ITERATED EXPONENTIALS

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1. Introduction. In recent papers E. T. Bell[1] and G. T. Williams[2] discussed some of the algebraic and number-theoretic properties of the coefficients of the expansion of e^{e^x-1} as a power series. The introduction to the second of the two papers contains a summary of the surprisingly meager results obtained by earlier writers.

In this note some of the results obtained in these papers are extended and the field of study is enlarged to include the coefficients of functions like

$$e^{e^{e^x}}, \quad e^{e^{\cdot^{\cdot^{e^x}}}}$$

which have been treated by E. T. Bell[3] and

$$\log[1+\log(1+x)],\ \log\{1+\log[1+\log(1+x)]\}, \ldots$$

etc., which do not seem to have been considered before. The results are believed to be new.

2. Notation and Fundamental Relationships. We shall denote by $e_n(x)$ the nth iterate of e^x, that is,

$$e_1(x) = e^x, \quad e_2(x) = e^{e^x}, \quad e_3(x) = e^{e^{e^x}}, \text{ etc.} \tag{1}$$

We shall have then

$$e_1(0) = 1, \quad e_2(0) = e, \quad e_3(0) = e^e, \quad e_4(0) = e^{e^e}, \text{ etc.}$$

Since every coefficient in the Taylor expansion of $e_n(x)$ contains $e_n(0)$ as a factor we may write

$$e_n(x) = e_n(0) + e_n(0)\sum_{k=1}^{\infty} A_{n,k}\frac{x^k}{k!} = e_n(0)\cdot E_n(x)$$

[1] E. T. Bell, "Exponential Polynomials," *Annals of Mathematics*, v. 35, 1934, p. 258-27? This article will be referred to as Bell I.

[2] G. T. Williams, "Numbers Generated by the Function e^{e^x-1}," *Am. M. Monthly*, v. 5? 1945, p. 323-327.

[3] E. T. Bell, "The Iterated Exponential Integers," *Annals of Matehmatics*, v. 39, 193? p. 539-557. This article will be referred to as Bell II.

where $E_n(x) = e_n(x) : e_0(x) = \sum_{k=1}^{\infty} A_{n,k} \frac{x^k}{k!}$. Thus

$$\left.\begin{aligned} E_1(x) &= e^x : e^0 = e^x \\ E_2(x) &= e_2(x) : e_2(0) = e^{e^x} : e = e^{e^x - 1} \\ E_3(x) &= e_3(x) : e_3(0) = e^{e^{e^x}} : e^e = e^{e^{e^x}-1} - 1, \text{ etc.} \end{aligned}\right\} \quad (2)$$

The following relationship can be easily verified:

$$E_{n+1}(x) = e^{E_n(x)-1} = E_n(e^x - 1) \qquad (3)$$

$$E_{n-1}(x) = 1 + \log E_n(x). \qquad (4)$$

Formula (4) allows us to interpret the meaning of $E_n(x)$ for $n = 0$, and for negative values of n. Starting with $E_2(x) = e^{e^x - 1}$, we have

$$E_1(x) = 1 + \log E_2(x) = 1 + (e^x - 1) = e^x$$

$$\left.\begin{aligned} E_0(x) &= 1 + \log E_1(x) = 1 + x \\ E_{-1}(x) &= 1 + \log E_0(x) = 1 + \log(1 + x) \\ E_{-2}(x) &= 1 + \log E_{-1}(x) = 1 + \log[1 + \log(1 + x)] \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ E_{-n}(x) &= 1 + \log E_{1-n}(x) = 1 + \log[1 + \ldots + \log(1 + x)] \end{aligned}\right\} \quad (5)$$

the operation of taking the logarithm being repeated n times.

