^k - 1}{p^{k(r+1)}} \tag{4.5.19}$$

From (4.5.15), (4.5.19),

$$P[GCD[X_1,\ldots,X_k] = g] = \prod_{p \le n} \frac{p^{k-1}}{p^{k(g_p+1)}} \tag{4.5.20}$$

$$= \prod_{p \le n} \frac{1}{p^{kg_p}} \prod_{p \le n} \frac{p^k-1}{p^k} = \frac{1}{g^k} \prod_{p \le n} \frac{p^k-1}{p^k} \quad \text{for } 0 < g \le n$$

Now (4.5.20) is based on the approximation (4.5.9), so we must normalize (4.5.20) to obtain the new probabilities

$$P[GCD[X_1,\ldots,X_k] = g] = \frac{\dfrac{1}{g^k} \displaystyle\prod_{p \le n} \frac{p^k-1}{p^k}}{\displaystyle\sum_{g=1}^{n} \frac{1}{g^k} \prod_{p \le n} \frac{p^k-1}{p^k}}$$

$$= \frac{1}{g^k}\left[\sum_{g=1}^{n} \frac{1}{g^k}\right]^{-1} \tag{4.5.21}$$

Then if $\lambda < k-1$,

$$E[[GCD(X_1,\ldots,X_k)]^{\lambda}] = \sum_{g=1}^{n} g^{\lambda} \frac{1}{g^k} \left[\sum_{g=1}^{n} \frac{1}{g^k}\right]^{-1}$$

$$= \sum_{g=1}^{n} \frac{1}{g^{k-\lambda}} \left[\sum_{g=1}^{n} \frac{1}{g^k}\right]^{-1}$$

$$\to \frac{\zeta(k-\lambda)}{\zeta(k)} \quad \text{as } n \to \infty \tag{4.5.22}$$

If $\lambda = k-1$, (4.5.22) shows that

$$E[[GCD(X_1,\ldots,X_k)]^{k-1}] \to \infty \qquad \text{as } n \to \infty$$

So we have proved

Theorem (4.5.23) For $\lambda \leq k - 1$, subject to the approximation (4.5.9),

$$E[[GCD(X_1,\ldots,X_k)]^\lambda] \to \frac{\zeta(k-\lambda)}{\zeta(k)} \quad \text{as } n \to \infty$$

Remarks (4.5.24)

1. As we mentioned in the introduction to this section, Theorem (4.5.23) has been proved rigorously by Gegenbauer ([19]), and this is an argument for the validity of (4.5.9) in this kind of analysis, and in particular in the calculation of the LCM to follow (Theorem (4.5.69)).

2. In the case k = 2,

$$\frac{1}{GCD(X_1,X_2)} = \frac{LCM(X_1,X_2)}{X_1 . X_2} \tag{4.5.25}$$

Setting $\lambda = -1$ in (4.5.23), we get

$$E\left[\frac{1}{GCD(X_1,X_2)}\right] = E\left[\frac{LCM(X_1,X_2)}{X_1 . X_2}\right] \to \frac{\zeta(3)}{\zeta(2)} = 0.73076 \tag{4.5.26}$$

a result due to Cesaro [9].

3. Another argument in favor of the validity of (4.5.9) is as follows.

Let $\omega(m)$ be the number of different prime factors of m, and let $\Omega(m)$ be its total number of prime factors, so that if

$$m = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}$$

then $\omega(m) = r$, $\Omega(m) = a_1 + a_2 + \ldots + a_r$.

Then it is known that $\omega(m)$ and $\Omega(m)$ are "nearly always" about log log m. More precisely

Theorem (4.5.27) [Hardy and Wright, [30] Theorem 431, p. 356]

The number of m, not exceeding x, for which

$$|f(m) - \log \log m| > (\log \log m)^{1/2 + \delta}$$

where $f(m)$ is $\omega(m)$ or $\Omega(m)$, is $o(x)$ for every positive δ.

Thus a number near 10^7 will usually have about three prime factors, and a number near 10^{80} about five or six. A number like

$$6092087 = 37.\ 229.\ 719$$

is in a sense a "typical number".

On the other hand one half of the numbers are divisible by 2, one third by 3, one quarter by 4, and so on.

Therefore, if two numbers X and Y are picked at random between 1 and n, we should expect them to have some small prime factors in common, but no large common prime factors.

So if we take care of the effect of small powers of the small primes 2,3,5,..., we might expect to get a good estimate of the LCM. We prove this in the case $k = 2$ by giving three successive approximations to $\frac{LCM(X,Y)}{XY}$, taking into account no common factors; common factors 2,4,8, ..., 2^R, where $2^R \ll n$; and finally common factors $2,4,8,\ldots,2^R,3,3^2,3^3,\ldots,3^S$, where $2^R \ll n$, $3^S \ll n$. It will be seen that the successive approximations rapidly approach the true value of $\frac{LCM(X,Y)}{XY}$. Now since our approximation (4.5.9) is most nearly true in the case of small powers of small primes,[†] we expect it to give an accurate estimate of

$$\frac{LCM(X_1,X_2,\ldots,X_k)}{X_1 . X_2 . \ldots . X_k} \quad \text{for } k > 2.$$

Successive Approximations to $\frac{LCM(XY)}{XY}$; $X,Y \varepsilon [1,2,\ldots,n]$

I. First Approximation

$$LCM_1(X,Y) = XY$$

$$E\left[\frac{LCM_1(X,Y)}{XY}\right] = 1$$

[†]As is seen by comparing (4.5.7) and (4.5.8).

II. Second Approximation is to take into account common factors $2, 4, 8, \ldots, 2^R$, where $2^R \ll n$. i.e.,

$$LCM_2(X,Y) \begin{cases} = \dfrac{XY}{2^{\min(a_2(X),a_2(Y))}} & \text{if } \min(a_2(X),a_2(Y)) < R \\ = \dfrac{XY}{2^R} & \text{otherwise} \end{cases}$$

$$\equiv \frac{XY}{2^{\overline{C}_2}} \quad \text{(say)}$$

Now $P[a_2(X) \geq r] = \dfrac{1}{2^r} - \dfrac{\varepsilon}{n}$, $0 \leq \varepsilon < 1$, by (4.5.1)

$$P[\overline{C}_2 \geq r] = P[a_2(X) \geq r,\ a_2(Y) \geq r]$$

$$= \left(\frac{1}{2^r} - \frac{\varepsilon}{n}\right)^2 \quad \text{if } 0 \leq r < R$$

and

$$P[\overline{C}_2 = R] = 1 - \sum_{r=0}^{R-1} P[\overline{C}_2 = r]$$

$$P[\overline{C}_2 = r] = \frac{3}{2^{2r+2}} + \frac{2\delta}{n}, \quad |\delta| < 1,\ r < R$$

$$P[\overline{C}_2 = R] = \frac{1}{2^{2R}} + \frac{2R\delta}{n}, \quad |\delta| < 1$$

$$E\left[\frac{LCM_2(X,Y)}{XY}\right] = \sum_{r=0}^{R} \frac{1}{2^r} P[\overline{C}_2 = r]$$

$$= \frac{6}{7} + \frac{1}{7}\frac{1}{2^{3R}} + \frac{4\delta}{n} - \frac{4\delta+2R\delta}{2^R n}$$

$$\to \frac{6}{7} \text{ as } n \to \infty,\ R \to \infty$$

$$= 0.857$$

III. Third Approximation In the same way it can be shown that by taking into account common factors $2, \ldots, 2^R, 3, \ldots, 3^S, 2^R \ll n, 3^S \ll n$, that

$$E\left[\frac{LCM_3(X,Y)}{XY}\right] \to 0.791 \quad \text{as } n \to \infty,\ R,\ S \to \infty.$$

IV. True Value From (4.5.25), $E\left[\frac{LCM(X,Y)}{XY}\right] \to 0.73076$ as $n \to \infty$.

4. As a final remark, we observe that (4.5.23) has two interesting corollaries:

(4.5.28) Cor. to Theorem (4.5.23)

If $\lambda < k-1$, $\lim_{n \to \infty} E[GCD(X_1,\ldots,X_k)^{\lambda}]$ is finite.

$$\lim_{n \to \infty} E[GCD(X_1,\ldots,X_k)^{k-1}] = \infty$$

(4.5.29) Cor. to Theorem (4.5.23)

$$E\left[\frac{LCM(X,Y)}{XY}\right]^2 \to \frac{\zeta(4)}{\zeta(2)}$$

$$\sigma^2\left[\frac{LCM(X,Y)}{XY}\right] \to \frac{\zeta(4)}{\zeta(2)} - \left(\frac{\zeta(3)}{\zeta(2)}\right)^2$$

We now derive our estimate for $E[LCM(X_1,\ldots,X_k)]$

Lower bound to Least Common Multiple

We would like to have an exact expression for

$$\frac{E[LCM(X_1,\ldots,X_k)]}{E[X_1X_2\ldots X_k]} \tag{4.5.30}$$

however we will only be able to obtain a lower bound.

From (4.5.12), (4.5.13),

$$E[LCM(X_1\ldots X_k)] \equiv E[LCM]$$

$$= E\left[\prod_{p \le n} p^{B_p}\right]$$

$$= \prod_{p \le n} E[p^{B_p}] \quad \text{since by (4.5.9) } B_p,\ p = 2,3,\ldots \text{ are independent}$$

Now for a convex function f, $E[f(X)] \ge f(E[X])$, (c.f. [3], p. 17). Since p^x is a convex function of x,

$$E[LCM] \ge \prod_{p \le n} p^{E[B_p]} \tag{4.5.31}$$

Next we consider the denominator of (4.5.30).

$$E(X_1,\ldots,X_k) = (E(X_1))^k \quad \text{since the } X_i \text{ are independent and identically distributed}$$

$$= \left(\frac{n}{2}\right)^k \tag{4.5.32}$$

Let $u_n \triangleq \sum_{p \le n} \frac{1}{p} \log p$. It is known ([51]) that

$$\log n > u_n > \log n - F\ , \quad n \ge 2 \tag{4.5.33}$$

where

$$F = 1.3326 + \frac{1}{2 \log 2} = 2.0540$$

Therefore

$$n < e^F \prod_{2 \le p \le n} p^{\frac{1}{p}} \tag{4.5.34}$$

$$E(X_1 \cdot \ldots \cdot X_k) = \left(\frac{n}{2}\right)^k$$

$$< \left(\frac{e^F}{2}\right)^k \prod_{2 \le p \le n} p^{\frac{k}{p}} \tag{4.5.35}$$

From (4.5.31), (4.5.35):

$$\frac{E[LCM]}{E[X_1 \cdot \ldots \cdot X_k]} \ge G^k \prod_{p \le n} p^{E[B_p] - \frac{k}{p}} \tag{4.5.36}$$

where $G = \dfrac{2}{e^F} = 0.256$

To find $E[B_p]$ we proceed as follows.

$$P[B_p \leq r] = P[\max_{i=1}^{k} a_p(X_i) \leq r] = P[a_p(X_i) \leq r]^k$$

$$= \left(1 - \frac{1}{p^{r+1}}\right)^k \quad \text{from (4.5.17)}$$

$$P[B_p = r] = \left(1 - \frac{1}{p^{r+1}}\right)^k - \left(1 - \frac{1}{p^r}\right)^k$$

$$E[B_p] = \sum_{r=1}^{\infty} r \left[\left(1 - \frac{1}{p^{r+1}}\right)^k - \left(1 - \frac{1}{p^r}\right)^k\right]$$

$$= \sum_{r=1}^{\infty} r \sum_{s=1}^{k} \binom{k}{s} (-1)^s \left(\frac{1}{p^{rs+s}} - \frac{1}{p^{rs}}\right)$$

$$= \sum_{s=1}^{k} \binom{k}{s} (-1)^s \frac{1 - p^s}{p^s} \sum_{r=1}^{\infty} \frac{r}{p^{rs}}$$

$$= \sum_{s=1}^{k} \binom{k}{s} \frac{(-1)^{s+1}}{p^s - 1} \qquad (4.5.37)$$

$$= W_{p,k} \quad \text{(say)} \qquad (4.5.38)$$

We next find a lower bound on $W_{p,k}$.

<u>Case I. $p \geq k$</u> In this case the terms in the series (4.5.37) decrease in magnitude. For let $W_{p,k} = \sum_{s=1}^{k} (-1)^{s+1} a_s$

so that

$$\frac{a_{s+1}}{a_s} = \frac{(k-s)\ (p^s - 1)}{(s+1)\ (p^{s+1} - 1)}$$

Then $\frac{a_{s+1}}{a_s} < 1$

Proof. Must show $(k-s)(p^s-1) < (s+1)(p^{s+1}-1)$ for $p \geq k$ and for $k \geq s \geq 1$

$$p \geq k$$

$$p > \frac{k - s}{s + 1}$$

$$p + \frac{p - 1}{p^s - 1} > \frac{k - s}{s + 1}$$

$$\frac{p^{s+1} - 1}{p^s - 1} > \frac{k - s}{s + 1}$$

$$(k - s)(p^s - 1) < (s + 1)(p^{s+1} - 1)$$ QED

Conclusion. For $p \geq k$, the first two terms give a lower bound, i.e.,

$$W_{p,k} > \frac{k}{p - 1} - \frac{k(k - 2)}{2(p^2 - 1)} \quad (4.5.39)$$

<u>Case II. $p \leq k$</u>

$$W_{p,k} = \sum_{s=1}^{k} \binom{k}{s} \frac{(-1)^{s+1}}{p^s - 1} \quad (4.5.37)$$

Now $$\frac{1}{p^s - 1} = \frac{1}{p^s}\left(\frac{1}{1 - \frac{1}{p^s}}\right) = \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots$$

$$W_{p,k} = \sum_{s=1}^{k} \binom{k}{s} (-1)^{s+1} \sum_{r=1}^{\infty} \frac{1}{p^{rs}} \quad (4.5.40)$$

Now $W_{p,k} = E[B_p] \leq E[A_p] = \frac{k}{p-1} < \infty$, so (4.5.40) converges.

$$W_{p,k} = \sum_{r=1}^{\infty} \left[1 - (1 - \frac{1}{p^r})^k\right] \quad (4.5.41)$$

The first term in this series is $1-(1-\frac{1}{p})^k \geq 1-(1-\frac{1}{p})^p \quad (k \geq p)$

$$\doteq 1 - e^{-1}$$

$$> \frac{1}{2}$$

Let r_o be the number of terms in this series that are $\geq \frac{1}{2}$, then $r_o \geq 1$, and

$$W_{p,k} \geq \frac{r_o}{2}. \qquad (4.5.42)$$

It is clear that the terms in (4.5.41) are monotonicly decreasing to zero as $r \to \infty$. Let r_1 be the solution of

$$1 - (1 - \frac{1}{p^{r_1}})^k = \frac{1}{2}, \qquad (4.5.43)$$

then $r_o = [r_1]$, and

$$r_o \geq r_1 - 1. \qquad (4.5.44)$$

From (4.5.43), solving for r_1 we get

$$r_1 = - \frac{\log (1 - 2^{-1/k})}{\log p} \qquad (4.5.45)$$

Now $\log 2^{-1/k} \leq 2^{-1/k} - 1$

$$1 - 2^{-1/k} \leq \frac{1}{k} \log 2$$

$$\log (1 - 2^{-1/k}) \leq - \log k + \log \log 2 < - \log k$$

$$r_1 > \frac{\log k}{\log p} \qquad (4.5.46)$$

From (4.5.42), (4.5.44), (4.5.46),

for $p \leq k$,

$$W_{p,k} \geq \frac{1}{2} \left(\frac{\log k}{\log p} - 1 \right) \qquad (4.5.47)$$

From (4.5.33), (4.5.39), (4.5.47), we have for $n \geq k$,

$$\frac{E[LCM]}{E[X_1 \cdot \cdots \cdot X_k]} \geq G^k \frac{\prod\limits_{2 \leq p \leq k} p^{\frac{1}{2}\left(\frac{\log k}{\log p} - 1\right)} \prod\limits_{k < p \leq n} p^{\frac{k}{p-1} - \frac{k(k-1)}{2(p^2-1)}}}{\prod\limits_{2 \leq p \leq n} p^{k/p}}$$

$$= G^k \frac{p_1 p_2}{p_3} \qquad \text{(say)} \qquad (4.5.48)$$

where

$$p_1 \triangleq \prod_{2 \leq p \leq k} p^{\frac{1}{2}\left(\frac{\log k}{\log p} - 1\right)}$$

$$p_2 \triangleq \prod_{k < p \leq n} p^{\frac{k}{p-1} - \frac{k(k-1)}{2(p^2-1)} - \frac{k}{p}}$$

$$p_3 \triangleq \prod_{2 \leq p \leq k} p^{\frac{k}{p}}$$

First we estimate

$$p_1 = \prod_{2 \leq p \leq k} p^{\frac{1}{2}\left(\frac{\log k}{\log p} - 1\right)} = \prod_{2 \leq p \leq k} k^{\frac{1}{2}} \prod_{2 \leq p \leq k} p^{-1/2}$$

$$= k^{\frac{1}{2}\Pi(k)} \left(\prod_{2 \leq p \leq k} p\right)^{-1/2} \qquad (4.5.49)$$

where $\Pi(X)$ is the number of primes $\leq X$, and

$$\Pi(X) \sim \frac{X}{\log X}. \qquad (4.5.50)$$

More precisely, from [51] we have

$$\Pi(X) > \frac{X}{\log X}, \quad X \geq 17$$

and then direct calculation shows

$$\Pi(X) > \frac{1}{3} \frac{X}{\log X}, \quad X \geq 2 \tag{4.5.51}$$

Again, $\log \prod_{2 \leq p \leq k} p = \sum_{2 \leq p \leq k} \log p \triangleq \theta(k)$, and

$$\theta(X) < X\left(1 + \frac{1}{2 \log X}\right) \quad X > 1 \qquad \text{(from [51])}$$

so that

$$\left(\prod_{2 \leq p \leq k} p \right)^{-1/2} > e^{-\frac{k}{2}\left(1 + \frac{1}{2 \log k}\right)} \tag{4.5.52}$$

From (4.5.49), (4.5.51), (4.5.52),

$$p_1 = \prod_{2 \leq p \leq k} p^{\frac{1}{2}\left(\frac{\log k}{\log p} - 1\right)}$$

$$> \frac{k^{\frac{k}{6 \log k}}}{e^{\frac{k}{2}\left[1 + \frac{1}{2 \log k}\right]}} = e^{-\frac{k}{3} - \frac{k}{4 \log k}}$$

$$\geq e^{-Dk}, \text{ for } k \geq 2, \tag{4.5.53}$$

where $D = \frac{1}{3} + \frac{1}{4 \log 2} \cong 0.694$

Secondly we estimate p_2. Now

$$\frac{k}{p - 1} - \frac{k(k-1)}{2(p^2 - 1)} - \frac{k}{p}$$

$$= -\frac{k^2}{2(p^2 - 1)} + \frac{k(3p + 2)}{2p(p^2 - 1)}$$

$$> \left(- \frac{k^2}{2} + \frac{3k}{2} \right) \frac{1}{p^2 - 1}$$

Therefore

$$\log p_2 > \left(- \frac{k^2}{2} + \frac{3k}{2} \right) \sum_{k<p\leq n} \frac{\log p}{p^2 - 1} \qquad (4.5.54)$$