3. Stirling Numbers. Most useful in the study of iterated exponential functions are the so-called Stirling Numbers of the first and second kind. A Stirling Number, n_k, of the first kind is the sum of the products of the numbers 1, 2, 3 ... n, taken as factors k at a time without repetition. Thus

$$3_2 = 1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3 = 2 + 3 + 6 = 11$$

$$4_2 = 1 \cdot 2 + 1 \cdot 3 + 2 \cdot 4 + 2 \cdot 3 + 2 \cdot 4 + 3 \cdot 4 = 35.$$

A Stirling Number, ${}_kn$, of the second kind is the sum of the products of the numbers 1, 2 ... n, taken k at a time, repetitions being allowed. Thus

$${}_22 = 1^2 + 1 \cdot 2 + 2^2 = 7, \quad {}_23 = 1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3 + 1^2 + 2^2 + 3^2 = 25$$

$${}_24 = 1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 2 \cdot 3 + 2 \cdot 4 + 3 \cdot 4 + 1^2 + 2^2 + 3^2 + 4^2 = 65.$$

For the sake of economy, and perhaps greater clarity, the accepted symbols A_k^n for Stirling Numbers of the first kind and B_k^n for the second kind are replaced in this paper by n_k and ${}_kn$, respectively.

Particularly important in this discussion are the Stirling Numbers of the second kind, given by Stirling as the coefficients of the formulas expressing x^n in terms of factorials:

$$\left.\begin{aligned}
x &= x \\
x^2 &= x(x-1) + x = x^{(2)} + x \\
x^3 &= x(x-1)(x-2) + 3x(x-1) + x = x^{(3)} + 3x^{(2)} + x^{(1)} \\
x^4 &= x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + \\
&\qquad 7x(x-1) + x = x^{(4)} + 6x^{(3)} + 7x^{(2)} + x^{(1)}.
\end{aligned}\right\} \quad (6)$$

In general $x^n = x^{(n)} + {}_1(n-1)x^{(n-1)} + \ldots {}_{n-3}3x^{(3)} + {}_{n-2}2x^{(2)} + x^{(1)}$, where $x^{(n)} = x(x-1)(x-2)\ldots(x-n+1)$.

Frequent use will be made of the following theorems and formulas:

$$(a)\quad \frac{\log^n(1+x)}{n!} = \frac{x^n}{n!} - n_1\frac{x^{n+1}}{n+1!} + (n+1)_2\frac{x^{n-2}}{n+2!} - (n+2)_3\frac{x^{n+3}}{n+3!} + \ldots \quad (7)$$

$$(b)\quad \frac{(e^x-1)^n}{n!} = \frac{x^n}{n!} + {}_1n\frac{x^{n+1}}{n+1!} + {}_2n\frac{x^{n+2}}{(n+2)!} + {}_3n\frac{x^{n+3}}{n+3!} + \ldots \quad (8)$$

(c) If $N = A_1x + A_2\frac{x^2}{2!} + A_3\frac{x^3}{3!} + \ldots$, $e^N = 1 + \sum_1^\infty B_k\frac{x^k}{k!}$ where

$$B_1 = A_1;\quad B_2 = \begin{vmatrix} A_1 & -1 \\ A_2 & A_1 \end{vmatrix};\quad B_3 = \begin{vmatrix} A_1 & -1 & 0 \\ A_2 & A_1 & -1 \\ A_3 & 2A_2 & A_1 \end{vmatrix},$$ in general

$$B_k = \begin{vmatrix} A_1 & -1 & 0 & . & 0 \\ A_2 & A_1 & -1 & . & 0 \\ A_3 & 2A_2 & A_1 & . & 0 \\ A_4 & 3A_3 & 3A_2 & . & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{k-1} & \binom{k-2}{1}A_{k-1} & \binom{k-2}{2}A_{k-3} & . & -1 \\ A_k & \binom{k-1}{1}A_{k-1} & \binom{k-1}{2}A_{k-2} & . & A_1 \end{vmatrix} \quad (9)$$

Proof. Differentiating $y = e^N$, we get

$$\frac{dy}{dx} = y\left(A_1 + A_2x + A_3\frac{x^2}{2!} + \ldots\right) = B_1 + B_2x + B_3\frac{x^2}{2!} + \ldots$$

or

$$\left(1 + B_1x + B_2\frac{x^2}{2} + \ldots\right)\left(A_1 + A_2x + A_3\frac{x^2}{2} + \ldots\right) =$$

$$B_1 + B_2x + B_3\frac{x^2}{2!} + \ldots \quad (9a)$$

which may be written symbolically

$$e^{Bx}\cdot Ae^{Ax} = Be^{Bx}, \text{ or } Ae^{(B+A)x} = Be^{Bx},$$

leading to the recurrent formula[4]

$$B_{n+1} = A(B + A)^n. \quad (10)$$

Expanding (10) for $n = 0, 1, 2, 3 \ldots$ and replacing the exponents by subscripts, we obtain the equations

$$\left.\begin{aligned} B_1 &= A_1 \\ B_2 &= B_1A_1 + A_2 \\ B_3 &= B_2A_1 + 2BA_2 + A_3, \text{ etc.} \end{aligned}\right\} \quad (10a)$$

Solving the first n equations for the B's, we obtain (9).