Now from [51]

$$\sum_{p\leq x} f(p) = \frac{f(X)\theta(X)}{\log X} - \int_2^X \theta(y) \frac{d}{dy} \left(\frac{f(y)}{\log y} \right) dy \qquad (4.5.55)$$

$$\sum_{p\leq X} \frac{\log p}{p^2 - 1} = \frac{\theta(X)}{X^2 - 1} + \int_2^X \frac{2\theta(y) y dy}{(y^2 - 1)^2} \qquad (4.5.56)$$

Therefore

$$\sum_{k<p\leq n} \frac{\log p}{p^2 - 1} = \left(\sum_{p\leq n} - \sum_{p\leq k} \right) \frac{\log p}{p^2 - 1}$$

$$= \frac{\theta(n)}{n^2 - 1} - \frac{\theta(k)}{k^2 - 1} + \int_k^n \frac{2\theta(y) y dy}{(y^2 - 1)^2} \qquad (4.5.57)$$

Again from [51],

$$\theta(X) < 1.01624\ X,\ X > 0 \qquad (4.5.58)$$

$$X - 2\sqrt{X} < \theta(X)\ ,\ 0 < X \leq 1400 \qquad (4.5.59)$$

$$X(1 - \frac{1}{\log X}) < \theta(X),\ 41 \leq X \qquad (4.5.60)$$

It is trivial that

$$0.231\ X < X - 2\sqrt{X},\ 7 \leq X \qquad (4.5.61)$$

$$0.73\ X < X(1 - \frac{1}{\log X})\ ,\ 41 \leq X \qquad (4.5.61)$$

So

$0.231\ X < \theta(X)$	$2 \leq X \leq 7$	(by direct calculation)
$0.231\ X < \theta(X)$	$7 \leq X \leq 41$	(by (4.5.59), (4.5.61))
$0.73\ X < \theta(X)$	$41 \leq X$	(by (4.5.60), (4.5.62))

Conclusion:

$$0.231\ X < \theta\ (X) < 1.01624\ X \qquad X \geq 2 \tag{4.5.63}$$

or

$$A\ X < \theta(X) < B\ X,\ X \geq 2,\ A = 0.231,\ B = 1.01624 \tag{4.5.64}$$

Now

$$2 \int \frac{y^2 dy}{(y^2 - 1)^2} = \frac{1}{2} \log \frac{y - 1}{y + 1} - \frac{y}{y^2 - 1} \tag{4.5.65}$$

From (4.5.57), (4.5.64), (4.5.65),

$$\sum_{k<p\leq n} \frac{\log p}{p^2 - 1} < \frac{Bn}{n^2 - 1} - \frac{Ak}{k^2 - 1} + B \left[\frac{1}{2} \log \frac{y - 1}{y + 1} - \frac{y}{y^2 - 1} \right]_k^n$$

$$= \frac{(B - A)k}{k^2 - 1} + \frac{B}{2} \log \frac{n - 1}{n + 1} - \frac{B}{2} \log \frac{k - 1}{k + 1}$$

$$< \frac{(B - A)k}{k^2 - 1} + \frac{B}{2} \log \frac{k + 1}{k - 1}$$

$$\leq \frac{(B - A)k}{k^2 - 1} + \frac{B}{k - 1} = \frac{(2B - A)\ k + B}{k^2 - 1} \tag{4.5.66}$$

From (4.5.54), (4.5.66),

$$\log p_2 > - \frac{k(k - 3)}{2(k^2 - 1)} \quad [(2B - A)\ k + B]$$

$$> - \frac{2B - A}{2} k - \frac{B}{2} \qquad \text{for } k \geq 1.$$

$$p_2 > e^{-Ck - D_1} \tag{4.5.67}$$

where $C = \frac{2B - A}{2} = 0.90$, $D_1 = \frac{B}{2} = 0.50812$

Thirdly we estimate p_3.

$$\log p_3 = k \sum_{p\leq k} \frac{1}{p} \log p$$

$< k \log k$ by (4.5.33)

$$p_3 < k^k \tag{4.5.68}$$

Now we return to (4.5.48): From (4.5.53), (4.5.67), (4.5.68) we have the final result:

(4.5.69) Theorem Assuming the approximation (4.5.9), for $1 \leq k < n$,

$$\frac{E[LCM(X_1,\ldots,X_k)]}{E[X_1.X_2.\ldots.X_k]} > \frac{C_1\ C_2^k}{k^k} \triangleq \frac{1}{\mu_k}$$

where $C_1 = e^{-D_1} = 0.60$

$$C_2 = Ge^{-C-D} = 0.052$$

4.6 LOWER BOUND TO EXPECTED VALUE OF CYCLE TIME LC UNDER RANDOM EXCITATION

We consider type 1A networks in steady state behavior, when only loop nodes are active. The period or cycle time LC of such a network is equal to the LCM of the periods of its components.

Suppose that a typical component contains a loop of length ℓ, and that in the steady state a of the ℓ loop nodes are active. Let e be the period of this component. We also call e the effective loop length of the component.

(4.6.1) Lemma $e|\ell$ and $\ell|ae$

Proof. If $a = o$, then $e = 1$ and $1|\ell$, $\ell|o$ for any ℓ.

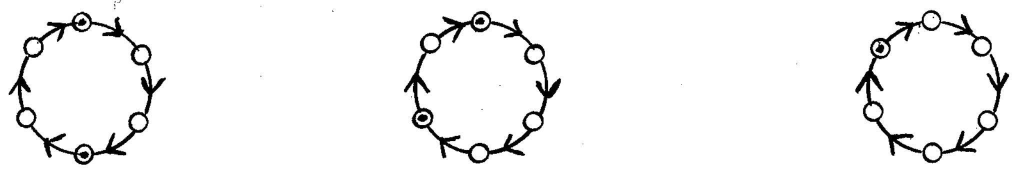
If $a > o$, then the a active nodes and $\ell - a$ inactive nodes must be arranged in a pattern with a period of e. It must be possible, therefore, to divide the loop in $\frac{\ell}{e}$ sectors of length e, each containing $\frac{ae}{\ell}$ active nodes. Consequently $e|\ell$ and $\ell|ae$.

(4.6.2) Lemma If $a > o$, and ℓ and a have no common factor greater than 1, then $e = \ell$

Proof. $\ell | ae$ then implies $\ell | e$, but $e | \ell$, therefore $e=\ell$.

Figure 4.6.1 gives examples of ℓ, e, a.

FIG. 4.6.1



$\ell = 6$, $a = 2$, $e = 3$ $\ell = 6$, $a = 2$, $e = 6$ $\ell = 6$, $a = 1$, $e = 6$

Initial Activity. It is convenient to choose the initial state by selecting

$$A = A(n) \triangleq \sqrt{n}(\log n)^{3/4} \tag{4.6.3}$$

active nodes out of the n nodes, independently and with replacement. (As will become apparent, this is a compromise between high initial activity, in which case many of the loops are saturated, and low initial activity, in which case many loops are not activated.)

Thus there are

$$n^{A(n)}$$

possible initial states, each equally likely. It is convenient to define the probability space Ω_7 to be the space of the $n^{A(n)}$ different initial states.

<u>Remark</u>. A(n) must be an integer, so we should really use

$$A(n) = [\sqrt{n}\ (\log n)^{3/4}] \tag{4.6.4}$$

instead of (4.6.3), but the error involved in using (4.6.3) is negligible.

<u>Distribution of Activity</u>. Our first two theorems deal with the distribution of the activity among the components of a typical type 1A network g containing k components of sizes $n_1, n_2, \ldots, n_k$.

(4.6.5) <u>Theorem</u> If $k < (\log n)^{5/4}$, then over the probability space Ω_7, with probability $P \to 1$ (as $n \to \infty$), all components of g of size

$$n_\nu \geq \sqrt{\frac{n}{k}}\ \frac{1}{\log k} \triangleq f_1(n,k) \tag{4.6.6}$$

are activated. Further, the approach of P to 1 is independent of k.

Proof. Without loss of generality we may assume

$$n_1, n_2, \ldots, n_r \geq f_1(n,k)$$

and

$$n_{r+1}, \ldots, n_k < f_1(n,k).$$

An equivalent problem is that of throwing A(n) balls into k boxes of sizes $n_1, \ldots, n_k$. Let S be the number of empty boxes among boxes $1,2,\ldots,r$. Define the random variables

$$\alpha_i = \begin{matrix} 1 & \text{if ith box is empty} \\ 0 & \text{if not} \end{matrix}$$

for $i = 1,\ldots,r$.

Then $P[\alpha_i = 1] = \left(1 - \frac{n_i}{n}\right)^A$

$$S = \sum_{i=1}^{r} \alpha_i$$

$$E[S] = \sum_{i=1}^{r} \left(1 - \frac{n_i}{n}\right)^A \leq k\left(1 - \frac{f_1}{n}\right)^A$$

$$\log E[S] \leq \log k + A \log\left(1 - \frac{f_1}{n}\right)$$

$$\leq \log k - \frac{(\log n)^{3/4}}{\sqrt{k}\,\log k} \qquad \text{by (4.6.3)}$$

$$< \frac{5}{4}\log\log n - \frac{(\log n)^{1/8}}{\frac{5}{4}\log\log n} \qquad (\text{since } k < (\log n)^{5/4})$$

$$\to -\infty \quad \text{as } n \to \infty$$

Therefore $E[S] \to 0$ as $n \to \infty$.

Similarly we find

$$E[S^2] < E[S] + k(k-1)\left(1 - \frac{2f_1}{n}\right)^A$$

$\to 0$ as $n \to \infty$. QED

(4.6.6) Theorem If $k < (\log n)^{5/4}$, then over the probability space Ω_7, with probability $P \to 1$ (as $n \to \infty$), no component of size $n_\nu \geq f_1$ contains more than

$$\frac{2A(n)\,n_\nu}{n} \triangleq f_4(n_\nu, n) \equiv f_4 \qquad (4.6.7)$$

initially active nodes. Further, the approach of P to 1 is independent of k.

Proof. We use the same notation as in the proof of Theorem (4.6.5).

Let N_i be the number of balls in box i, i = 1,...,r.

We must show

$$P \equiv P[N_i > f_4 \text{ for some } i = 1,\ldots,r] \to 0 \text{ as } n \to \infty.$$

Now

$$P \leq \sum_{i=1}^{r} P[N_i > f_4] \qquad (4.6.8)$$

First let us observe that if $n_i \geq \frac{n}{2}$, then

$$N_i \leq A \leq \frac{2n_i A}{n}$$

and so

$$P[N_i > \frac{2n_i A}{n} \equiv f_4] = 0.$$

So in (4.6.9) we need only consider i for which

$$f_1 \leq n_i < \frac{n}{2} \qquad (4.6.9)$$

Let us therefore consider such an i, and define

$$X_u = \begin{array}{ll} 1 & \text{if the } u^{th} \text{ ball falls into box } i \\ 0 & \text{if not} \end{array}$$

Then

$$P[X_u = 1] = \frac{n_i}{n} = p \quad \text{(say)}$$

$$N_i = \sum_{u=1}^{A} X_u$$

$$\frac{f_1}{n} \leq p < \frac{1}{2} \qquad \text{by (4.6.9).} \qquad (4.6.10)$$

Let $q = 1 - p$.

We will bound $P[N_i > f_4]$ by the Chernov method (see Jelinek [34]), which in the present context gives us the

<u>Theorem</u>

Let $\gamma(s) \triangleq \log E[e^{sX_i}] = \log (pe^s + q)$, (4.6.11)

Then for $p < x < 1$,

$$P[N_i \geq Ax] \leq e^{-A(s_1\gamma'(s_1) - \gamma(s_1))} \quad (4.6.12)$$

where s_1 is the unique solution of

$$x = \gamma'(s) \quad (4.6.13)$$

We apply this theorem as follows:

From (4.6.11), (4.6.13),

$$s_1 = \log \frac{xq}{(1 - x)\ p} \quad (4.6.14)$$

From (4.6.12), (4.6.14),

$$P[N_i \geq Ax] \leq \exp - A\left[X \log \frac{X}{p} + (1 - X) \log \frac{1 - X}{1 - p}\right]$$

Let $X = ap$ where $a \triangleq \frac{3}{2}$. Then

$$P[N_i > f_4] = P[N_i > \frac{2An_i}{n}]$$

$$\leq P[N_i \geq \frac{3}{2} \frac{An_i}{n}] = P[N_i \geq AX]$$

$$\leq \exp [- Af(p)] \quad (4.6.15)$$

where

$$f(p) = ap \log a + (1 - ap) \log \frac{1 - ap}{1 - p}$$

It follows that $f'(o) = a \log a - a + 1 \cong 0.1$ and

$$f''(p) = \frac{(a - 1)^2}{(1 - p)^2(1 - ap)} > 0 \qquad \text{by (4.6.10)}$$

Therefore $f(p)$ is monotonically increasing for $0 < p < 1$ and f is a minimum at

$$P = \frac{f_1}{n}$$

and there

$$f\left(\frac{f_1}{n}\right) = \frac{f_1}{n} (a \log a - a + 1) + 0\left(\frac{f_1}{n}\right)^2$$

$$\cong \frac{f_1}{n} 0.1 + 0 \left(\frac{f_1}{n}\right)^2$$

$$> \frac{f_1}{20n} \text{ for n sufficiently large}$$

Therefore from (4.6.15),

$$P[N_i > f_4] \leq \exp\left(- \frac{A f_1}{20n} \right)$$

$$< \exp\left(\frac{-(\log n)^{1/8}}{25 \log \log n} \right) \quad \text{(by (4.6.3), (4.6.6))} \quad (4.6.16)$$

$$\text{since } k < (\log n)^{5/4} \quad (4.6.17)$$

From (4.6.8), (4.6.16),(4.6.17), for n sufficiently large,

$$P \leq (\log n)^{5/4} \exp\left(\frac{-(\log n)^{1/8}}{25 \log \log n} \right)$$

$$\to 0 \text{ as } n \to \infty.$$ QED

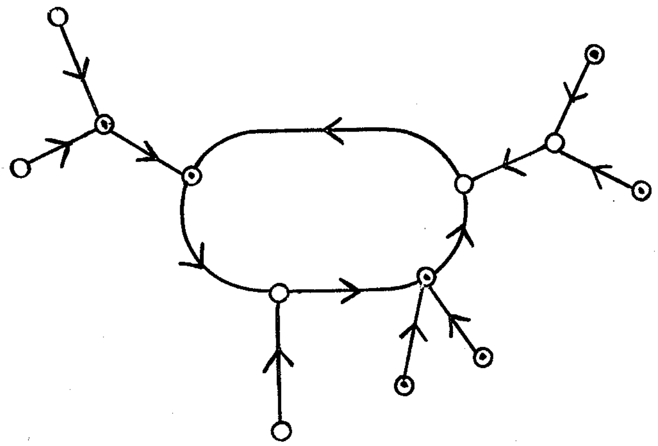
Next let us look at a particular component of size n_i (say), whose initial state contains a (say) active nodes.

Let b be the activity that finally enters the loop in this component. More precisely, if at time t there are x active nodes in this component at distance one from the loop, we say that at time t + 1 an activity of X enters the loop. For example, see Fig. (4.6.2).

Clearly $b \leq a$.

FIG. 4.6.2

ACTIVITY OF 3 ENTERS THE LOOP AT TIME $t + 1$, ACTIVITY OF 2 ENTERS THE LOOP AT TIME $t + 2$



(4.6.18) Assumption. We assume that the b nodes are randomly distributed over the length of the loop; in other words, if the loop is of length ℓ_i, then the final activity c is obtained by picking b nodes at random, with replacement, from the ℓ_i nodes. Again, c is the number of occupied boxes when b balls are thrown into ℓ_i boxes (of equal size).

This assumption is justified simply by the fact that the initial activity is chosen at random.

Clearly

$$c \leq \ell_i \leq n_i; \quad c \leq b \leq a \leq n_i.$$

The loop is <u>saturated</u> if $c = \ell_i$. Our next two theorems give some conditions under which saturation will not occur.

(4.6.19) <u>Theorem</u> If $n = N(\log N)^{\alpha}$ ($\alpha < 1$) balls are thrown into N boxes of equal size, then the number of empty boxes, S, has mean

$$E[S] = Ne^{-\lambda}\ [1 + 0(\frac{\lambda}{N})] \tag{4.6.20}$$

and variance

$$\sigma^2[S] = Ne^{-\lambda} + o(Ne^{-\lambda})$$

where $\lambda = \frac{n}{N} = (\log N)^{\alpha}$

Proof. See David and Barton, [12], page 243.

(4.6.21) <u>Cor.</u>

If $n \leq N\ (\log N)^{7/8}$, then with probability $\to 1$ (as $N \to \infty$) there will be at least one empty box.

(4.6.22) <u>Cor.</u>

If $b \leq \ell_i\ (\log \ell_i)^{7/8}$, then with probability $\to 1$ (as $\ell_i \to \infty$)the loop will not be saturated. (This depends on the truth of Assumption (4.6.18).)

(4.6.23) <u>Theorem</u>*

Provided $n \geq \alpha_2 = e^{e^{33}}$ and

$$(\log n)^{5/4} > k \tag{4.6.24}$$

then with probability $\to 1$ (as $n_i \to \infty$), the loop will not be saturated if $n_i \geq f_1(n,k)$ and

$$\ell_i \geq f_2(n_i) \equiv \frac{\sqrt{n_i}}{(\log n_i)^{1/16}} \tag{4.6.25}$$

Proof. From Theorem (4.6.6), with probability $\to 1$ (as $n \to \infty$)

---

*This theorem depends on Assumption (4.6.18).

we have

$$b \leq a \leq f_4(n_i,n) = \frac{2(\log n)^{3/4} n_i}{\sqrt{n}} \tag{4.6.26}$$

Since $n_i \leq n$ it follows that (4.6.26) is true with probability $\to 1$ (as $n_i \to \infty$). Then from Cor. (4.6.22) and (4.6.26) it is sufficient to show that (4.6.25) implies

$$\frac{2(\log n)^{3/4} n_i}{\sqrt{n}} \leq \ell_i \ (\log \ell_i)^{7/8}$$

i.e., to show that

$$\frac{2(\log n)^{3/4} n_i}{\sqrt{n}} \leq \frac{\sqrt{n_i}}{(\log n_i)^{1/16}} \left[\frac{1}{2} \log n_i - \frac{1}{16} \log \log n_i\right]^{7/8}$$

i.e., that

$$\frac{2^{15/8}}{(\log n_i)^{1/16}} \ \frac{\sqrt{n_i}}{(\log n_i)^{3/4}} \ \frac{1}{\left(1 - \frac{1}{8} \frac{\log \log n_i}{\log n_i}\right)^{7/8}} \leq \frac{\sqrt{n}}{(\log n)^{3/4}}$$

which is true, since $n_i \leq n$, as soon as

$$2^{15/8} < (\log n_i)^{1/16} \left(1 - \frac{1}{8} \frac{\log \log n_i}{\log n_i}\right)^{7/8}$$

i.e., as soon as

$$4^{16} < \log n_i \left(1 - \frac{7}{4} \ \frac{\log \log n_i}{\log n_i}\right)$$

which is true when

$$n_i > e^{e^{32}} \tag{4.6.27}$$

Because of the condition $n_i \geq f_1(n,k) = \sqrt{\frac{n}{k}} \frac{1}{\log k}$

and (4.6.24), (4.6.27) is satisfied if

$$\frac{\sqrt{n}}{\frac{5}{4}(\log n)^{5/8} \log \log n} > e^{e^{32}}$$

and this is certainly true if

$$n \geq e^{e^{33}} \qquad (4.6.28)$$

QED

(4.6.29) Theorem. There exists* a constant α_3 such that for $n \geq \alpha_3$ and $k < (\log n)^{5/4}$ the following is true:

Let η denote the event that all components of size $n_i \geq f_1$ contain initial activity $\underline{a}_i$ satisfying

$$1 \leq a_i \leq f_4(n,n_i) = \frac{2(\log n)^{3/4} n_i}{\sqrt{n}} \qquad (4.6.30)$$

Let us consider a typical component of size $n_i \geq f_1$, of loop length ℓ_i and effective loop length e_i. Then (as usual over the probability space Ω_7)

$$E_7[e_i | n_i \text{ and event } \eta] \geq \sqrt{\frac{2n_i}{\Pi}} \frac{3}{e\Pi^2} \qquad (4.6.31)$$

Proof. We assume that the event η occurs. Then with probability $\to 1$ (as $n_i \to \infty$) the loop is not saturated if

$$\ell_i \geq f_2(n_i)$$

(by Theorem (4.6.23)).

i.e., there exists an α_4 such that

$$P[\text{loop is not saturated if } \ell_i \geq f_2] \geq \frac{1}{2} \text{ for } n_i \geq \alpha_4 \qquad (4.6.32)$$

Let c_i be the final activity in the loop. If ℓ_i and c_i have no common factor, then by lemma (4.6.2), $e_i = \ell_i$. Now the probability that two numbers chosen at random have no common factor is asymptotically $\frac{6}{\Pi^2}$ (Hardy and Wright, [30], page 269). In fact ℓ_i and c_i are not chosen at

---

* α_3 is specified in (4.6.40) below.

random, but again we will invoke the randomness of the inial state and make the

(4.6.33) Assumption

ℓ_i and c_i have no common factor with probability $\frac{6}{\Pi^2}$.

From (4.6.32), (4.6.33),

$$E[e_i | n_i \text{ and event } \eta] \geq \sum_{\ell_i = f_2}^{n_i} \ell_i \frac{6}{\Pi^2} \cdot \frac{1}{2} \cdot P[\ell_i | n_i] \tag{4.6.34}$$

We have shown earlier (equation (4.4.5) that

$$P[\ell_i = \ell | n_i = m] = \frac{m^{m - \ell}}{\Psi (m - \ell)!} \tag{4.6.35}$$

where

$$\Psi = \sum_{r = o}^{m - 1} \frac{m^r}{r!} \tag{4.6.36}$$

From (4.6.34), (4.6.35),

$$E[e_i | n_i = m, \text{ and event } \eta] \geq \frac{3}{\Pi^2 \Psi} \sum_{\ell = f_2}^{m} \frac{\ell m^{m-\ell}}{(m - \ell)!} \tag{4.6.37}$$

The last sum is

$$\sum_{r = o}^{m - f_2} \frac{(m - r) m^r}{r!} = \frac{m^{m - f_2 + 1}}{(m - f_2)!} \tag{4.6.38}$$

and by lemma A5 of Appendix 1,

$$\Psi < \frac{e^m}{2} \tag{4.6.39}$$

From (4.6.37) - (4.6.39) and lemma A9,

$$E[e_i | n_i = m, \eta] \geq \frac{3}{\Pi^2} \sqrt{\frac{2}{\Pi}} \left[\frac{m^{m - f_2 + 1}}{(m - f_2)^{m - f_2 + \frac{1}{2}} e^{f_2}} \right]$$

The logarithm of expression inside the parentheses is

$$(m - f_2 + 1) \log m - (m - f_2 + \tfrac{1}{2}) \log m - (m - f_2 + \tfrac{1}{2}) \log (1 - \frac{f_2}{m}) - f_2$$

$$> \frac{1}{2} \log m - \frac{f_2^{\,2}}{m} + \frac{f_2}{2m}$$

$$= \frac{1}{2} \log m - R \qquad \text{(say)}$$

so that

$$E[e_i | n_i = m, n] \geq \frac{3}{\Pi^2} \sqrt{\frac{2m}{\Pi}}\; e^{-R}$$

where

$$R = \frac{f_2^{\,2}}{m} - \frac{f_2}{2m} = \frac{1}{(\log m)^{1/8}} - \frac{1}{2\sqrt{m}\,(\log m)^{1/16}} \qquad \text{by (4.6.25)}$$

$$< \frac{1}{(\log m)^{1/8}} < 1 \text{ for } m > e$$

which completes the proof of (4.6.31), provided the following conditions are satisfied

(a) $n \geq \alpha_2$ (see Theorem (4.6.23))

(b) $n_i \geq e$

(c) $n_i \geq \alpha_4$

Let $\alpha_5 = \max(\alpha_4, e)$, then (b), (c) are satisfied if

$$f_1(n,k) = \sqrt{\frac{n}{k}}\, \frac{1}{\log k} \geq \alpha_5$$

But by assumption,

$$k < (\log n)^{5/4}$$

therefore we require

$$\begin{cases} \dfrac{\sqrt{n}}{\frac{5}{4}(\log n)^{5/8}\,(\log \log n)} \geq \alpha_5 \\[2ex] n \geq \alpha_2 \end{cases} \qquad (4.6.40)$$

i.e., for some $\alpha_3 = \alpha_3(\alpha_2, \alpha_5)$,

$$n \geq \alpha_3.$$

QED

Remarks (4.6.41).

It is shown in section 4.4, Remark (4.4.18), that ℓ_i depends only on n_i. What we have done in the above theorem is to give a lower bound to

$$E_{\Omega_1 x \Omega_7}[e_i | n_i, \eta]$$

which depends only on n_i. In fact, let $e_i^* = 1$ if $\ell_i < f_2$, $e_i^* = \frac{3}{\Pi^2} \ell_i$ if $\ell_i \geq f_2$, then

$$E_{\Omega_1 x \Omega_7}[e_i | n_i, \eta] \geq E_{\Omega_1}[e_i^* | n_i, \eta] = \frac{3}{e\Pi^2}\sqrt{\frac{2n_i}{\Pi}}$$

(4.6.42) From Theorems (4.6.5), (4.6.6), it follows that given $\varepsilon > o$ there exists $\Omega_6(\varepsilon)$ such that (over the probability space Ω_7)

$$P[\text{event } \eta] > 1 - \varepsilon \quad \text{for } n > \alpha_6(\varepsilon).$$

(4.6.43) Theorem (4.6.29) depends on Assumptions (4.6.18) and (4.6.33).

(4.6.44) Theorem

There exists a constant α_{16} such that for

$$n \geq \alpha_8 \triangleq \max\left(\alpha_3, \alpha_6\left(\tfrac{1}{2}\right)\right) \tag{4.6.45}$$

then

$$E[LC] = E_{\Omega_1 x \Omega_7}[LC] > \exp\left[\frac{1}{2}(\log n)^{9/4} - \alpha_{16}(\log n)^{5/4} \log \log n\right].$$

Proof. $E_{\Omega_1 x \Omega_7}[LC]$

$$= E_{\Omega_1 x \Omega_7}[LCM(e_1, e_2, \ldots, e_k)] \tag{4.6.46}$$

where e_i is the effective loop length of the i^{th} component.

Now in section 4.5 we showed that

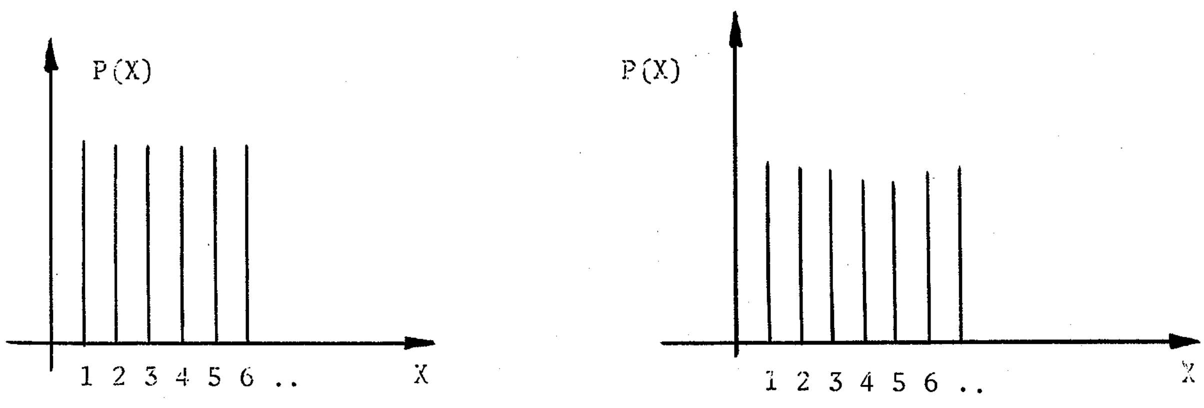
$$\frac{E_*[LCM(X_1,...,X_k)]}{E_*[X_1.X_2.....X_k]} > \frac{1}{\mu_k} \qquad (4.6.47)$$

where

$$\frac{1}{\mu_k} = \frac{\alpha_{10}.\alpha_{11}^k}{k^k} \ , \qquad \alpha_{10} = 0.60, \ \alpha_{11} = 0.052$$

and where the asterisk * indicates that this expectation is for X_i taken from a uniform distribution. In (4.6.46) the e_i do not have a uniform distribution, nor are they independent. However, we argue that the constraint between them is "weak" and that the distribution of any one e_i is "flat", i.e., smooth but nonuniform. Now it is the small primes 2,3,5,... that dominate the calculation of the LCM as we saw in section 4.5, and Figure (4.6.3) shows that

FIG 4.6.3



UNIFORM DISTRIBUTION

"FLAT" DISTRIBUTION

$$\underset{\text{uniform distribution}}{P}[2 \text{ divides } X] \cong \underset{\text{"flat" distribution}}{P}[2 \text{ divides } X]$$

In other words changing from a uniform distribution to a smooth nonuniform distribution does not increase the probability that X is even (or divisible by 3, etc.).

Therefore in (4.6.46) we make the

(4.6.48) <u>Assumption</u>

In (4.6.46),

$$E[\mathrm{LCM}(e_1,e_2,\ldots,e_k)] > \frac{E[e_1.e_2.\cdots.e_k]}{\mu_k} .$$

Therefore (4.6.46) becomes

$$E[LC] > E_{\Omega_1 x \Omega_7}\left[\frac{\prod_{i=1}^{k} e_i}{\mu_k}\right] \tag{4.6.49}$$

$$> E_{\Omega_1 x \Omega_7}\left[\frac{\Pi e_i}{\mu_k} \middle| \eta\right] P[\eta] \qquad \text{where } \eta \text{ is defined in Theorem (4.6.29)}$$

$$> E_{\Omega_1 x \Omega_7}\left[\frac{\Pi e_i^*}{\mu_k} \middle| \eta\right] P[\eta] \qquad \text{where } e_i^* \text{ is defined in Remark (4.6.41)}$$

$$= \sum_{k=1}^{(\log n)^{5/4}} \sum_{\substack{n_1,n_2,\ldots,n_k \\ \text{such that} \\ n_1+\ldots+n_k=n}} P[n_1,n_2,\ldots,n_k] \frac{\prod E[e_i^*|n_i,\eta]}{\mu_k} P(\eta) \tag{4.6.50}$$

by Remark (4.6.41), where $P[n_1,\ldots,n_k]$ is the probability that a type 1A network has k components of sizes $n_1,\ldots,n_k$, and is given in section 3.2.

We decrease (4.6.50) by restricting ourselves to the type 1A networks whose component sizes $n_1,n_2,\ldots,n_k$ are all distinct, for which case from (3.2.9),

$$P[n_1,\ldots,n_k] = \frac{n!}{n^n} \prod_{\nu=1}^{k} \frac{C_{n_\nu}}{n_\nu!} \tag{4.6.51}$$

where $C_r = \sum_{s=o}^{r-1} \frac{(r-1)!r^s}{s!}$

By (3.2.7) and lemma A9 of Appendix 1,

$$\frac{C_m}{m!} > \frac{1}{2me^{-m}} \left(1 - \sqrt{\frac{2}{\Pi m}}\right) e^{-\frac{1}{12m}}$$

$$\geq \frac{k_1}{me^{-m}} \quad \text{for } m \geq 1$$

where $k_1 = \frac{1}{2}\,(1 - \sqrt{\frac{2}{\Pi}})\,(1 - \frac{1}{12}) = 0.0926$

Therefore

$$P[n_1,\ldots,n_k] > \frac{\sqrt{2\Pi n}}{K_1^k \prod_{i=1}^{k} n_i} \quad \text{if the } n_i \text{ are distinct,} \tag{4.6.52}$$

From (4.6.50), (4.6.52), and Theorem (4.6.29),

$$E[LC] > \sum_{k=1}^{(\log n)^{5/4}} \frac{P(n)\sqrt{2\Pi n}}{\mu_k K_1^k} \sum_{n_1,n_2,\ldots n_k} \frac{\prod_{n_j > f_1} K_2\sqrt{n_j}}{\prod_{i=1}^{k} n_i}$$

where $K_2 = \frac{3}{e\Pi^2}\sqrt{\frac{2}{\Pi}} = 0.088 < 1.$ (4.6.53)

$$E[LC] > \sum_{k=1}^{(\log n)^{5/4}} \frac{P(n)\sqrt{2\Pi n}}{\mu_k} \left(\frac{K_2}{K_1}\right)^k \sum_{\substack{n_1,n_2,\ldots,n_k \\ \text{such that} \\ 1 \leq n_1 < n_2 < \ldots < n_k \leq n \\ \sum n_i = n}} \frac{\prod_{n_j > f_1} \sqrt{n_j}}{\prod_{i=1}^{k} n_i} \tag{4.6.54}$$

Let

$$\theta_{n,k} = \sum \prod_{n_j > f_1} \left(\sqrt{n_j}\right) \Big/ \prod_{i=1}^{k} \left(n_i \right) \qquad (4.6.55)$$

where the summation extends over all k-tuples $n_1, n_2, \ldots, n_k$ such that

$$1 \leq n_1 < n_2 < \ldots n_k \leq n$$

and

$$n_1 + \ldots + n_k = n \qquad (4.6.56)$$

and where f_1 is given by (4.6.6).

We find a lower bound to θ by finding the smallest term of (4.6.55). Equivalently, it is desired to maximize

$$H = \prod_{n_j \leq f_1} \left(n_j \right) \prod_{i=1}^{k} \left(n_i \right) \qquad (4.6.57)$$

subject to $1 \leq n_i \leq n$ and (4.6.56). It is clear that there are k essentially distinct local maxima given by, for $r = 0, 1, \ldots, k - 1$,

$$n_1 = \ldots = n_r = f_1$$

$$n_{r+1} = \ldots n_k = (n - rf_1) / (k - r)$$

giving rise to

$$H_r = f_1^{2r} \left[(n - rf_1) / (k - r)\right]^{k - r} \qquad (4.6.58)$$

It is easily shown that if $k \geq 10$, then

$$H_o \geq H_i, \quad i = 1, \ldots, k - 1.$$

Therefore

$$\theta(n,k) \geq Q(n,k) / \sqrt{H_o} \qquad (4.6.59)$$

where Q(n,k) is the number of partitions of n into exactly k unequal parts.

From [58] we have for all n,k

$$Q(n,k) \geq \frac{1}{k!}\binom{n - \binom{k}{2} - 1}{k - 1}$$

Using Stirling's approximation it follows easily that

$$Q(n,k) \geq \frac{n^{k-1}}{2k!\ (k-1)!} \qquad \text{for } k \leq n^{1/5} \qquad (4.6.60)$$

From (4.6.58) - (4.6.60), for $10 \leq k \leq n^{1/5}$,

$$\theta(n,k) \geq \frac{1}{2} n^{\frac{k}{2}-1} k^{\frac{k}{2}+1} (k!)^{-2} \qquad (4.6.61)$$

From (4.6.42), (4.6.47), (4.6.54), (4.6.55), (4.6.61), if n satisfies (4.6.45) then

$$E[LC] > \sum_{k=10}^{(\log n)^{5/4}} \alpha_{12}\, n^{(k-1)/2}\, \alpha_{13}^{k}\, k^{(2-k)/2}\, (k!)^{-2}$$

$$= \sum_{k} t_k \quad \text{(say)} \qquad (4.6.62)$$

where $\alpha_{12} = \sqrt{2\pi}\ \alpha_{10}/4 = 0.376 \ldots$

$\alpha_{13} = \alpha_{11}\, k_2/ k_1 = 0.049 \ldots$

Which is the greatest term of (4.6.62)?

We have $t_{k+1}/t_k \approx \alpha_{13} \sqrt{n} / [\sqrt{e}\, k(k+1)^{3/2}]$ for k large (4.6.63)

≥ 1 when

$10 \leq k \leq (\log n)^{5/4}$

and so the last term of (4.6.62) is the greatest.

Using Lemma A9 and (4.6.45) the desired result follows immediately.

Q.E.D.

Remark

This theorem depends on the approximations (4.5.9), (4.6.18), (4.6.33), and (4.6.48).

4.7 LOWER BOUND TO EXPECTED VALUE OF LCM OF LOOP LENGTHS.

In the previous section we gave a lower bound to the expected value of the cycle time of type 1A networks under random excitation. However the maximum cycle time for a particular network is obtained when just one node in each component is activated. Then saturated and inactive loops are avoided, and for each loop

effective loop length e_i = loop length l_i (4.7.1)

and therefore

$$LC = LCM\ (e_1, e_2, \ldots, e_k) = LCM\ (l_1, \ldots, l_k) \qquad \ldots (4.7.2)$$

In this section we give a lower bound to the expectation of the latter quantity.