(*d*) $\log\left(1 + B_1x + B_2\frac{x^2}{2!} + B_3\frac{x^3}{3!} + \ldots\right) = \sum_1^\infty A_k\frac{x^k}{k!}$, where

$$A_k = (-1)^{k+1}\begin{vmatrix} B_1 & 1 & 0 & \cdot\ \cdot & 0 \\ B_2 & B_1 & 1 & \cdot\ \cdot & 0 \\ B_3 & B_2 & 2B_1 & \cdot\ \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ B_{k-1} & B_{k-2} & \binom{k-2}{1}B_{k-3} & \cdot & \binom{k-2}{1}B_1 & 1 \\ B_k & B_{k-1} & \binom{k-1}{1}B_{k-2} & \cdot & \binom{k-1}{2}B_2 & \binom{k-1}{1}B_1 \end{vmatrix} \quad (11)$$

Proof. Differentiating we get (9*a*), (10) and the series of equations (10*a*), the solution of which for the A's leads to (11).

Theorems *a* and *b* are known. The same may probably be said about *c* and *d*.

[4] Cf. Bell I, equation 4, 1.

Stirling Numbers of the First Kind

$2_0 = 1$
$2_1 = 3$
$2_2 = 2$

$3_0 = 1$
$3_1 = 6$
$3_2 = 11$
$3_3 = 6$

$4_0 = 1$
$4_1 = 10$
$4_2 = 35$
$4_3 = 50$
$4_4 = 24$

$5_0 = 1$
$5_1 = 15$
$5_2 = 85$
$5_3 = 225$
$5_4 = 274$
$5_5 = 120$

$6_0 = 1$
$6_1 = 21$
$6_2 = 175$
$6_3 = 735$
$6_4 = 1624$
$6_5 = 1764$
$6_6 = 720$

$7_0 = 1$
$7_1 = 28$
$7_2 = 322$
$7_3 = 1960$
$7_4 = 6769$
$7_5 = 13132$
$7_6 = 13068$
$7_7 = 5040$

$8_0 = 1$
$8_1 = 36$
$8_2 = 546$
$8_3 = 4536$
$8_4 = 22449$
$8_5 = 67284$
$8_6 = 118124$
$8_7 = 109584$
$8_8 = 40320$

$9_0 = 1$
$9_1 = 45$
$9_2 = 870$
$9_3 = 9450$
$9_4 = 63273$
$9_5 = 269325$
$9_6 = 723680$
$9_7 = 1172700$
$9_8 = 1026576$
$9_9 = 362880$

$10_0 = 1$
$10_1 = 55$
$10_2 = 1320$
$10_3 = 18150$
$10_4 = 157773$
$10_5 = 902055$
$10_6 = 3416930$
$10_7 = 8409500$
$10_8 = 12753576$
$10_9 = 10628640$
$10_{10} = 3628800$

Stirling Numbers of the Second Kind

${}_02 = 1$
${}_12 = 3$
${}_22 = 7$
${}_32 = 15$
${}_42 = 31$
${}_52 = 63$
${}_62 = 127$
${}_72 = 255$
${}_82 = 511$
${}_92 = 1023$
${}_{10}2 = 2047$

${}_03 = 1$
${}_13 = 6$
${}_23 = 25$
${}_33 = 90$
${}_43 = 301$
${}_53 = 966$
${}_63 = 3025$
${}_73 = 9330$
${}_83 = 28501$
${}_93 = 86526$

${}_04 = 1$
${}_14 = 10$
${}_24 = 65$
${}_34 = 350$
${}_44 = 1701$
${}_54 = 7770$
${}_64 = 34105$
${}_74 = 145750$
${}_84 = 611501$

${}_05 = 1$
${}_15 = 15$
${}_25 = 140$
${}_35 = 1050$
${}_45 = 6951$
${}_55 = 42525$
${}_65 = 246730$
${}_75 = 1379400$