We have

$$E_3[\ LCM\ (l_1, \ldots, l_k)] \geq E_3[\ v_k \prod_{i=1}^{k} l_i]$$

where $v_k = 1 / \mu_k$, with a justification similar to that given for (4.6.49),

$$\geq \sum_k v_k \sum P[n_1, \ldots, n_k] \prod_{i=1}^{k} E(l_i / n_i) \qquad (4.7.3)$$

where the second summation is over all partitions of n into k distinct parts. From (4.4.8),

$$E\ (l_i / n_i) > K_1 \sqrt{n_i} \qquad (4.7.4)$$

where $K_1 = 11\sqrt{2} / 12\sqrt{\pi} = 0.732 \ldots$

and from (4.6.52),

$$P[n_1, \ldots, n_k] > \sqrt{2\pi n} \quad k_2^{-k} / \prod n_i \qquad (4.7.5)$$

if the n_i are distinct, where

$$K_2 = 11/24 - 11 / (12 \sqrt{2\pi}) = 0.0926 \ldots$$

From (4.7.3) - (4.7.5),

$$E[LCM] \geq \sum_k \sqrt{2\pi n} \quad v_k \, k_3^k \, \phi_{n,k} \qquad (4.7.6)$$

where $K_3 = K_1 / K_2$ and

$$\phi_{n,k} = \sum (n_1 n_2 \ldots n_k)^{-1/2} , \qquad (4.7.7)$$

the latter summation extending over all partitions of n into k distinct parts. A lower bound to the smallest term of (4.7.7) is given by

$$n_1 = \ldots = n_k = n/k$$

and therefore

$$\phi_{n,k} > Q(n,k) \, k^{k/2} \, n^{-k/2} \qquad (4.7.8)$$

From (4.6.60), (4.5.69), (4.7.6), (4.7.8),

$$E[LCM] > \sum_{k=1}^{n^{1/5}} t_k \qquad (4.7.9)$$

where

$$t_k = c_3 n^{(k-1)/2} \, c_4^k \, k^{(2-k)/2} \, (k!)^{-2}$$

$$c_3 = c_1 \sqrt{\pi} / \sqrt{2} = 0.76 \ldots$$

$$c_4 = K_3 c_2 = 0.41 \ldots$$

Now (4.7.9) has the same form as (4.6.62), and from (4.6.63) it follows that the greatest term of (4.7.9) is located very close to

$$k_0 = n^{1/5} c_5 < n^{1/5}$$

where $c_5 = c_4^{2/5} e^{-1/5} = 0.57 \ldots$

and certainly

$$E[LCM] > t_{k_\circ} = \exp[c_7 n^{1/5} - \frac{1}{2} \log n - c_6]$$

where

$$c_6 = \log (2\pi e^{1/6}/ c_3)$$

$$c_7 = -5c_5 \log c_5 /2 + c_5 (2 + \log c_4) = 0.80 \ldots$$

so we have proved

(4.7.12) <u>Theorem</u>. For $n \geq 32$, there exist constants c_6, $c_7 = 0.80 \ldots$ such that

$$E[LCM] > \exp[c_7 n^{1/5} - \frac{1}{2} \log n - c_6],$$

subject to the validity of approximations (4.5.9) and (4.6.48).

4.8 UPPER BOUND TO EXPECTED VALUE OF CYCLE TIME

Suppose there are k components, of sizes $n_1\ n_2 \ldots n_k$, and containing loops of lengths $l_1\ l_2 \ldots l_k$ respectively;

where

$$\sum_{i=1}^{k} n_i = n, \qquad 1 \leq n_i \leq n \text{ for all } i. \qquad (4.8.1)$$

Then $E[l_i/n_i] < c_1\sqrt{n_i}$ by equation (4.4.8), where $c_1 = 1.33 \ldots$

(4.8.2)

so that

$$E_1[LC] \leq E_3 [LCM (l_1, l_2, \ldots, l_k)] \qquad (4.8.3)$$

$$\leq E_3 [l_1\ l_2 \ldots l_k]$$

$$\leq E_2 [c_1^k \prod_{i=1}^{k} \sqrt{n_i}] \qquad (4.8.4)$$

Now the maximum of $n_1 \, n_2 \ldots n_k$ subject to the constraint (4.8.1) is $(n/k)^k$, when $n_1 = n_2 = \ldots = n_k = n/k$. So (4.8.4) is

$$\leq E_4 \, [\, c_1^k \, (n/k)^{k/2}]$$

$$= \sum_{k=1}^{n} c_1^k \, P \, [k] \, (n/k)^{k/2} \qquad (4.8.5)$$

where P[k] = Prob. [Type 1A network has k components] is given by (3.3.1). Now from (3.3.12), (3.3.17), (3.3.18), for $n \geq 3$,

$$E[k] < \frac{1}{2} \log n + c_2 \qquad (4.8.7)$$

$$\sigma^2 \, [k] < (\log^2 n) \, /4 \, + c_3 \log n \qquad (4.8.8)$$

where $c_2 = 1.003 \ldots$, $c_3 = 3.471 \ldots$

Therefore for any $t > 0$ and $n \geq 100$,

$$E[k] + t\sigma \, [k] < \frac{1}{2} \log n + c_2 + t \, (\frac{1}{4} \log^2 n + c_3 \log n)^{1/2}$$

$$< \frac{1}{2} (1 + c_4 \, t) \log n + c_2$$

where $c_4 = \exp \, (2 \, c_3 \, / \, \log 100) = 4.53 \ldots$

$$< \frac{1}{2} (\, 1 + c_4 \, t + 2 \, c_2) \log n$$

so that, with $A = \frac{1}{2} (1 + 2 \, c_2 + c_4 \, t)$,

$$P \, [\, k < A \log n \,] \geq P[\, k < E[k] + t \, \sigma[k] \,]$$

$$\geq 1 - t^{-2}$$

by Chebyshev's inequality

i.e. for any $A > c_2 + \frac{1}{2} \approx 1.503$, and $n \geq 100$, with at most

$$\frac{c_4^2 \, n^n}{4 \, (A - (c_2 + \frac{1}{2}))^2}$$

exceptions, all type 1A networks with n nodes have fewer than A log n components. Further we shall now prove

(4.8.9) Theorem. The expected value of the cycle time of all type 1A networks with n nodes and fewer than A log n components is less than

$$\exp \left[\frac{1}{2} A \log^2 n + A \log n \log c_1 \right]$$

for $n \geq 100$ and $n/(2 \log n) > A > c_2 + 0.5$

Proof. Let Ω_4 be the sample space defined in section 3.2, and let $\Omega_4^* \subset \Omega_4$ be the subspace of Ω_4 consisting of all type 1A networks with less than A log n components. Then the theorem asserts that

$$E_{\Omega_4^*} [LC] < \exp \left[\frac{1}{2} A \log^2 n + A \log n \log c_1 \right]$$

From (4.8.5),

$$E_{\Omega_4^*} [LC] \leq \sum_{k=1}^{A \log n} c_1^k \left(\frac{n}{k}\right)^{k/2} P_* [k] \qquad (4.8.10)$$

where

$$P_* [k] = P [k] / P [\Omega_4^*]$$

is the probability distribution of Ω_4^*. Let

$$b_k = c_1^k \, n^{k/2} \, k^{-k/2}$$

It is easy to see that

$$b_k \leq b_{k+1}$$

for $k < \frac{n}{2}$; so that if $A < n/(2 \log n)$ it follows from (4.8.10) that

$$E_{\Omega_4^*} [LC] < \exp [\frac{1}{2} A \log^2 n + A \log n \log \alpha]$$

which proves the theorem.

Q.E.D.

Remark

This upper bound over Ω_4^* is less than the lower bound over Ω_4 obtained in theorem (40.41), for large n. This implies that the networks in $\Omega_4 - \Omega_4^*$, i.e. those with $\geq A \log n$ components, must make a large contribution to E [LC]. This remark is expanded in Chapter 2. The next two theorems give upper bounds to E [LC] over Ω_4, i.e. when this contribution is included.

(4.8.11) Theorem. There exist constants A,B such that for $n > B$,

$$E [LC] < \exp [c_1 \sqrt{n} \log n/4 + A \sqrt{n}]$$

where c_1 is given by (4.8.2).

Proof. From (4.8.3), (4 8.5),

$$E [\text{Cycle Time}] \leq \sum_{k=1}^{n} \alpha^k (\tfrac{n}{k})^{k/2} P [k] \text{ where } \alpha = c_1$$

$$\leq \sum_{r=1}^{n} \binom{n-1}{r-1} n^{-r} \sum_{k=1}^{r} \alpha^k n^{k/2} C(r,k) \qquad \text{using (4.8.6)}$$

$$= \sum_{r=1}^{n} \begin{matrix} n-1 \\ r-1 \end{matrix} n^{-r} \Gamma (x + r) / \Gamma (x)$$

where $x = \alpha \sqrt{n}$, by (for instance) equation 6.1.22 of [2] and page 71 of [45],

$$= \sum_r a_r \quad \text{(say)}$$

Now $a_{r+1} / a_r = (n - r)(x + r) / nr$

so the greatest term of (4.8.12) occurs when r is the nearest integer to

$$\alpha^{1/2} n^{3/4}$$

for n sufficiently large. Using Stirling's approximation A9, one finds therefore that

$$\log a_r < \frac{1}{4} \alpha \sqrt{n} \log n + A \sqrt{n}$$

for some constant A and n > B. Theorem (4.8.11) follows.

Q.E.D.

In the next theorem we use a different method to obtain a better bound than Theorem (4.8.11). (We give two upper bounds because neither agrees with the present lower bound, so that this question is still open, and the more methods of attack that are available the better.)

(4.8.13) Theorem. There exists a constant A such that for all positive n,

$$E\,[\text{Cycle Time}] < \exp\left[\, 2\sqrt{2n} - \frac{1}{2} \log n + \log A\right]$$

Proof. Like the preceeding theorems, this is based on

$$\text{Cycle Time} \leq \text{LCM} \leq \text{Product}.$$

We have

$$E\,[\text{Cycle Time}] \leq E\,[\text{ Product of cycle lengths}]$$

$$= \sum n!\, w/[(n - w)!\, n^{w+1}\, a_1! \ldots a_n!] \qquad (4.8.15)$$

using (3.2.14), where the summation is over all n-tuples of integers $(a_1, \ldots, a_k)$ such that

$$1 \leq a_i \leq n\,, \text{ for all } i \qquad (4.8.16)$$

and

$$1 \leq w = \sum_{i=1}^{n} i a_i \leq n, \qquad (4.8.17)$$

$$\leq \sum n!w \,/\, [(n-w)!\, n^{w+1}]$$

over the same range of summation

$$= \sum_{w=1}^{n} p(w)\, n!\, w/ \,[(n-w)\,!\, n^{w+1}] \qquad (4.8.18)$$

where $p(w)$ = number of partitions of w

Now ([29]) for some constant K, and $n \geq 1$,

$$p(n) < Kn^{-1} \quad \exp\,[2\sqrt{2n}\,] \qquad (4.8.19)$$

The desired result now follows from (4.8.18), (4.8.19) and Lemmas A5 and A9.

Q.E.D.

4.9 A THIRD METHOD FOR OBTAINING AN UPPER BOUND TO E [LC]

If a good upper bound to $\dfrac{E\,[LCM\,(x_1 \ldots x_k)\,]}{E\,[\,x_1\, x_2 \ldots x_k]}$ were known, analogous to the lower bound obtained in Section 4.5, say

$$\frac{E\,[LCM\,(x_1 \ldots x_k)]}{E\,[x_1\, x_2 \ldots x_k]} \leq u_k \qquad (4.9.1)$$

then the method of Section 4.6 would lead to an upper bound for E [LC], as follows.

$$E\,[LC] = E_{\Omega_1 x \Omega_7}\,[LCM\,(e_1, \ldots, e_k)] \qquad (4.9.2)$$

where e_i = effective loop length of i^{th} loop

$\leq E_{\Omega_1 x \Omega_7}$ [LCM $(l_1, \ldots, l_k)$]

where l_i is the length of the i^{th} loop (with equality here if the initial activity were chosen so that each component contains exactly one active node).

$$= E_{\Omega_1} [\text{LCM } (l_1, \ldots, l_k)]$$

since the activity is no longer relevant

$$\leq E_2 [u_k \prod l_i] \tag{4.9.3}$$

with a justification similar to that of (4.6.49).

Now once the size n_i of a component is given, the length l_i of the loop in that component is a random variable depending only on n_i, and in fact from equation (4.4.8),

$$E [l_i / n_i] < c_1 n_i^{1/2} \tag{4.9.4}$$

where c_1 is given by (4.8.2).

(4.9.3) becomes

$$= \sum_{k=1}^{n} \sum \left[P[n_1 \ldots n_k m_1 \ldots m_k] \, u_k \prod_{i=1}^{k_1} E[P_i / n_i]^{m_i} \right] \tag{4.9.5}$$

where the second summation is over all partitions of n into exactly k components, m_i of size n_i, $i = 1, \ldots, k_1$, so that

$$\sum_{i=1}^{k_1} m_i = k, \qquad \sum_{i=1}^{k_1} m_i n_i = n \tag{4.9.6}$$

and where $P[n_1 \ldots n_k \; m_1 \ldots m_k]$ is the probability that a type 1A network has such a decomposition into components, which is given by (3.2.9)

From (3.2.7), (3.2.9), (4.9.6),

$$P[n_1 \ldots n_k \; m_1 \ldots m_k] \leq \sqrt{2\pi n} \; e^{\frac{1}{12n}} K_1^k \prod_{i=1}^{k_1} \frac{1}{n_i^{m_i} \; m_i!} \qquad (4.9.8)$$

where $K_1 = \frac{1}{2} e^{1/12}$

Using (4.9.4) and (4.9.8), (4.9.5) is

$$\leq \sqrt{2\pi n} \; e^{\frac{1}{12n}} \sum_{k=1}^{n} u_k K_2^k \; \Phi_{n,k} \qquad (4.9.9)$$

where
$$\Phi_{n,k} = \sum (a_1 \; a_2 \ldots a_k)^{-1/2} \qquad (4.9.10)$$

the summation extending over all partitions of n into exactly k components, and where $K_2 = c_1 K_1$.

From (4.9.10) it is clear that $\Phi_{n,k}$ has recurrence relation given by

$$\Phi_{n,k+1} = \sum_{a=1}^{n-k} \Phi_{n-a,k} \, a^{-1/2}, \quad n \geq k+1 \geq 2$$

$$\Phi_{n,1} = n^{-1/2} \qquad (4.9.11)$$

A trivial calculation shows that

$$\Phi_{n,2} \leq \pi - 2/n \quad \text{for} \quad n \geq 2 \tag{4.9.12}$$

(4.9.13) Theorem

For all $n \geq k \geq 2$,

$$\Phi_{n,k} \leq A^k B n^{(k-2)/2} [(k-2)!]^{-1/2} \tag{4.9.14}$$

where $A = (2\pi)^{1/2} e^{1/6}$, $B = 1/2 \; e^{-1/3}$

Proof. We use induction on k. For k = 2 the desired result follows from (4.9.12).

Now let us assume (4.9.14) is true for some $k \geq 2$. Then we must prove

$$\Phi_{n,\, k+1} \leq u_{k+1} \, n^{(k-1)/2} \quad \text{for all} \quad n \geq k+1 \tag{4.9.15}$$

where $u_k = A^k B [(k-2)!]^{-1/2}$ (4.9.16)

Now

$$\Phi_{n,k+1} \leq \sum_{a=1}^{n-k} u_k (n-a)^{(k-2)/2} a^{-1/2} \text{ , from (4.9.11) and (4.9.14)}$$

$$\leq u_k \int_0^{n-k} (n-x)^{(k-2)/2} x^{-1/2} \, dx$$

which becomes, using the substitution $x = n \cos^2 \theta$ and the result that

$$\int_0^{\pi/2} \sin^n \theta \, d\theta = \pi n!/[2^{2m+1} (m!)^2] \qquad \text{if } n = 2m$$

$$= 2^{2m} (m!)^2/n! \qquad \text{if } n = 2m + 1$$

$$\Phi_{n,\, k+1} \leq 1 \qquad \text{if } n = 1$$

$$\leq \sqrt{\frac{\pi}{2n}}\; e^{\frac{1}{3(n-1)}} \qquad \text{if } n \text{ odd, } n \geq 3$$

$$\leq \sqrt{\frac{\pi}{2n}}\; e^{\frac{1}{12n}} \qquad \text{if } n \text{ even}$$

Therefore

$$\Phi_{n,k+1} \leq \sqrt{\frac{\pi}{2n}}\; e^{1/6} \qquad \text{for all } n \geq 1 \tag{4.9.18}$$

From (4.9.17), (4.9.18)

$$\Phi_{n,k+1} \leq u_{k+1} \; n^{(k-1)/2}$$

QED

From Theorem (4.9.13) and (4.9.9) we then obtain

$$E[LC] \leq \sqrt{2\pi n}\; e^{1/(12n)} \sum_{k=1}^{n} u_k \, (AK_2)^k \, B \, n^{(k-2)/2} \, (k-2)!^{-1/2}$$

which is as far as we can go without knowing u_k.

4.10 RESULTS OF COMPUTER SIMULATION

LBF, LC and LF were measured experimentally by simulating Type 1A networks on an IBM 7090 computer. The results were generally incondusive, since only networks with 100 nodes or less were analized. The results of one series of experiments, with initial activity equal to 30%, are shown in Table 4.10.1 and Figure 4.10.1.

TABLE 4.10.1

MEANS OF LBF, LC, LF AS FUNCTIONS OF NETWORK SIZE

Type 1A networks, 30% initial activity

N	LBF	LC	LF	No. of Samples
5	.78	1.95	2.73	60
10	2.43	2.37	4.80	130
15	3.49	2.70	6.19	180
20	4.20	4.49	8.69	190
25	5.26	3.82	9.11	160
30	5.94	4.64	10.58	110
35	6.55	5.84	12.28	190
40	7.56	6.61	14.18	267
45	8.30	6.42	14.72	90
50	8.84	6.94	15.78	196
55	8.90	8.40	17.31	42
60	9.67	6.21	15.88	75
65	10.85	9.90	20.75	40
70	10.48	6.86	17.34	29
75	11.44	10.28	21.71	108
80	12.66	9.44	22.10	41
100	14.08	11.16	25.24	82

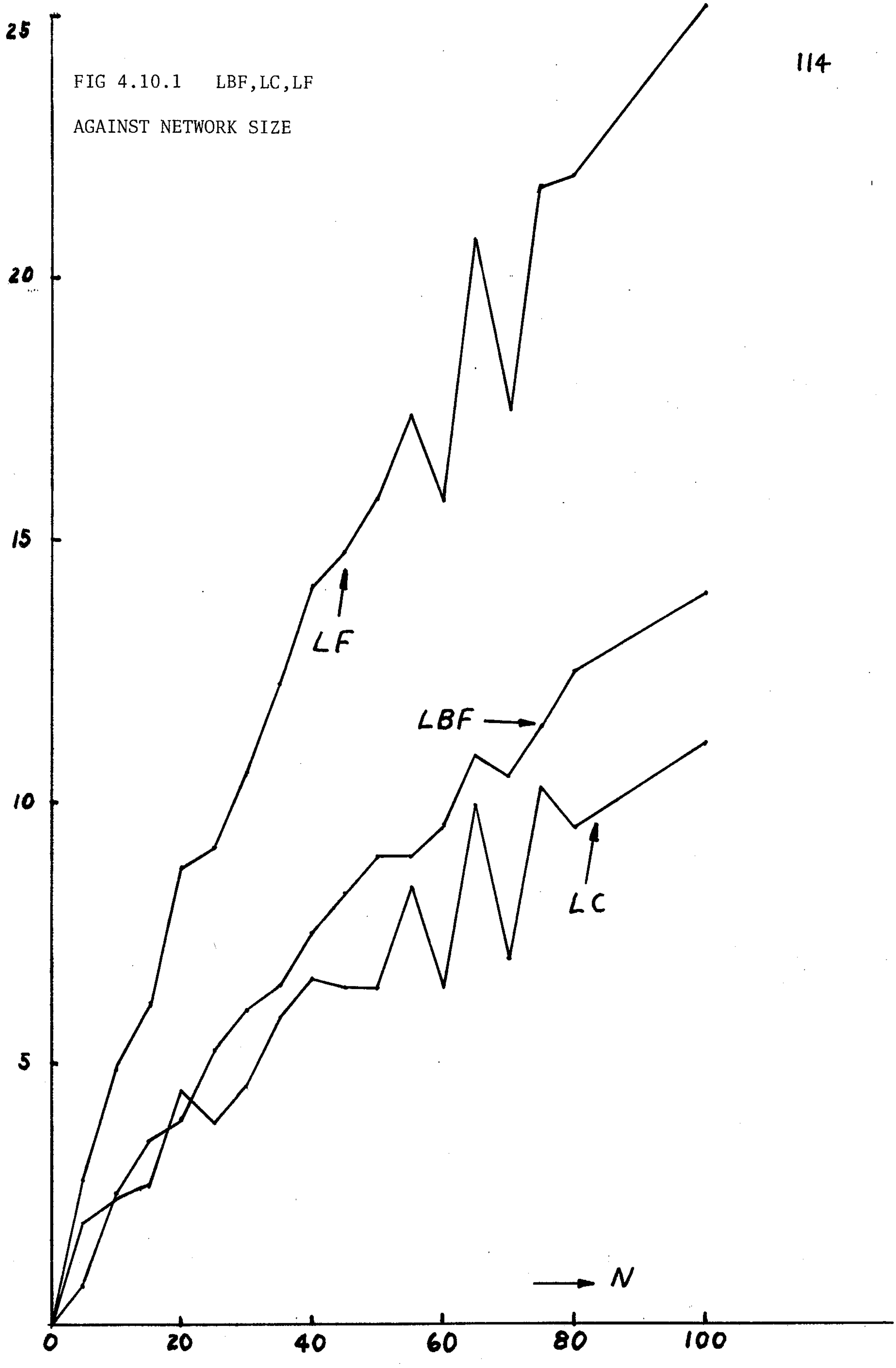


FIG 4.10.1 LBF,LC,LF
AGAINST NETWORK SIZE
LF
LBF
LC
N
0
5
10
15
20
25
20
40
60
80
100

CHAPTER 5

FURTHER PROPERTIES OF TYPE 1A NETWORKS

5.1 INTRODUCTION

This chapter contains further discussion of Type 1A networks, arranged as follows.

5.2 A brief note on the history of the study of type 1A networks.

5.3 The average height of a node in a tree is $\sim\sqrt{\frac{\Pi n}{2}}$.

5.4 Further results on structure of Type 1A networks.

5.5 The expected number of nodes with k incoming branches.

5.6 The relation between type 1A and type 4 networks.

5.7 A description of the problem for type 1A networks in terms of semi-groups, and applications to Markov chains, cryptanalysis, etc.

5.2 A NOTE ON THE HISTORY OF THE STUDY OF TYPE 1A NETWORKS

Previous work on Type 1A networks is as follows:

N. Metropolis and S. Ulam in 1953 ([41]) raised the question of the expected number of components of a type 1A network. This was answered by M.D. Kruskal in 1954 ([37]).

Kruskal's paper was followed by the contributions of H. Rubin and R. Sitgreaves in 1954 [52], J.E. Folkert in 1955 [17], L. Katz in 1955 [36], B. Harris in 1960 [31], and J. Riordan in 1962 [47].

5.3 THE AVERAGE HEIGHT OF A NODE IN A TREE IS $\sim\sqrt{\frac{\Pi n}{2}}$.

This section is an extension of section 4.2, and uses the same notation. $V_{n,1}$ is the average height of a node in an average rooted tree with $n + 1$ nodes, and, we feel, is of independent interest. As far as type 1A networks are concerned this gives us a lower bound to the average time the activity will take to die out in particular tree of a type 1A network.

Let $L_{n,m}$ be the number of rooted labelled trees with n nodes and weight m, with generating function

$$L(x,y) = \sum_{n,m=o}^{\infty} L_{n,m} \frac{x^n y^m}{n!} \tag{5.3.1}$$

(5.3.2) <u>Theorem</u>† L satisfies the functional equation

$$L(x,y) = x \exp L(xy,y) \tag{5.3.3}$$

<u>Proof</u>. This is an application of Polya's theorem, ([45], p. 133):

Consider the family F_n of rooted trees with n branches at the root node. The store consists of the rooted trees which may be connected to these branches, and has the generating function

$$L(xy,y) = \sum_{n,m=o}^{\infty} L_{n,m} \frac{x^n y^{n+m}}{n!}$$

since the n nodes all have their distance from the root increased by one.

The nodes are labelled, so no tree may be used more than once; so the generating functions $S_2, S_3, \ldots, S_n$ in the statement of Pólya's theorem, (p. 131 of [45]) are all 0.

Any permutation of the n branches at the root leaves the same tree, so the cycle index is that of the symmetric group S_n, i.e.,

$$\frac{C_n(t_1, t_2, \ldots, t_n)}{n!}$$

---

†Theorem (5.3.2) and the recurrence relation (5.3.13) are due to John Riordan, to whom I am very grateful.

Then by Polya's theorem, since the root node contributes one node, the generating function for F_n is

$$\frac{X\, C_n(L(xy,y),\, 0,\, \ldots,\, 0)}{n!}$$

and for all rooted trees is

$$L(x,y) = \sum_{n=1}^{\infty} \frac{X\, C_n(L(xy,y),\, 0,\, \ldots,\, 0)}{n!}$$

$$= x \exp L(xy,y) \qquad \text{by } (6.1.1)$$

which proves (5.3.3).

Let $L_n(y)$ be the generating function for rooted labelled trees with n nodes and weight j, so that

$$L_n(y) = \sum_{j=0}^{\infty} L_{nj}\, y^j \tag{5.3.4}$$

and

$$L(x,y) = \sum_{n=0}^{\infty} \frac{x^n}{n!} L_n(y) \tag{5.3.5}$$

Therefore

$$L_n'(1) = \sum_{j=1}^{\infty} j\, L_{nj} = q_n \quad \text{(say)} \tag{5.3.6}$$

$$W_{n,1} = \frac{L_{n+1}'(1)}{n+1} = \frac{q_{n+1}}{n+1} \tag{5.3.7}$$

By the preceding theorem (5.3.2),

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} L_n(y) = x \exp \sum_{n=0}^{\infty} \frac{x^n y^n L_n(y)}{n!} \tag{5.3.8}$$

Differentiate w.r.t. y:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} L_n'(y) = \sum_{n=o}^{\infty} \frac{x^n}{n!} L_n(y) \left[\sum_{n=1}^{\infty} \frac{x^n y^{n-1}}{(n-1)!} L_n(y) + \right.$$

$$+ \sum_{n=0}^{\infty} \frac{x^n}{n!} y^n L_n'(y) \Bigg] \tag{5.3.9}$$

Now

$$L_n(1) = \sum_{j=0}^{\infty} L_{nj} = n^{n-1} \text{ by theorem (A.2.1) of Appendix 2} \tag{5.3.10}$$

Setting $y = 1$ in (5.3.9), equating coefficients of x^n, and using (5.3.7), (5.3.10), we obtain

$$\frac{q_n}{n!} = \sum_{r=1}^{n-1} \frac{r^{r-1}(n-r)^{n-r-1}}{r!(n-r-1)!} + \sum_{r=1}^{n} \frac{r^{r-1}}{r!\ (n-r)!}\ q_{n-r} \tag{5.3.11}$$

Now the first term on the right-hand side

$$= \frac{1}{N!} \sum_{s=0}^{N} \binom{N}{s} (s+1)^{s-1} (n-s-1)^{N-s}, \text{ putting } s=r-1,\ N=n-2$$

$$= \frac{1}{(n-2)!} n^{n-2}, \text{ by (A10.1) of Appendix 1.} \tag{5.3.12}$$

From (5.3.7), (5.3.11), (5.3.12), $w_{n,1}$ satisfies the recurrence relation

$$W_{0,1} = 0$$

$$W_{n,1} = n(n+1)^{n-1} + \sum_{r=1}^{n} \binom{n}{r} r^{r-1} W_{n-r,\,1}, \quad n \geq 1 \tag{5.3.13}$$

which is an unpublished result of John Riordan.

(5.3.13) may be solved as follows. Let $w_n = w_{n,1}$, $a_n = n(n+1)^{n-1}$, $\alpha_n = n^{n-1}$, so that

$$W_o = 0$$

$$W_n = a_n + \sum_{k=1}^{n-1} \binom{n}{k} \alpha_{n-k} W_k, \quad \text{~~k ≥ 1~~} \quad n \geq 2. \tag{5.3.14}$$

Let

$$W(u) = \sum_{n=1}^{\infty} w_n \frac{u^n}{n!} \tag{5.3.15}$$

$$a(u) = \sum_{n=0}^{\infty} a_n \frac{u^n}{n!} = \sum_{n=0}^{\infty} \frac{n(n+1)^{n-1}}{n!} u^n \tag{5.3.16}$$

$$A(u) = \sum_{n=1}^{\infty} \alpha_n \frac{u^n}{n!} = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} u^n \equiv R(u) \tag{5.3.17}$$

where R(u) is the generating function for rooted labelled trees, and satisfies the functional equation (see [45], p. 128)

$$R(u) = u \exp R(u), \; R(o) = 0 \tag{5.3.18}$$

Therefore

$$R'(u) = u^{-1} R(u) + R(u) R'(u)$$

$$R'(u) = \frac{R(u)}{u(1 - R(u))} \tag{5.3.19}$$

(5.3.14) becomes

$$W(u) = a(u) + A(u) W(u)$$

$$W(u) = \frac{a(u)}{1-A(u)} \tag{5.3.20}$$

Now from (5.3.16)

$$a(u) = \sum_{n=0}^{\infty} \frac{(n+1)^{n+1}}{(n+1)!} u^n - \sum_{n=0}^{\infty} \frac{(n+1)^n}{(n+1)!} u^n$$

$$= R'(u) - \frac{R(u)}{u} \tag{5.3.21}$$

From (5.3.17), (5.3.19)-(5.3.21),

$$W(u) = u(R'(u))^2 \tag{5.3.22}$$

$$= u + \frac{8u^2}{2!} + \frac{78u^3}{3!} + \frac{944u^4}{4!} + \frac{13800u^5}{5!} + \frac{237432\,u^6}{6!} + \ldots \tag{5.3.23}$$

which has been verified by direct calculation of W_n.

From (5.3.22),

$$\sum_{n=1}^{\infty} W_n \frac{u^n}{n!} = u \left[\sum_{n=0}^{\infty} \frac{(n+1)^n u^n}{n!} \right]^2 \qquad (5.3.24)$$

Equating coefficients,

$$\frac{W_{n+1}}{(n+1)!} = \sum_{r=0}^{n} \frac{(r+1)^r (n-r+1)^{n-r}}{r!\,(n-r)!} \qquad (5.3.25)$$

$$= \frac{1}{n!} \sum_{r=0}^{n} \binom{n}{r} (r+1)^r \, (n-r+1)^{n-r}$$

$$= \sum_{\nu=0}^{n} \frac{(n+2)^\nu}{\nu!} \qquad \text{by (A10.4) of Appendix 1}$$

Therefore

$$W_{n-1} = (n-1)! \sum_{\nu=0}^{n-1} \frac{n^\nu}{\nu!} - n^{n-1} \qquad (5.3.26)$$

From lemmas A5, A9, for $n \geq 2$

$$W_{n-1} < \sqrt{2\Pi}\, n^{n-1/2} e^{\frac{1}{12n}} \frac{1}{2}\left(1 - \frac{11}{18\sqrt{2\Pi n}}\right) - n^{n-1} \qquad (5.3.27)$$

Now for $x > 0$,

$$e^x < 1 + xe^x$$

$$e^{\frac{1}{12n}} < 1 + \frac{1}{12n} e^{\frac{1}{12n}} \leq 1 + \frac{1}{12n} e^{\frac{1}{24}}, \quad n \geq 2$$

$$W_{n-1} < \sqrt{\frac{\Pi}{2}}\, n^{n-\frac{1}{2}} \left(1 + \frac{1}{12n} e^{\frac{1}{24}}\right)\left(1 - \frac{11}{18\sqrt{2\Pi n}}\right) - n^{n-1}, \quad n \geq 2 \qquad (5.3.28)$$

from which it follows that

$$W_{n-1} < \sqrt{\frac{\Pi}{2}}\, n^{n-\frac{1}{2}} \left(1 - \frac{A}{\sqrt{n}}\right), \quad n \geq 2, \qquad (5.3.29)$$

where $$A = \frac{11}{18\sqrt{2\Pi}} - \frac{e^{1/24}}{12\sqrt{2}} + \frac{2}{\Pi} > 0 \tag{5.3.30}$$

Also

$$W_{n-1} > \sqrt{2\Pi}\, n^{n-\frac{1}{2}} \frac{1}{2}\left(1 - \frac{1}{\sqrt{2\Pi n}}\right) - n^{n-1}$$

$$= \sqrt{\frac{\Pi}{2}}\, n^{n-\frac{1}{2}} - \frac{3}{2} n^{n-1} \tag{5.3.31}$$

Finally from (5.3.29), (5.3.31),

$$W_{n-1} \sim \sqrt{\frac{\Pi}{2}}\, n^{n-\frac{1}{2}} \quad \text{as } n \to \infty$$

Also from (4.2.5),

$$V_{n,1} = \frac{W_{n,1}}{(n+1)^n} = \frac{W_n}{(n+1)^n} \tag{5.3.32}$$

so that from (5.3.29), (5.3.32),

$$V_{n,1} < \sqrt{\frac{\Pi(n+1)}{2}} - A\sqrt{\frac{\Pi}{2}}$$

$$\leq \sqrt{\frac{\Pi n}{2}} + \sqrt{\Pi} - \sqrt{\frac{\Pi}{2}} - A\sqrt{\frac{\Pi}{2}} \qquad \text{for } n \geq 1 \tag{5.3.33}$$

$$V_{n,1} > \sqrt{\frac{\Pi(n+1)}{2}} - \frac{3}{2} > \sqrt{\frac{\Pi n}{2}} - \frac{3}{2} \qquad \text{for } n \geq 1$$

and

$$V_{n,1} \sim \sqrt{\frac{\Pi n}{2}}$$

So we have proved these three theorems:

(5.3.34) <u>Theorem</u> $W_{n,1}$ has the following properties:

$$W_{0,1} = 0$$

$$W_{n,1} = n(n+1)^{n-1} + \sum_{r=1}^{n} \binom{n}{r} r^{r-1} W_{n-r,1} \qquad n \geq 1$$

$$W_{n,1} \sim \sqrt{\frac{\Pi}{2}}\,(n+1)^{n+\frac{1}{2}} \quad \text{as } n \to \infty$$

$$\sqrt{\frac{\Pi}{2}}\, n^{n-\frac{1}{2}} - \frac{3}{2}\, n^{n-1} < W_{n-1,1}$$

$$< \sqrt{\frac{\Pi}{2}}\, n^{n-\frac{1}{2}} - A\sqrt{\frac{\Pi}{2}}\, n^{n-1}\,, \quad \text{for } n \geq 1,$$

where A is given by (5.3.30).

The generating function of $W_{n,1}$,

$$W(u) = \sum_{n=1}^{\infty} W_{n,1} \frac{u^n}{n!}\,,$$

is given by

$$W(u) = u(R'(u))^2$$

(5.3.35) Theorem $V_{n,1}$ has the following properties:

$$V_{n,1} \sim \sqrt{\frac{\Pi n}{2}}$$

$$\sqrt{\frac{\Pi n}{2}} - \frac{3}{2} < V_{n,1} < \sqrt{\frac{\Pi n}{2}} - B, \qquad n \geq 1$$

where $B > 0$ is given by (5.3.33).

(5.3.36) Theorem

Let G be the family of all rooted trees with n labelled nodes, and let

$$D_n = \underset{g \,\varepsilon\, G}{\text{average}} \left[\underset{\text{nodes of } g}{\text{average}} \left(\text{distance of a node from the root} \right) \right]$$

Then $\quad D_n \sim \sqrt{\frac{\Pi n}{2}}$ and

$$\sqrt{\frac{\Pi n}{2}} - \frac{3}{2} < D_n < \sqrt{\frac{\Pi n}{2}} - A\sqrt{\frac{\Pi}{2}}, \quad n \geq 1$$

where A is given by (5.3.30)

Proof. D_n is just $V_{n-1,1}$.

5.4 FURTHER RESULTS ON STRUCTURE OF TYPE 1A NETWORKS

Let a type 1A network be chosen at random and let a node β be chosen at random in this network. In the notation of section 3.2 this is the probability space $\Omega_1 \times \Omega_6$ with $\alpha = 1$.

Define s, the number of descendants of β, to be one more than the number of distinct nodes that can be reached from β.

Define p, the number of ancestors of β, to be one more than the number of distinct nodes from which β can be reached.