${}_06 = 1$
${}_16 = 21$
${}_26 = 266$
${}_36 = 2646$
${}_46 = 22827$
${}_56 = 179487$
${}_66 = 1323652$

${}_07 = 1$
${}_17 = 28$
${}_27 = 462$
${}_37 = 5880$
${}_47 = 63987$
${}_57 = 627396$

${}_08 = 1$
${}_18 = 36$
${}_28 = 750$
${}_38 = 11880$
${}_48 = 159027$

${}_09 = 1$
${}_19 = 45$
${}_29 = 1155$
${}_39 = 22275$

${}_010 = 1$
${}_110 = 55$
${}_210 = 1705$

${}_011 = 1$
${}_111 = 66$

${}_012 = 1$

4. Expansion of $E_2(x)$. Summary of known results. For the computations of the coefficients of $E_2(x) = e^{e^x - 1}$, Bell uses the explicit formula

$$B_n = \left(\frac{\Delta}{1!} + \frac{\Delta^2}{2!} + \frac{\Delta^3}{3!} + \ldots + \frac{\Delta^n}{n}\right) 0^n \ldots \qquad (12)$$

which, in view of the combinational properties of the Δ's, may be written as

$$B_n = [1 + {}_{n-2}2 + {}_{n-3}3 + {}_{n-4}4 + \dots + {}_2(n-2) + {}_1(n-1) + 1] \dots \quad (13)$$

As far as the writer knows, this is the only explicit formula given in the literature for the computation of the B's.

Formulas (13) and (6) yield the following:

THEOREM. The coefficient of $\frac{x^n}{n!}$ in the expansion of e^{e^x-1} is equal to the sum of the coefficients in Stirling's factorial expansion of x^n.

By the use of Cayley's table of Stirling Numbers Bell compiled the following table of the first 20 B's.

n	B_n	n	B_n	n	B_n
0	1	7	877	14	190899322
1	1	8	4140	15	1382958545
2	2	9	21147	16	10480142147
3	5	10	115975	17	82864869804
4	15	11	678570	18	682076806159
5	52	12	4213597	19	5832742205057
6	203	13	27644437	20	51724158235372

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Williams computed the first 14 B's from the symbolic recurrence formula

$$B_{n+1} = (1 + B)^n \quad (14)$$

which will be shown to be a special case, for $n = 2$, of a general recurrence formula for $E_n(x)$.

H. W. Becker,[5] in the solution of a problem proposed by D. H. Browne, uses an algorithm based on (14) to find

$$B_{25} = 4\ 638\ 590\ 332\ 229\ 999\ 353$$
$$B_{30} = 846\ 749\ 014\ 511\ 809\ 332\ 450\ 147$$
$$B_{35} = 286\ 600\ 203\ 019\ 560\ 266\ 563\ 340\ 426\ 570$$

One of the most significant propositions in Williams' paper is Dobinski's[5] ingenious definition of the B's, namely,

$$B_n = \frac{1}{e}\sum_{r=0}^{\infty}\frac{r^n}{r!} \quad (15)$$

from which he derives, among other things, the beautiful theorem

$$\sum_{r=0}^{\infty}\frac{(ar+b)^n}{r!} = (aB+b)^n e \quad (16)$$

where the exponents of the B's are to be taken as subscripts.

[5] *Am. M. Monthly*, v. 48, 1941, p. 701–702.

5. Extension of Above Results. Formulas (15)–(16), comb with Stirling's formula (6) enable us to derive and prove the follo propositions in a less cumbersome way than would otherwise be sible. In all of these formulas the operation of multiplicatio symbolic; that is, the exponents of the B's are to be replaced by scripts.