Let ℓ be the length of the loop in the component containing β. Then it is known that (see [52]), over the probability space $\Omega_1 \times \Omega_6$,

$$P[s = k] = \frac{(n-1)!\, k}{(n-k)!\, n^k}$$

$$P[\ell = j] = \sum_{k=j}^{n} \frac{(n-1)!}{(n-k)!\, n^k}$$

The asymptotic density of $\frac{s}{\sqrt{n}}$ is, for large n,

$$\begin{array}{ll} xe^{-x^2/2} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{array}$$

and of $\frac{\ell}{\sqrt{n}}$ is

$$\begin{array}{ll} \sqrt{2\Pi}(1 - \Phi(y)) & \text{for } y > 0 \\ 0 & \text{for } y \leq 0 \end{array}$$

where Φ is defined by (4.1.13).

Also

$$P[p = j] = \frac{n!\, j^{j-1}\, (n-j)^{n-j}}{(n-j)!\, j!\, n^n} \qquad (5.4.1)$$

Harris ([31]) and Riordan ([47]) collected and rederived these results, and extended them for type 4 networks, type 5A networks, and type 4 networks with loops of length 1 excluded.

Number of Trees in a Type 1A Network

If a type 1A network contains W loop nodes, not all of them may have trees leading into them. Let $B_{n,k}$ be the number of elements of T_n with exactly k trees, and let

$$B_n = \sum_{k=1}^{n} \frac{k}{n^n} B_{n,k} = \text{expected number of trees.}$$

B_n is of interest to us because it gives some information about the way in which activity enters the loops. If

$$B_n << \text{expected number of loop nodes,}$$

then it might be expected that activity enters the loops at a small number of nodes, and therefore that the distribution of activity in the loops would be irregular, some loops not being activated at all. On the other hand, if

$$B_n \sim A.\ \text{expected number of loop nodes, A constant,}$$

then it would be expected that there are "plenty" of nodes at which the activity enters the loops. The latter statement agrees with our intuitive picture, and in fact we prove

(5.4.2) Theorem

$$B_n \sim \sqrt{\frac{\Pi n}{2}}\ (1 - \frac{1}{e})$$

Proof. Let a node β be chosen as at the beginning of this section, over the sample space $\Omega_1 \times \Omega_6$. It is clear (although we do not give a proof) that the following properties of β are independent:

(I) β is a loop node

(II) there is at least one node, not a loop node, that is directly connected to β.

Then the expected number of trees, B_n =

$$= n\, P[\beta \text{ has properties I and II}]$$

$$= n\, P[\beta \text{ has property I}]\, P[\beta \text{ has property II}] \qquad (5.4.3)$$

Now $P[\beta \text{ has property I}] \sim \sqrt{\frac{\Pi}{2n}}$ by lemma (3.4.4)

and

$$P[\beta \text{ has property II}] = 1 - P[p = 1] \quad \text{(in the above notation)}$$

$$\sim 1 - \frac{1}{e} \quad \text{(by (5.4.1))}$$

From (5.4.3),

$$B_n \sim \sqrt{\frac{\Pi n}{2}}\,(1 - \frac{1}{e})$$ QED

Remark

The number of trees is also of interest to mathematical sociologists. (See [10], p. 73 and [57].)

5.5 NUMBER OF NODES WITH k INCOMING BRANCHES

Another parameter related to type 1A networks is $C_{n,k}$ = the expected number of nodes with k incoming branches in a type 1A network of n nodes. In particular $C_{n,0}$ is the expected number of end points.

Let there be n nodes labelled 1,2,...,n and let us consider the family of type 1A networks with these nodes.

Let $X_i = 1$ if the i^{th} node has k incoming branches
 0 if not

Then $$P[X_i = 1] = \binom{n}{k}\left(\frac{1}{n}\right)^k (1 - \frac{1}{n})^{n-k}$$

$$\sim \frac{e^{-1}}{k!} \text{ for fixed k, as } n \to \infty \text{, since}$$

this is just the Poisson approximation to the binomial distribution.

Also

$$S_{n,k} = \sum_{i=1}^{n} X_i = \text{number of nodes with k incoming branches, so that}$$

Conclusion

$$C_{n,k} = E[S_{n,k}] = \sum_{i=1}^{n} E[X_i] \sim \frac{ne^{-1}}{k!}, \text{ for fixed k, as } n \to \infty.$$

5.6 RELATION BETWEEN TYPE 1A NETWORKS AND TYPE 4 (PERMUTATION) NETWORKS

Let S_n be the family of all n! type 4 networks with n nodes, or equivalently all n! permutations of n objects. Equation (3.2.11) suggests that there is a rough correspondence between T_n and S_n, and that we may expect properties concerning the decomposition of permutations into cycles to be roughly the same as properties concerning the decomposition of type 1A networks into components. This is useful because of the work of Erdos and Turan [15] on the structure of permutations, a summary of which is given in Chapter 6.

Distribution of Loop Sizes

The distribution of loop sizes is obviously important for the calculation of cycle times. An explicit formula for the probability of occurrence of a_1 cycles of length 1, a_2 of length 2,..., a_m of length m, is given in equation (3.2.14), but this does not tell us very much. However, in section 4.4 it is shown that the size k of a loop, considered as a random variable, depends only on the size n of the component containing it, and in fact

$$E[k|n] \sim \sqrt{\frac{2n}{\Pi}} . \qquad (5.6.1)$$

For type 4 networks it is shown in [15] (see Chapter 6) that the component sizes are roughly uniformly distributed (on a logarithmic scale). By the argument of the preceding paragraph this suggests that components of type 1A networks are also roughly uniformly distributed, and therefore by (5.6.1) so are the loop sizes. This would seem to be a promising line for future research.

5.7 DESCRIPTION OF THE PROBLEM FOR TYPE 1A NETWORKS IN TERMS OF SEMI-GROUPS

Erdos and Turan call their basic paper ([15]) on the structure of random permutations "On Some Problems of a Statistical Group Theory". Type 1A networks do not form a group, but they do form a semi-group.

Let T_n be the set of all type 1A networks of n nodes. T_n can be identified with the set of all mappings of the set

$$[1, 2, \ldots, n]$$

into itself. We define a multiplication of elements $s, t \in T_n$ to be the composition $s \circ t$ of the corresponding mappings. The following properties are then satisfied:

(1) $s, t \in T_n \Rightarrow s \circ t \in T_n$

(2) $s \circ (t \circ u) = (s \circ t) \circ u$ for all $s, t, u \in T_n$

(3) there exists an identity element $e \in T_n$ (the identity mapping) such that

$$e \circ s = s \circ e = s \text{ , for all } s \in T_n .$$

Thus $(T_n, \circ)$ is a semi-group with unit element (see e.g., [39]). For convenience we will call it T_n.

Some additional properties of T_n are:

(4) the number of elements of T_n is n^n

(5) if β_n is the group of all permutations of n objects, then

$$\beta_n \subset T_n$$

(6) T_n is non-commutative

i.e., $\exists\, s, t \in T_n \ni s \circ t \neq t \circ s$

(7) T_n does not have the property of right cancellation

i.e., $\exists\, s, t, u \in T_n \ni$

$$s \circ u = t \circ u \text{ and } s \neq t$$

(8) T_n does not have the property of left cancellation

(9) Inverses do not always exist in T_n

i.e., $\exists s \in T_n$ such that

$$s \circ t \neq e, \ \forall t \in T_n$$

In fact if $s \in T_n$ is such that

$$s(x) = s(y) \quad , x \neq y$$

then s does not have an inverse.

Our problem for type 1A networks may be stated in terms of semi-groups as follows:

Let X be a subset of [1,2,...,n]. Pick an element s at random from the semi-group T_n. What is the behavior of

$$s^m(X)$$

as $m \to \infty$?

This work could therefore be subtitled "On Some Problems of a Statistical Semi-Group Theory".

Two Examples where Semi-groups T_n occur in Science

Example 1. Taken from Ljapin ([39]): "Let us consider the following very common situation in physics. Let $\mathcal{A}$ be a physical system of states for which it is known that the system can occur in one of the states $n_\alpha, n_\beta, \ldots, n_\xi, \ldots$, the totality of which we will denote by Ω. The system $\mathcal{A}$ is now subjected to certain actions. Each of these actions S is defined by stating that if the system is in the state n_ξ, then as a result of the action in question the system will pass to a new state n_η (which may of course coincide with the original state n_ξ if the given state is stable with respect to this action). ...". Clearly if Ω has n elements, then the family of all actions S is just T_n.

Example 2.

Let $M = (S, I, O, \delta, \lambda)$ be a finite automaton, where S is the finite non-empty set of states,

I is the set of inputs

O is the set of outputs

δ: $S \times I \rightarrow S$ is the next state function

λ: $S \times I \rightarrow O$ is the output function

Suppose there are n states. Then for any input symbol i, $\delta(.,i)$ defines a mapping from S into S, and the set of all such mappings is just T_n.

Application to Directed Graphs

Directed graphs with weights (labels, numbers) attached to the branches are of widespread occurence (see [5], [26]). Examples are given in Table 5.7.1. Provided such a directed weighted graph has the property that at least one branch leaves every node, we may condense this graph to a type 1A network by omitting all branches leaving each node except that with the highest weight.*

In other words, the condensed graph is a subgraph which is a type 1A network in which any node i is followed by that node to which it is closest, or which is most likely to follow it.

As an example, Figure 5.7.1 shows the condensation of the graph of a random walk with absorbing barriers ([16] Volume I, p. 310) with six states.

Application to Cryptanalysis

For additional examples, it is amusing to construct the condensed graphs corresponding to the last example in the above table. Thus, the nodes of the graph correspond to the letters of the alphabet, and node i is connected to node j if the letter i is most likely to be followed by the letter j. In Figures (5.7.2-3) this is done for 2 languages. The data is taken from [18], [54], [62], and [63]. For each language, two graphs

---

*Provided such a branch exists. If there are two branches with the same, highest, weight leaving a node, there are two possible condensations.

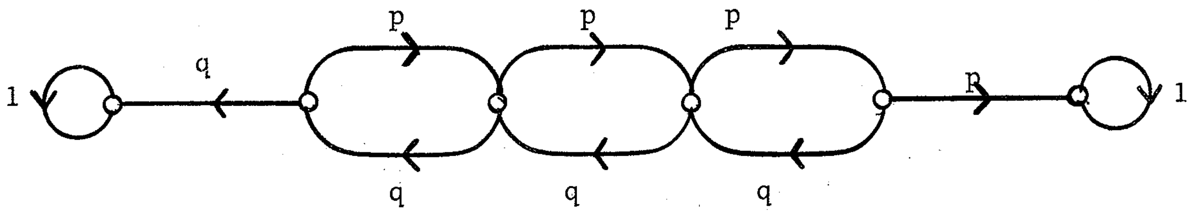
Table 5.7.1. Examples of Directed Graphs

Field of Application	Nodes Represent	Branch from Node i to Node j Represents	The Weight of this Branch Represents
Communication systems	Towns	Communication channels (telephone, road, railway, etc.)	Traffic (calls per hour, cars per hour, etc.)
Social structure	People	Political or social contact	Degree of political or social equality
" "	Football teams	Team i played team j	Equality of ability
" "	People	Friendship	Strength of friendship
Automata theory	States of an automata or Markov chain, etc.	Possibility of transition from state i to state j	Probability of this transition
Transmission of disease, spread of rumours, etc.	People	Contact	Measure of contact
Cryptanalysis	Letters of alphabet	Letter j has been observed to follow letter i	Frequency of this event

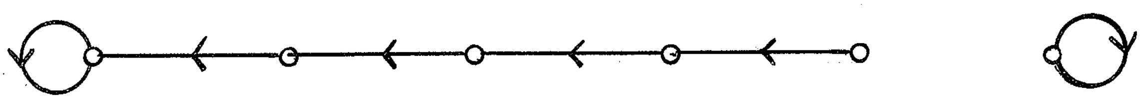
Fig. 5.7.1

Example of Condensing a Graph

$p + q = 1;\ p < q$



Original directed weighted graph



Condensed graph (Type 1A network)

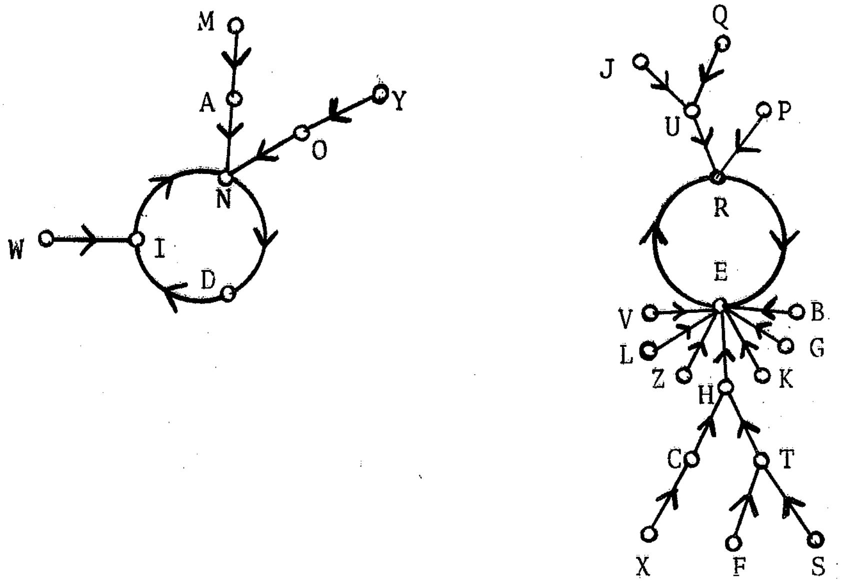
are obtained, depending on whether or not spaces between words are omitted. In the former case the data is taken from text in which the words have been run together,* while in the latter case the "space" counts as a 27^{th} letter.

An obvious application is that given a sufficiently long message coded by a simple substitution cipher, one could in this way determine the original language.

Thus, our thesis problem may be stated in terms of weighted directed graphs with at least one branch coming out of every node. For instance, in terms of Markov chains: given a Markov chain, and an arbitrary subset X of its states. Assuming that at each time instant a state goes to the most likely next state, what is the behavior of the "descendents" of the set X?

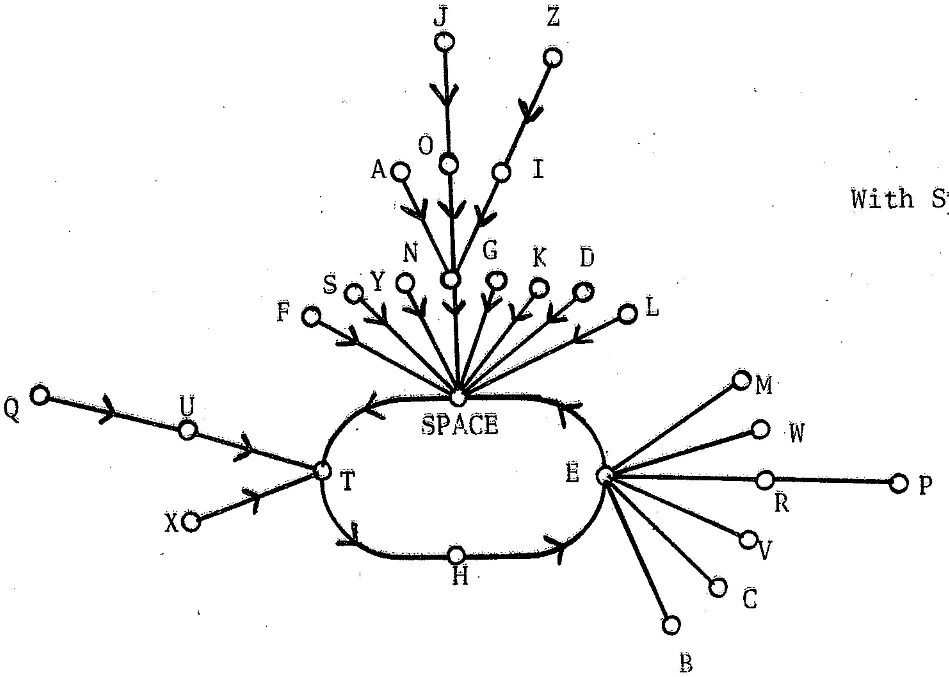
---

*Which seems to be common practice in cryptanalysis.



Without Spaces

Fig. 5.7.2 English



With Spaces

Without Spaces

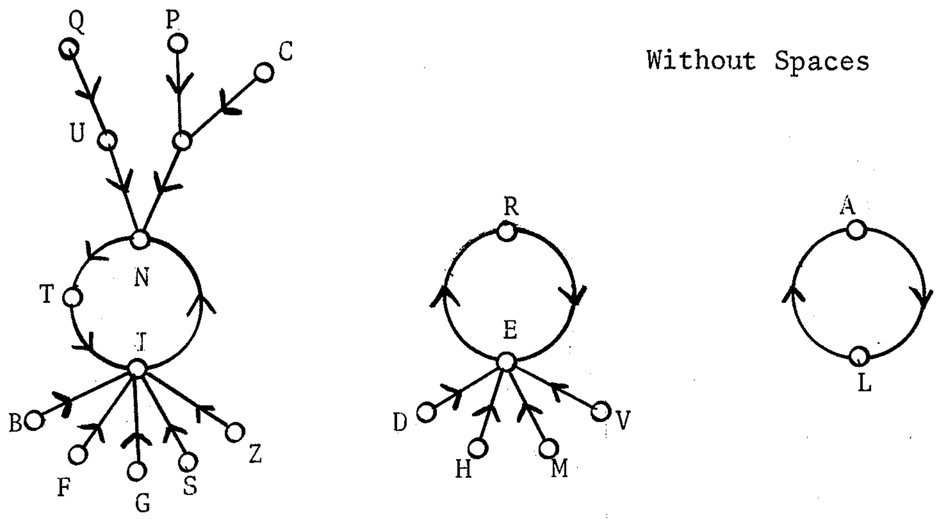
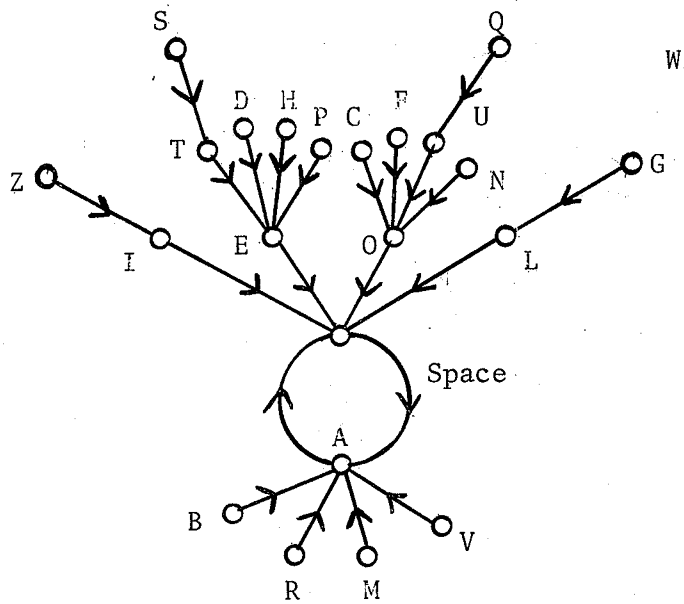


Fig. 5.7.3. Italian

(J, K, W, X, Y, Not Used)

With Spaces



CHAPTER 6

TYPE 4 NETWORKS

The family of type 4 networks with n nodes is by definition the family of all graphs of the n! permutations of n objects, and will be denoted by S_n. In 6.1 and 6.2 we collect known properties of S_n, including the work of Erdos and Turan ([15]), and 6.3 give an upper bound a_n to the average order of an element of S_n (= average value of LCM of cycle lengths of a type 4 network).