LEMMA 1. $$B_n = (B + 2)^{n-2} + (B + 1)^{n-2}. \quad (17$$

Proof. By (10) and (11)

$$B_n = \frac{1}{e}\sum_0^\infty \frac{r^n}{r!} = \frac{1}{e}\sum \frac{r^{n-2}}{r!}\cdot r^2.$$

By Stirling's formula (6), $r^2 = r(r-1) + r$. Hence

$$B_n = \frac{1}{e}\sum \frac{r^{n-2}}{r!}\cdot r^{(2)} + \frac{1}{e}\sum \frac{r^{n-2}}{r!}\cdot r = \frac{1}{e}\sum \frac{r^{n-2}}{(r-2)!} + \frac{1}{e}\sum \frac{r^{n-2}}{(r-1)!} =$$
$$\frac{1}{e}\sum_2^\infty \frac{(r-2+2)^{n-2}}{(r-2)!} + \frac{1}{e}\sum_1^\infty \frac{(r-1+1)^{n-2}}{(r-1)!} = (B+2)^{n-2} + (B+1)^{n-2}.$$

LEMMA 2. $$B_n = (B + 3)^{n-3} + 3(B + 2)^{n-3} + (B + 1)^{n-3}. \quad (17a)$$

Proof. As before, $B_n = \frac{1}{e}\sum_0^\infty \frac{r^n}{r!} = \frac{1}{e}\sum \frac{r^{n-3}}{r!}\cdot r^3 = \frac{1}{e}\sum \frac{r^{n-3}}{r}\times$

$$(r^{(3)} + 3r^{(2)} + r) = \frac{1}{e}\sum \frac{r^{n-3}}{(r-3)!} + \frac{3}{e}\sum \frac{r^{n-3}}{(r-2)!} + \frac{1}{e}\sum \frac{r^{n-3}}{(r-1)!} =$$
$$\frac{1}{e}\sum \frac{(r-3+3)^{n-3}}{(r-3)!} + \frac{3}{e}\sum \frac{(r-2+2)^{n-3}}{(r-2)!} + \frac{1}{e}\sum \frac{(r-1+1)^{n-3}}{(r-1)!} =$$
$$(B+3)^{n-3} + 3(B+2)^{n-3} + (B+1)^{n-3}.$$

In a similar manner we get

$$B_4 = (B+4)^{n-4} + 6(B+3)^{n-4} + 7(B+2)^{n-4} + (B+1)^{n-4} \quad (17b)$$

$$B_5 = (B+5)^{n-5} + 10(B+4)^{n-5} + 25(B+3)^{n-5} + 15(B+2)^{n-5} + (B+1)^{n-5}. \quad (17c)$$

All of these relationships are special cases of the following gener theorem:

$$B_n = (B+k)^{n-k} + {}_1(k-1)(B+k-1)^{n-k} + \dots + {}_{n-2}2(B+2)^{n-k} + (B+1)^{n-k}. \quad (18)$$

Proof. $B_n = \frac{1}{e}\sum_0^\infty \frac{r^n}{r!}$. By Stirling's formula

$$\frac{r^n}{r!} = \frac{r^{n-k}}{r!}r^k = \frac{r^{n-k}}{r!}[r^{(k)} + {}_1(k-1)r^{(k-1)} + \ldots + {}_{k-2}2r^{(2)} + r] =$$
$$\frac{r^{n-k}}{(r-k)!} + {}_1(k-1)\frac{r^{n-k}}{(r-k+1)!} + \ldots + {}_{k-2}2\frac{r^{n-k}}{(r-2)!} + \frac{r^{n-k}}{(r-1)!}.$$

Substituting in the expression for B_n and applying (16) we obtain the proposition.[6]

For $n = k$, (18) becomes

$$B_k = 1 + {}_1(k-1) + {}_2(k-2) + \ldots + {}_{k-2}2 + 1. \quad (18a)$$

Hence (13) is a special case of (18).

Putting $n = k + 1$, we get

$$B_{k+1} = 2 + 3\cdot{}_{k-2}2 + 4\cdot{}_{k-3}3 + \ldots + k\cdot{}_1(k-1) + (k+1), \quad (18b)$$

an explicit formula giving the B next to the one obtained from (13).

Subtracting (18b) from (18c), we get

$$1 + 2\cdot{}_{k-2}2 + 3\cdot{}_{k-3}3 + \ldots + (k-1)\cdot{}_1(k-1) + k = B_{k+1} - B_k. \quad (19)$$

This is one of a series of formulas obtained by putting, in (18), $n = k, k+1, k+2, k+3, k+4 \ldots$ Thus

$$1^2 + 2^2\cdot{}_{k-2}2 + \ldots + (k-1)^2\cdot{}_1(k-1) + k^2 = B_{k+2} - 2B_{k+1}$$
$$1^3 + 2^3\cdot{}_{k-2}2 + \ldots + (k-1)^3\cdot{}_1(k-1) + k^3 = B_{k+3} - 3B_{k+2} + B_k$$
$$1^4 + 2^4\cdot{}_{k-2}2 + \ldots + (k-1)^4\cdot{}_1(k-1) + k^4 = B_{k+4} - 4B_{k+3} + 4B_{k+1} + B_k$$
$$1^5 + 2^5\cdot{}_{k-2}2 + \ldots + (k-1)^5\cdot(k-1) + k^5 = B_{k+5} - 5B_{k+4} + 10B_{k+2} + 5B_{k+1} - 2B_k$$
$$1^6 + 2^6\cdot{}_{k-2}2 + \ldots + (k-1)^6\cdot(k-1) + k^6 = B_{k+6} - 6B_{k+5} + 20B_{k+3} + 15B_{k+2} - 12B_{k+1} - 9B_k.$$

6. Other Relationships. A more compact explicit formula for computing B_n is obtained from (18a) by substituting for the various Stirling Numbers their values according to the well-known formula

[6] Another proof of (18) is given by John Riordan in, "The Number of Impendances of an n Terminal Network," *Bell System Technical Journal*, v. 18, 1934, p. 312.

$$_{n-k}k = \sum_{i=0}^{i=k} \binom{k}{i} (k-1)^n. \tag{20}$$

Simplifying the result we get

$$n!B_n = n^n + \binom{n}{2}(n-2)^n + 2\binom{n}{3}(n-3)^n + 9\binom{n}{4}(n-4)^n + \ldots$$

or

$$n!B = \sum_{k=0}^{k=n-1} \alpha_k \binom{n}{k} (n-k)^n, \tag{21}$$

where $\alpha_0 = 1$, $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_k = k\alpha_{k-1} + (-1)^k$.

For the convenience of computation a few of the α's are listed:

$$\alpha_0 = 1,\ \alpha_1 = 0,\ \alpha_2 = 1,\ \alpha_3 = 2,\ \alpha_4 = 9,\ \alpha_5 = 44,\ \alpha_6 = 265,\ \alpha_7 = 1854.$$

α_k will be easily recognized as the number of permutations of the first k natural numbers in which no one of the numbers $1, 2 \ldots k$ is in its right place.[7]

Examples: $5!B_5 = 5^5 + \binom{5}{2}3^5 + 2\binom{5}{3}2^5 + 9\binom{5}{4}$.

$$6!B_6 = 6^6 + \binom{6}{2}4^6 + 2\binom{6}{3}3^6 + 9\binom{6}{4}2^6 + 44\binom{6}{5}.$$

Simplified forms of (21) can be derived from (18*a*) by transposing some of the terms $1, {}_1(k-1) \ldots$ before substituting for the others their values from (20). Thus, we have

$$(n-1)!(B_n - 1) = (n-1)^n + \binom{n-1}{2}(n-3)^n + \ldots = \sum_{k=0}^{k=n-2} \alpha_k \binom{n-1}{k}(n-k-1)^n.$$

$$(n-2)!\left[B_n - 1 - \frac{n(n-1)}{2}\right] = \sum_{0}^{n-3} \alpha_k \binom{n-2}{k}(n-2-k)^n.$$

[7] E. Netto, *Lehrbuch der Combinatorik*, 2nd edition, Leipzig-Berlin, 1927, p. 67. Another recurrence given by Netto is $\alpha_n = (n-1)(\alpha_{n-1} + \alpha_{n-2})$. "A formula equivalent to (21) was given by Ugo Broggi in Instituto Lombardo *Rend.*, v. 61, 1933, p. 196–202." (Mr. John Riordan in a letter to the editor.)