6.1 KNOWN PROPERTIES OF S_n

One of the n! permutations is chosen at random, each having the probability of $\frac{1}{n!}$ of being chosen. We study the probable structure of this permutation.

Let it have a_1 cycles of length 1, a_2 of length 2,..., and a_n of length n, so that

$$\sum_{r=1}^{n} ra_r = n;$$

then the permutation is said to be of type $(a_1, a_2, \ldots, a_n)$.

The probability that a permutation has this type is easily seen to be ([45], p. 67)

$$P(a_1, a_2, \ldots, a_n) = \frac{1}{1^{a_1} a_1! \, 2^{a_2} a_2! \ldots n^{a_n} a_n!}$$

Incidentally we verify by Lemma A3 that

$$\sum P(a_1, a_2, \ldots, a_n) = 1.$$

A generating function for P is

$$C_n(t_1, t_2, \ldots, t_n) = \sum n! \, P(a_1, \ldots, a_n) \, t_1^{a_1} t_2^{a_2} \ldots t_n^{a_n}$$

$$= \sum \frac{n!}{a_1!\, a_2! \ldots a_n!} \left(\frac{t_1}{1}\right)^{a_1} \left(\frac{t_2}{2}\right)^{a_2} \cdots \left(\frac{t_n}{n}\right)^{a_n}$$

which satisfies the functional equation ([45], p. 68)

$$\sum_{n=o}^{\infty} \frac{C_n(t_1, t_2, \ldots, t_n)\, u^n}{n!} = \exp(ut_1 + \frac{u^2 t_2}{2} + \frac{u^3 t_3}{3} + \ldots) \quad (6.1.1)$$

from which most of the following results have been deduced.

($C_n(t_1, t_2, \ldots, t_n)$ is the <u>cycle indicator</u> of the symmetric group and was used earlier in the proof of Theorem (5.3.2).)

The total number of cycles, (of any size) is

$$m = \sum_{r=1}^{n} a_r,$$

and for $n \to \infty$,

$$E(m) \sim \log n$$

$$\sigma^2(m) \sim \log n$$

(see [45] p. 72, [22], [23], [24], [16] Vol. I, p. 242, and [31]).

Also

$$P[m = k] = \frac{c(n,k)}{n!}$$

where $c(n,k)$ is defined in section 3.3, and

$$\lim_{n \to \infty} P[\alpha < \frac{m - \log n}{\sqrt{2 \log n}} < \beta] = \frac{1}{\sqrt{\Pi}} \int_{\alpha}^{\beta} e^{-t^2}\, dt$$

([22]).

Also, having selected a permutation, let a node x be chosen at random, and let ℓ be the length of the cycle containing x. Then it is shown in [31] that

$$P\left[\ell = j\right] = \frac{1}{n}.$$

The probability that there are k cycles of length r is ([23])

$$P[a_r = k] = \frac{1}{k!\, r^k} e^{-\frac{1}{r}}$$

The most likely type is ([23])

$$a_1 = a_{n-1} = 1$$

$$a_2 = a_3 = \ldots = a_{n-2} = a_n = 0,$$

that is, with one cycle of length one, and one cycle of length n - 1.

The number of permutations without cycles of length 1 is investigated in [45] p. 72, and [31].

The ordering of the objects within the cycles is investigated in [45] p. 74, and [59].

The number of odd and even permutations is considered in [45] p. 78.

The density function for the length of the longest cycle is obtained in [23], and this has been considerably generalized in [56].

6.2 KNOWN PROPERTIES OF THE ORDER OF AN ELEMENT OF S_n; RESULTS OF ERDOS AND TURAN

Let Π be a typical element of S_n, and let its cycles have lengths $\ell_1, \ell_2, \ldots, \ell_k$. Then the <u>order</u> of Π is defined to be

$$LCM[\ell_1, \ell_2, \ldots, \ell_k]$$

Landau ([38]) showed that if

$$G(n) = \max_{\Pi \in S_n} [\text{order } (\Pi)] \qquad (6.2.1)$$

then

$$\log G(n) \sim \sqrt{n \log n} \quad \text{as } n \to \infty.* \qquad (6.2.2)$$

---

*Therefore, $e^{\sqrt{n \log n}}$ is an asymptotic upper bound to the cycle time LC and to LF, for both type 1A and type 4 networks.

On the other hand, Π's of order as low as n are common; all Π's consisting of a single cycle are of order n, and there are

$$(n - 1)! = \frac{1}{n} n! \qquad (6.2.3)$$

of them, which is relatively large. In spite of the large difference between (6.2.2) and (6.2.3), Erdos and Turan ([45]) prove

(6.2.4) <u>Theorem</u>

For any positive ε, δ and $n > n_o(\varepsilon, \delta)$

$$e^{(\frac{1}{2} - \varepsilon) \log^2 n} \leq \text{order } (\Pi) \leq e^{(\frac{1}{2} + \varepsilon) \log^2 n}$$

holds, apart from at most

$$\delta \; n!$$

exceptional Π's.

What can be said about the expected value of the order of Π? Turan ([60]) has announced

$$E[\text{order } (\Pi)] < e^{\frac{C \sqrt{n}}{\log n}} \qquad (6.2.5)$$

but this has not yet been published. In the next section we prove the weaker result

$$E[\text{order } (\Pi)] < \frac{e^{2\sqrt{n}}}{2\sqrt{e\Pi}\; n^{3/4}} \left(1 + 0(\frac{1}{\sqrt{n}})\right) \qquad (6.2.6)$$

In the course of proving (6.2.4), Erdos and Turan derive the following result about S_n.

<u>Theorem</u> Apart from o(n!) Π's the remaining ones have these properties ($w_1(n)$, $w_2(n)$ and $w(n)$ are any functions which tend monotonically to ∞ with n, e.g., $\sqrt{\log \log n}$)

(6.2.7) Let $k(\Pi)$ be the number of different cycle lengths of Π, then

$$|k(\Pi) - \log n| \leq w(n)\sqrt{\log n}$$

(6.2.8) No two cycles of length $\geq w_1(n)$ in Π have the same length

(6.2.9) At most $w_2(n)$ cycles in Π can have the same length $\leq w_1(n)$

(6.2.10) The different cycle - lengths in Π are "equi-distributed" in the following sense: Define N by

$$N = [k(\Pi)^{1/3}].$$

For each $\Pi \varepsilon S_n$ we can determine uniquely the nonnegative integers

$$S'_1, S'_2, \ldots, S'_N$$

such that, if $n_1, n_2, \ldots, n_{k(\Pi)}$ are the different cycle lengths,

$$1 \leq n_1 < n_2 < \ldots < n_{s'_1} \leq n^{1/N} < n_{s'_1+1} < \ldots < n_{s'_1+s'_2} \leq n^{2/N} <$$

$$< n_{s'_1+s'_2+1} < \ldots < n_{s'_1+s'_2+\ldots+s'_N} = n_k \leq n;$$

if there is no n_j e.g., in $n^{1/N} < x \leq n^{2/N}$ we have $S'_2 = 0$, etc.

Of course $S'_\nu = S'_\nu(\Pi)$ and

$$S'_1 + S'_2 + \ldots + S'_N = k$$

Let Π_3 be the subset of S_n whose Π's satisfy the inequality

$$\max_{\mu = 1, \ldots, N} \left| S_\mu - \frac{k}{N} \right| \leq \left(\frac{k}{N}\right)^{4/5}$$

and let $|\Pi_3|$ be the number of its Π's. Then

$$\lim_{n \to \infty} \frac{1}{n!} |\Pi_3| = 1$$

(6.2.11) For any cycle length n_ν, the contribution to the LCM of the prime factors not exceeding $\log^6 n$ cannot exceed $\exp((\log \log n)^4)$.

6.3 UPPER BOUND TO THE AVERAGE ORDER OF AN ELEMENT OF S_n

(6.3.1) <u>Theorem</u>

The expected value of the order of an element Π of S_n is bounded above by†

---

†In (6.3.2), Π = 3.14159... . It will always be clear from the context which Π is meant.

$$a_n = \frac{e^{2\sqrt{n}}}{2\sqrt{e\Pi}\ n^{3/4}}\left[\ 1 + 0\left(\frac{1}{\sqrt{n}}\right)\right] \tag{6.3.2}$$

<u>Proof</u>

$$E[\text{order of } \Pi] = \sum_{[k_i]} \frac{1}{1^{k_1}\ k_1!\ \ldots\ n^{k_n}\ k_n!}\ .\ [\text{l.c.m. of all } i,\ 1 \leq i \leq n, \text{ such that } k_i > 0]$$

where the summation extends over all n-tuples of nonnegative integers $(k_1, k_2, \ldots, k_n)$ such that

$$k_1 + 2k_2 + \ldots + nk_n = n \tag{6.3.3}$$

Now certainly the lcm is less than or equal to the product of the cycle lengths, i.e., [l.c.m. of all i, $1 \leq i \leq n$, such that $k_i > 0$]

$$\leq 1^{k_1}\ 2^{k_2}\ \ldots\ n^{k_n}$$

Therefore

$$E[\text{order of } \Pi] \leq \sum_{[k_i]} \frac{1}{1^{k_1}\ k_1!\ \ldots\ n^{k_n} k_n!}\ .\ 1^{k_1}\ 2^{k_2}\ \ldots\ n^{k_n}$$

$$= \sum_{[k_i]} \frac{1}{k_1!\ k_2!\ \ldots\ k_n!} \tag{6.3.4}$$

$$= a_n \text{ (say)}$$

Now it is easy to see from (6.3.3) and (6.3.4) that a_n is the coefficient of t^n in

$$\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right)\left(1 + \frac{t^2}{1!} + \frac{t^4}{2!} + \frac{t^6}{2!} + \ldots\right)\left(1 + \frac{t^3}{1!} + \frac{t^6}{2!} + \frac{t^9}{3!} + \ldots\right)$$

$$\ldots\left(1 + \frac{t^n}{1!} + \frac{t^{2n}}{2!} + \frac{t^{3n}}{3!} + \ldots\right) \tag{6.3.5}$$

or, what is the same thing, the coefficient of t^n in

$$\prod_{i=1}^{\infty} \left(1 + \frac{t^i}{1!} + \frac{t^{2i}}{2!} + \frac{t^{3i}}{3!} + \ldots \right)$$

$$= \prod_{i=1}^{\infty} e^{t^i} = e^{\sum_{i=1}^{\infty} t^i}$$

$$= e^{\frac{t}{1-t}}$$

So it remains to find the power series expansion of

$$e^{\frac{t}{1-t}} = \frac{1}{e} \cdot e^{\frac{1}{1-t}} = \sum_{n=o}^{\infty} a_n t^n . \qquad (6.3.6)$$

From (6.3.5) it is easy to get $a_0 = a_1 = 1$

The expansion follows from a result of MacIntyre and Wilson,([40]):

Let
$$e^{\frac{\gamma}{1-z}} = \sum_{n=0}^{\infty} C_n z^n$$

If $|\arg \gamma| < \Pi$, then

$$C_n = \frac{\gamma^{1/4} e^{\gamma/2}}{2\sqrt{\Pi}\, n^{3/4}} e^{2\sqrt{\gamma n}} \left[1 + 0\left(\frac{1}{\sqrt{n}}\right)\right] \qquad (6.3.7)$$

Setting $\gamma = 1$, and substituting (6.3.7) into (6.3.6) gives (6.3.2).

CHAPTER 7

TYPE 5A NETWORKS

Type 5A networks are defined in section 1.5, and as the following theorems will show, have properties very similar to type 1A networks. We first prove various structure theorems and then show that most of the bounds for E[LBF] and E[LC] obtained in Part II for type 1A networks with n nodes apply also for type 5A networks with n nodes, for n large.

(7.1) Theorem (c.f. equation (4.1.1))

If the initial state contains exactly one active node, then

$$P[LC = j,\ LF = k] = \frac{(n-1)!}{(n-k)!\ (n-1)^k} \tag{7.2}$$

Proof. Similar to that for (4.1.1).

(7.3) Theorem (c.f. Theorem (4.4.16))

$$E[\text{number of loop nodes in a component} \mid \text{component has n nodes}] \sim \sqrt{\frac{2n}{\Pi}}$$

Proof. Let $C_{n,k}$ be the number of possible different components with n nodes and k loop nodes. Then as in 4.4 we obtain

$$C_{n,k} = 0 \text{ if } k = 1$$

$$C_{n,k} = \phi_n(k)\ n^{n-1} \quad \text{if } k > 1 \tag{7.4}$$

The theorem then follows from (7.4) in the same way that theorem (4.4.16) follows from (4.4.1). In particular we obtain

$$C_n \triangleq \sum_{k=2}^{n} C_{n,k} \sim \sqrt{\frac{\Pi}{2}}\ n^{n-\frac{1}{2}} \tag{7.5}$$

QED

(7.6) Theorem (c.f. equation (3.2.9))

$P[m_1$ components of size 1, ..., m_n components of size n]

$$= \frac{n!}{(n-1)^n} \prod_{i=1}^{n} \frac{C_i^{m_i}}{i!^{m_i}\, m_i!}$$

where C_n is given by (7.5).

Proof. See [31].

(7.7) Theorem (c.f. equation (3.2.14))

P[a_2 loops of length 2, ..., a_m loops of length m]

$$= \frac{n!\ w\ n^{n-w-1}}{(n-w)!\ (n-1)^n} \quad \frac{1}{\prod_{i=2}^{m} a_i!\ i^{a_i}}$$

$$\sim \frac{n!\ w}{en^{w+1}(n-w)!} \cdot \frac{1}{\prod_{i=2}^{m} a_i!\ i^{a_i}}$$

where $w = \sum_{i=2}^{m} ia_i$

Proof. Similar to that for (3.2.14).

(7.8) Theorem (c.f. equations (3.3.5) and (3.3.12)

Let E_n be the expected number of components of a type 5A network with n nodes. Then

$$E_n = \sum_{s=2}^{n} \frac{n!}{(n-s)!\ s\ (n-1)^s} \quad , n \geq 1;$$

and, for $n \geq 2$,

$$\frac{1}{2} \log n + A \leq E_n \leq 2 \log n + B$$

where $A = \frac{1}{2} (\log 2 + \gamma) - 2$

$B = 2 \log 2 + 2\gamma + 4e^{-1} - 1$

γ = Euler's constant

Proof.

We use Kruskal's method [37]. Let G be a type 5A network with n nodes labelled 1, 2, ..., n. Let a function f be defined in terms of G as follows: if node i is connected to node j in G, define j = f(i). Since each node is connected to exactly one other node, f is well-defined. We may determine the components of G systematically in the following way. Starting with an arbitrary node X_1, let $X_2 = f(X_1)$. Then by definition of a type 5A network, $X_2 \neq X_1$. If $f(X_2) \neq X_1$ we choose $X_3 = f(X_2)$; if $f(X_3) \neq X_1$, and $f(X_3) \neq X_2$ we choose $X_4 = f(X_3)$, and so on until we come to the first node X_j whose image $f(X_j)$ is one of the j nodes already defined. We then call these nodes the first partial component, and start over, beginning with an arbitrary node X_{j+1} different from the j distinct nodes chosen so far. The sequence starting with X_{j+1} is continued until we reach the first node whose image is already either in the sequence or in the first partial component. If this image is in the sequence then we form the second partial component out of the sequence elements and start over. If it is in the first partial component, however, then we enlarge the first partial component by the addition of the sequence elements, and again start over. In either case, another arbitrary starting node generates a sequence which grows either until it runs into itself, thereby forming a new partial component, or else until it runs into an already existant partial component, in which case it is added on to that partial component. This process is continued until the nodes are exhausted. The components of G are then just the partial components at the end.

Let $p_m(s)$ be the probability that the current growing sequence has exactly s elements just after the m^{th} node X_m has been chosen. It is easy to see that the initial conditions are

$$p_1(1) = 1, \; p_1(s) = 0, \; 2 \leq s \leq n$$

$$p_2(2) = 1, \; p_2(s) = 0, \; s \neq 2$$

and

$$p_m(1) = \frac{m-2}{n-1}, \quad 3 \leq m \leq n;$$

and that the recurrence relation is

$$p_m(s) = \frac{n-m+1}{n-1} \quad p_{m-1}(s-1), \; 2 \le m \le n, \; 2 \le s \le n \qquad (7.9)$$

Clearly also for $2 \le m \le n$

$$p_{m-1}(n) = 0$$

Now let us define, for $1 \le i \le n$, the random variables

$X_i = 1$ if a new partial component is formed during the step of going to the image of X_i

$= 0$ if not

Also let

$$P_i \triangleq P[X_i = 1]$$

so that

$$E_n \triangleq \text{expected number of components}$$

$$= E[\sum_{i=1}^{n} X_i]$$

$$= \sum_{i=1}^{n} EX_i = \sum_{i=1}^{n} P_i \qquad (7.10)$$

Clearly

$$P_m = 0, \; m = 1$$

and

$$P_m = \sum_{s=2}^{n} \frac{s-1}{n-1} \, p_m(s) \qquad (7.11)$$

and

$$\sum_{s=1}^{n} p_m(s) = 1, \quad 1 \le m \le n$$

From (7.9) and (7.11) for $2 \le m \le n$

$$P_m = \sum_{s=2}^{n} \frac{s-1}{n-1} \frac{n-m+1}{n-1} p_{m-1}(s-1)$$

$$= \frac{n-m+1}{n-1} \left[\sum_{r=1}^{n} \frac{r-1}{n-1} p_{m-1}(r) + \sum_{r=1}^{n} \frac{1}{n-1} p_{m-1}(r) \right]$$

$$= \frac{n-m+1}{n-1} \left[P_{m-1} + \frac{1}{n-1} \right] \tag{7.12}$$

We will next solve this recurrence relation by means of generating functions.

$$\text{Let } g(y) = \sum_{m=1}^{n} P_m y^m \tag{7.13}$$

To agree with (7.12) define $P_{n+1} = 0$. Then multiplying (7.12) by y^m and summing over m from 2 to n + 1 gives after some simplification,

$$g'(y) + \frac{n-1-ny}{y^2} g(y) = \frac{n-1-ny+y^n}{(n-1)(1-y)^2}$$

This has integrating factor

$$\mu = e^{-\frac{n-1}{y}} y^{-n}$$

and the solution is

$$g(y) = e^{\frac{n-1}{y}} y^n \int_0^y e^{-\frac{n-1}{u}} u^{-n} \frac{n-1-nu+u^n}{(n-1)(1-u)^2} du,$$

the constant of integration being determined by the condition that g(y) be regular at y = 0. Then the expected number of components

$$E_n = g(1) \quad \text{from (7.10), (7.13)}$$

$$= \int_0^1 e^{(n-1)(1-\frac{1}{u})} u^{-n} \frac{n-1-nu+u^n}{(n-1)(1-u)^2} du$$

Let $-z = (n - 1)(1 - \frac{1}{u})$, then this becomes

$$E_n = \int_0^\infty dz \frac{e^{-z}}{z^2} \left[(z - 1) \left(1 + \frac{z}{n - 1}\right)^{n - 1} + 1 \right]$$

Expanding $\left(1 + \frac{z}{n - 1}\right)^{n - 1}$ by the binomial theorem,

and using

$$\int_0^\infty z^m e^{-z} dz = m!$$

we obtain

$$E_n = \sum_{s = 2}^{n} \frac{n!}{(n - s)!\ s\ (n - 1)^s} \tag{7.14}$$

Now the corresponding formula for type 1A networks is given by (3.3.5) above, and is

$$E_n^* = \sum_{s = 1}^{n} \frac{n!}{(n - s)!\ s\ n^s} \tag{7.15}$$

Also in 3.3 it is shown that

$$0 \leq E_n^* - \frac{1}{2}(\log 2n + \gamma) \leq e^{-1} \tag{7.16}$$

and we will use this to obtain bounds on E_n.

From (7.14), (7.15) we have

$$E_n^* \frac{n}{n - 1} - \frac{n}{n - 1} \leq E_n \leq E_n^* \left(\frac{n}{n - 1}\right)^n - \frac{n}{n - 1} \tag{7.17}$$

Now ([28]) $(1 - \frac{1}{n})^{-n}$ decreases for $n > 1$, as $n \to \infty$, so for $n \geq 2$, from (7.16), (7.17),

$$\frac{1}{2}(\log 2n + \gamma) - 2 \leq E_n \leq 2(\log 2n + \gamma) + 4e^{-1} - 1$$

QED

The following theorems now follow from the above results in exactly the same way that the corresponding theorems for type 1A networks were proved in Part II.

(7.18) Theorem (c.f. Theorem (4.1.9))

If the initial state contains exactly one active node,

$$E[LBF] = \sqrt{\frac{\Pi n}{8}} + 0(1) \tag{7.19}$$

$$\sigma^2[LBF] = n\left(\frac{2}{3} - \frac{\Pi}{8}\right) + 0(\sqrt{n}) \tag{7.20}$$

and for arbitrary initial activity (7.19) is a lower bound on E[LBF]

(7.21) Theorem (c.f. equation (4.2.18))

For arbitrary initial activity,

$$E[LBF] \leq u_{n-1, 1}$$

where $u_{m, 1}$ is defined in (4.2.6).

(7.22) Theorem (c.f. Theorem (4.6.44))

There exist constants α_8, α_{16} such that if the initial state is formed by choosing $\sqrt{n}(\log n)^{3/4}$ active nodes (with replacement), then

$$E[LC] > \exp\left[\frac{1}{2}(\log n)^{9/4} - \alpha_{16}(\log n)^{5/4} \log \log n\right]$$

for $n \geq \alpha_8$. (This depends on the approximations (4.5.9), (4.6.18), (4.6.33) and (4.6.48).)

(7.23) Theorem (c.f. Theorem (4.7.12))

There exists a constant α_{17} such that

$$E[\text{LCR of loop lengths}] > \exp\left[0.80\, n^{1/5} - \alpha_{17} \log n\right]$$

for $n \geq 32$. (This depends on approximations (4.5.9) and (4.6.48).)

(7.24) Theorem (c.f. Theorem (4.8.9))

For $n \geq 100$ and any A such that

$$\frac{n}{2 \log n} > A > 1.6,$$

the expected value of the cycle time of all networks with n nodes and fewer than A log n components is less than

$$\exp\left[\frac{A}{2}\log^2 n + A \log c_1 \log n\right]$$

where $c_1 \doteq 1.33$ is given by (4.8.2).

(7.25) Theorem (c.f. Theorem (4.8.13))

There exists a constant A such that for $n \geq 1$,

$$E[LC] < e^{2\sqrt{2n} - \frac{1}{2}\log n + \log A} = \frac{A\, e^{2\sqrt{2n}}}{\sqrt{n}}$$

CHAPTER 8

"BIRTHDAY MODEL" FOR FINDING LC AND LF*

The name is derived from the well-known problem (see [16] Vol. I, p. 31): given p people in a room, what is the probability that two of them will have the same birthday?

In the simplest birthday model for networks with n nodes the first state is given, the second and later states are chosen at random, independently and with replacement, from the 2^n possible states.

Let $p = 2^n$. Then for $0 \le i \le p-1$, $1 \le j \le p$, $1 \le i + j \le p$,†

$$P[LBF = i, LC = j, LF = i + j] = \frac{(p - 1)(p - 2) \ldots (p - (i + j - 1))}{p^{i + j}}$$

$$= \frac{p!}{p^{i + j + 1}(p - (i+j))!},$$

$$P[LF = k] = \sum_{i = 0}^{k - 1} P[LBF = i, LC = k - i, LF = k]$$

$$= \frac{kp!}{p^{k+1}(p-k)!}$$

$$= \frac{k}{p} \exp\left[- \frac{k(k-1)}{2p} + 0\left(\frac{k^3}{p^2}\right) \right]$$

for $k = o(p^{2/3}) = o(2^{2n/3})$ by Lemma (A2)

---

*This idea is due to Mr. Mike Stitelman.

†Because this is the probability that the first i + j states chosen are all different, and that the next state chosen is the $(i + 1)^{th}$.

$$E[LF] = \sum_{k=1}^{p} \frac{k^2 \, p!}{p^{k+1}(p-k)!} = \frac{p!}{p^{p+1}} \sum_{r=0}^{p-1} (p-r)^2 \frac{p^r}{r!}$$

$$\sim \sqrt{\frac{\Pi p}{2}} = \sqrt{\frac{\Pi}{2}} \, 2^{n/2} \qquad \text{by Lemma A7} \tag{8.1}$$

$$\sigma^2[LF] \sim \sum_{k=1}^{n} \frac{k^3 \, p!}{(p-k)! \, p^{k+1}} - \frac{\Pi}{2} \, 2^n$$

$$\sim 2^n\left(2 - \frac{\Pi}{2}\right) \qquad \text{by Lemma A8}$$

The main trouble here of course is the assumption that the states are chosen out of the 2^n possible states each time. As a result E[LF] is far too high, in fact we show in section 6.2 that for type 1A and type 4 networks

$$\max LF < Ae^{\sqrt{n \log n}}$$

which is smaller than (8.1) for large n.

APPENDIX 1. LEMMAS FREQUENTLY USED

Lemma A1

The number of m-trees with p labelled points is

$$\binom{p}{m} m\, p^{p-m-1}$$

The number of m-trees with p labelled points, for which the m root nodes are specified in advance, is

$$m\, p^{p-m-1}$$

Proof: see [32], Appendix

Lemma A2

Let

$$\Phi_n(p) = n!\big/[(n-p)!\, n^p]$$

For $p = o(n^{2/3})$, $\Phi_n(p) = \exp[-p(p-1)/2n + O(p^3/n^2)]$

For $p = o(n^{3/4})$, $\Phi_n(p) = \exp[-p(p-1)/2n - p(p-1)(2p-1)/6n^2 + O(p^4/n^3)]$

The proofs are elementary. These results are stated, without proof, in [14].

Lemma A3. Cauchy's Identity

$$\sum \frac{1}{1^{a_1}\, a_1!\; 2^{a_2}\, a_2! \ldots n^{a_n}\, a_n!} = 1$$

the summation extending over all non-negative n-tuples $(a_1, a_2, \ldots, a_n)$ for which

$$\sum_{i=1}^{n} i a_i = n$$

Proof: see [45] page 69.

Lemma A4

$$\sum_{s=o}^{n-1} \frac{n^s}{s!} = \psi = e^n \left[\frac{1}{2} - \frac{1}{\sqrt{2\pi n}} \left(\frac{1}{3} + \frac{1}{540n} - \frac{25}{6048n^2} + O(\frac{1}{n^3}) \right) \right]$$

$$= e^n \left| 0.5 - \frac{0.1328}{\sqrt{n}} - \frac{0.00738}{n} \ldots \right|$$

Proof. Follows from

$$\sum_{s=o}^{n-1} \frac{n^s}{s!} = \frac{e^n}{2} - \theta_n \frac{n^n}{n!}$$

where

$$\theta_n \sim \frac{1}{3} + \frac{4}{135n} - \frac{8}{2835n^2} + \ldots$$

as $n \to \infty$ (see [11], p. 230, ex. 18),

and

$$n! \sim \sqrt{2\pi}\ n^{n+1/2}\ e^{-n} \left[1 + \frac{1}{12n} + \frac{1}{288n^2} + \ldots \right]$$

Lemma A5. For $n \geq 1$,

$$\frac{e^n}{2} \left(1 - \frac{1}{\sqrt{2\pi n}} \right) < \sum_{s=o}^{n-1} \frac{n^s}{s!} < \frac{e^n}{2} \left(1 - \frac{11}{18\sqrt{2\pi n}} \right) < \frac{e^n}{2}$$

Proof. It is known from [11] that θ_n above satifies

$$\frac{1}{3} \leq \theta_n \leq \frac{1}{2}$$

and the result follows from lemma A4.

Lemma A6

$$\sum_{r=o}^{n-1} (n - r) \frac{n^r}{r!} = \frac{n^n}{(n-1)!} \sim \sqrt{\frac{n}{2\pi}}\ e^n$$

Lemma A7

$$\sum_{r=o}^{n-1} (n - r)^2 \frac{n^r}{r!} = n \psi$$

Proof. Expand $(n - r)^2$ as

$$(n - r)^2 = r(r - 1) + r(1 - 2n) + n^2.$$

Lemma A8

$$\sum_{r=o}^{n-1} (n - r)^3 \frac{n^r}{r!} = - n\psi' + \frac{2n^{n+1}}{(n-1)!}$$

$$= e^n \left[\sqrt{\frac{2}{\pi}} n^{3/2} - \frac{n}{2} + \sqrt{\frac{2n}{\pi}} \cdot \frac{1}{12} + O(1) \right]$$

Proof. Use

$$(n - r)^3 = -(r)_3 + 3(n - 1)(r)_2 - (3n^2 - 3n+1)r + n^3$$

where

$$(r)_m = r(r - 1)(r - 2)\ldots(r - m+1)$$

and lemmas A4, A6, A7.

Lemma A9 Stirling's Approximation For $x \geq 1$

$$\sqrt{2\pi}\, x^{x-1/2} e^{-x} < \Gamma(x) < \sqrt{2\pi}\, x^{x-1/2} e^{-x + \frac{1}{12x}}$$

Proof. See [16], Vol. 1, p. 52.

Lemma A10 (Formulae of Abel, Jensen, Hölder and Cauchy)

$$\sum_{\nu=o}^{n} \binom{n}{\nu} (x + \nu a)^{\nu-1} (y - \nu a)^{n-\nu} = \frac{(x + y)^n}{x} \qquad \text{(A10.1)}$$

$$\sum_{\nu=o}^{n} \binom{n}{\nu} (x + \nu a)^{\nu} \quad (y - \nu a)^{n-\nu-1} = \frac{(x + y)^{n}}{y - na} \qquad \text{(A10.2)}$$

$$\sum_{\nu=o}^{n} \binom{n}{\nu} (x + \nu a)^{\nu-1} \quad (y - \nu a)^{n-\nu-1} = \frac{(x+y - na) \; (x + y)^{n-1}}{x(y - na)} \qquad \text{(A10.3)}$$

$$\sum_{\nu=o}^{n} \binom{n}{\nu} (x + \nu a)^{\nu} \quad (y - \nu a)^{n-\nu} = n! \sum_{\nu=o}^{n} \frac{(x + y)^{\nu} \; a^{n-\nu}}{\nu!} \qquad \text{(A10.4)}$$

Proof. See Salié [55], and Abel [1].

APPENDIX 2. DEFINITIONS AND RESULTS FROM GRAPH THEORY

A directed graph $G(N,E,I)$ consists of a set N whose elements are called nodes or vertices; a set E whose elements are called edges or branches; and a mapping

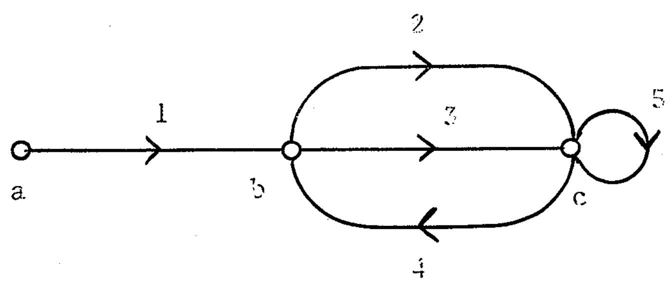
$$I: E \to N \times N$$

which associates with each edge $e \in E$ an ordered pair of not necessarily distinct nodes (a,b) called the endpoints of e, thus

$$I(e) = (a,b) \tag{1}$$

The edge e is then said to be directed from its originating node a to its terminating node b; and nodes a, b are said to be incident with e. For example, see Fig. (A.2.1).

FIG. A. 2. 1 EXAMPLE OF A DIRECTED GRAPH

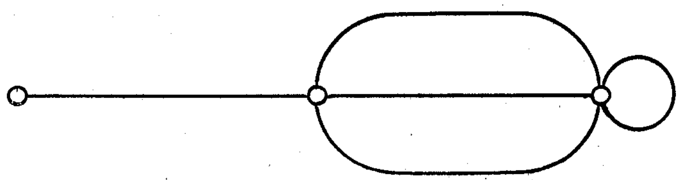


Graph $G(N,E,I)$ with $N = [a,b,c]$, $E = [1,2,3,4,5]$, $I(1) = (a,b)$, $I(2) = I(3) = (b,c)$, $I(4) = (c,b)$, $I(5) = (c,c)$.

The undirected graph $G'(N, E, I')$ corresponding to the directed graph $G(N, E, I)$ is obtained by removing the "arrows" from $G(N, E, I)$; or formally by replacing the phrase "ordered pair" by "unordered pair" in the above definition.

Example. The unordered graph corresponding to Fig. (A.2.1) is shown in Fig. (A.2.2).

FIG. A. 2 2 EXAMPLE OF AN UNDIRECTED GRAPH



In the remainder of this appendix we shall only be concerned with undirected graphs (which we usually abbreviate to graph).

The degree of a node is the number of edges having the node as an endpoint.

A path in a graph $G(N, E, I)$ is a sequence

$$a_1, x_1; a_2, x_2; \ldots; x_{n-1}, a_n \quad (2)$$

with $a_i \varepsilon N$, $x_i \varepsilon E$ for all i, satisfying

(i) $a_1, a_2, \ldots, a_{n-1}$ are distinct, and

(ii) $I(x_i) = (a_i, a_{i+1})$ for $1 \leq i \leq n-1$.

It will sometimes be convenient to represent the path (2) just by the sequence of edges

$$x_1, x_2, \ldots, x_{n-1}$$

A connected graph has the property that there exists a path $a_1, \ldots, a_n$ between any pair of nodes.

A loop is a path (2) whose first and last nodes coincide.

A subgraph H(N', E', I) of a graph consists of a subset E' of the edges E together with a subset N' of the nodes N, such that the endpoints of E' are contained in N'.

A connected component is a maximally connected subgraph.

If G has n nodes, e edges and p connected components, then the rank of G = n - p, and the nullity of G = e - n + p.

A graph G is finite if the sets N and E are finite.

A weighted graph is a graph G (N, E, I) together with a function W on E (which associates a weight, for example an electric current, with each edge).

An undirected graph is complete if there is one edge between every pair of nodes.

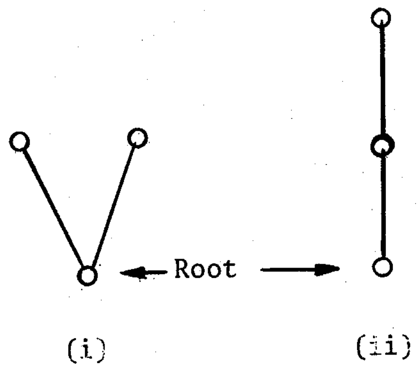
A tree is a connected subgraph containing no loops.

An m-tree is a subgraph with m connected components and containing no loops. (Thus a 1-tree is the same as a tree). A rooted tree is a tree in which one particular node called the root is distinguished. A rooted m-tree is an m-tree in which m particular nodes called the roots, one in component, are distinguished. Theorem (A.2.1) (See [45] p. 128.)
In a complete graph with n nodes 1, 2,..., n , the number of trees with n nodes is n^{n-2}; the number of rooted trees with n nodes and with node 1 as the root is n^{n-2}, and the number of rooted trees with n nodes and an unspecified root node is n^{n-1}.

Remark. Labelled versus Unlabelled Graphs

Since our networks are made up of physical neurons we are concerned throughout this work with graphs with labelled (as opposed to unlabelled) nodes. The difference is best illustrated by an example. Consider rooted trees with 3 nodes. There are two such unlabelled trees, shown in Fig. (A.2.3).

FIG. A. 2.3 UNLABELLED ROOTED TREES WITH 3 NODES



but 9 labelled trees, since (i) can be labelled in 3 ways and (ii) in 6 ways. The average height of the unlabelled trees is

$$\frac{1}{2}(1.3 + 1.2) = 3/2$$

but for the labelled trees it is

$$(1.3 + 6.2)/9 = 5/3$$

For further information, see for example [42] or [45] Ch. 6.

APPENDIX III. LIST OF SYMBOLS

SYMBOL	MEANING	PAGE
C_n	Usually the number of Type 1A Networks with n nodes and 1 component	62
c(n,m)	Signless Stirling number of 1^{st} kind	45
dist(x)		55
$E_1, E_2, \ldots$	Expectation over $\Omega_1, \Omega_2, \ldots$	45
graph		A-5
height(x)		56
k	Usually the number of components in a Type 1A network	45
LBF	Transient time	20
LC	Cycle time	20
LCM	Least common multiple	39
LF	= LC + LBF	20
l_w	Probability that a Type 1A network has w loop nodes	50
m-tree		A-7
n	Usually the number of nodes in a Type 1A network	29
T_n	Family of all Type 1A networks with n nodes	29
tree		A-7
$U_{n,m}$		56
$V_{n,m}$		56
$W_{n,m}$		56
$Y_{n,m}$		56
γ	Euler's constant	46

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