7. B's as Determinants. Expanding (14) for $n = 1, 2, 3 \ldots$, and solving the resulting equations, we get

$$B_2 = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}, \quad B_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix},$$

$$B_{n+1} = \begin{vmatrix} 1 & -1 & 0 & 0 & 0 & . & 0 \\ 1 & 1 & -1 & 0 & 0 & . & 0 \\ 1 & 2 & 1 & -1 & 0 & . & 0 \\ 1 & 3 & 3 & 1 & -1 & . & 0 \\ . & . & . & . & . & . & . \\ 1 & n & n & n & n & . & -1 \\ 1 & 1 & 2 & 3 & 4 & & 1 \end{vmatrix} \qquad (22)$$

Another form is obtained by applying (9) to

$$E_2(x) = e^{e^x - 1} = e^{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots} =$$

$$1 + x + \frac{x^2}{2!}\begin{vmatrix} 1 & -1 \\ \frac{1}{1!} & 1 \end{vmatrix} + \frac{x^3}{3!}\begin{vmatrix} 1 & -1 & 0 \\ \frac{1}{1!} & 1 & -2 \\ \frac{1}{2!} & \frac{1}{1!} & \frac{1}{1!} \end{vmatrix} + \cdots$$

Hence

$$B_n = \begin{vmatrix} 1 & -1 & 0 & . & 0 \\ \frac{1}{1!} & 1 & -2 & . & 0 \\ \frac{1}{2!} & \frac{1}{1!} & 1 & . & 0 \\ . & . & . & . & 0 \\ . & . & . & . & -(n-1) \\ \frac{1}{\overline{n-1}!} & \frac{1}{\overline{n-2}!} & \frac{1}{\overline{n-3}!} & . & 1 \end{vmatrix} \qquad (22a)$$

Since $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \log\left(1 + B_1 x + B_2 \frac{x^2}{2!} + \ldots\right) =$

$$B_1 x - \frac{1}{2}x^2 \begin{vmatrix} B_1 & 1 \\ B_2 & B_1 \end{vmatrix} + \frac{1}{3}x^3 \begin{vmatrix} B_1 & 1 & 0 \\ \frac{B_2}{1!} & \frac{B_1}{1!} & 1 \\ \frac{B_3}{2!} & \frac{B_2}{2!} & \frac{B_1}{1!} \end{vmatrix} + \cdots$$

we must have

$$\overline{n-1}! \begin{vmatrix} B_1 & 1 & 0 & \cdot & \cdot & 0 \\ \frac{B_2}{1!} & \frac{B_1}{1!} & 1 & \cdot & \cdot & 0 \\ \frac{B_3}{2!} & \frac{B_2}{2!} & \frac{B_1}{1!} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ & & & \cdot & \cdot & 1 \\ \frac{B_n}{\overline{n-1}!} & \frac{B_{n-1}}{\overline{n-1}!} & \frac{B_{n-2}}{\overline{n-2}!} & \cdot & \cdot & B_1 \end{vmatrix} = (-1)^{n-1} \qquad (22b)$$

8. The General Case. Let $E_n(x) = 1 + \alpha_1 x + \alpha_2 \frac{x^2}{2!} + \ldots$ and $E_{n+1}(x) = 1 + \beta_1 x + \beta_2 \frac{x^2}{2} + \ldots$. By (3), $E_{n+1}(x) = e^{E_n(x)-1}$, or

$$1 + \beta_1 x + \beta_2 \frac{x^2}{2!} + \ldots = e^{\alpha_1 x + \alpha_2 \frac{x^2}{2!} + \alpha_3 \frac{x^3}{3!} + \ldots}$$

Differentiating both sides we get

$$\beta_1 + \beta_2 x + \beta_3 \frac{x^2}{2!} + \ldots =$$
$$\left(1 + \beta_1 x + \beta_2 \frac{x^2}{2!} + \ldots\right)\left(\alpha_1 + \alpha_2 x + \alpha_3 \frac{x^3}{3!} + \ldots\right).$$

Multiplying and collecting terms we obain a result which may be expressed symbolically as

$$\beta_{k+1} = \alpha(\beta + \alpha)^k \qquad (23)$$

a formula yielding the means of computing the coefficients of any exponential $E_n(x)$ from those of the exponential of the immediately lower order. This formula is equivalent to (2.1) of Bell II.

For the sake of easier reference we shall adopt $A_i, B_i, C_i \ldots$ as symbols of the coefficients of $E_1(x)$, $E_2(x)$, $E_3(x)$, etc. That is

$$E_1(x) = e^x = 1 + \sum_{k=1}^{\infty} A_k \frac{x^k}{k!}, \text{ or, symbolically, } e^{Az}$$

$$E_2(x) = e^{e^x-1} = 1 + \sum_{k=1}^{\infty} \frac{B^k}{k!} \text{ or, symbolically, } e^{Bz}.$$

Similarly $E_3(x) = e^{Cx}$, $E_4(x) = e^{Dx}$, $E_5(x) = e^{Ex}$. We shall have then by (23)

$$B_{k+1} = A(B+A)^k = (B+1)^k, \text{ since } A_1 = A_2 = A_3 = \dots = 1$$
$$C_{k+1} = B(C+B)^k;\ D_{k+1} = C(D+C)^k,\ E_{k+1} = D(E+D)_k. \quad (23a)$$

We have seen how Williams used the first of these formulas for the computation of the first 14 B's, that is, the coefficients of $E_2(x)$. To compute the C's, or the coefficients of $E_3(x)$, we can use the second formula, namely

$$C_{k+1} = B(C+B)^k$$

obtaining the coefficients 1, 3, 12, 60, 358, 2471, 19302,. . . ← sequence (258) ✓

The same method is used in computing the coefficients of $E_4(x)$, $E_5(x)$, and $E_6(x)$, given in the table below.

We shall now derive explicit formulas for the coefficients of $E_{n+1}(x)$ in terms of those of $E_n(x)$.

By the second part of (3)

$$E_{n+1}(x) = E_n(e^x - 1)$$

or

$$1 + \beta_1 x + \beta_2\frac{x^2}{2!} + \beta_3\frac{x^3}{3!} + \dots = 1 + \alpha_1(e^x - 1) + \alpha_2\frac{(e^x-1)^2}{2!} + \dots$$

Expanding and comparing coefficients we get

$$\beta_1 = \alpha_1,\ \beta_2 = \alpha_1 + \alpha_2,\ \beta_3 = \alpha_1 + {}_1 2\alpha_2 + \alpha_3,\ \dots$$

In general

$$\beta_k = \sum_{i=0}^{k} {}_{k-i}i\,\alpha_i \quad (24)$$

that is

$$C_k = \sum_{1}^{k} {}_{k-i}iB_i,\quad D_k = \sum_{i=1}^{k} {}_{k-i}iC_i.$$

To develop formulas leading from $E_{n+1}(x)$ to $E_n(x)$ we observe that $E_n(x) = E_{n+1}[\log(1+x)]$, or

$$1 + \alpha_1 x + \alpha_2\frac{x^2}{2!} + \dots = 1 + \beta_1 \log(1+x) + \frac{\beta_2}{2!}\log^2(1+x) + \dots =$$
$$1 + \beta_1 x + (\beta_2 - \beta_1)\frac{x^2}{2!} + (\beta_3 - 2_1\beta_2 + 2_2\beta_1)\frac{x^3}{3!} + \dots$$

Hence

$$\alpha_k = \sum_{i=0}^{k} (-)^i k_i \beta_{k-1}$$

or

$$\alpha_k = \beta(\beta-1)(\beta-2)\ldots(\beta-k) = \beta^{(k)} \tag{25}$$

or, recalling that $E_1(x) = e^{Ax}$, $E_2(x) = e^{Bx}$, $E_3(x) = e^{Dx}, \ldots$

$$A_k = B^{(k)} \tag{25a}$$
$$B_k = C^{(k)} \tag{25b}$$
$$C_k = D^{(k)} \tag{25c}$$

It should be noted here that formula (25) is identical with (2.3) in Bell's second paper, while the special case (25*a*) is equivalent to Theorem 5 in Williams's paper. In fact, since $A_1 = A_2 = \ldots = A_k = 1$, (25*a*) becomes $B^{(k)} = 1$, or

$$B(B-1)\ldots(B-n+1) = 1.$$

Again, since $A_i = A_2 = \ldots = A_n = 1$, $A^{(k)} = 0$, for all values of k except $k = 1$. Hence, if we put $E_0(x) = e^{hx}$ we shall have $h_1 = 1$, $h_2 = h_3 = \ldots = h_4 = 0$ so that $E_0(x) = 1 + x$.

9. Expansion of $E_{-n}(x)$. Using the symbolic notation

$$E_{-1}(x) = e^{ax} = 1 + a_1 x + a_2 \frac{x^2}{2!} + \ldots = 1 + \log(1+x)$$

$$E_{-2}(x) = e^{bx} = 1 + b_1 x + b_2 \frac{x^2}{2!} + \ldots = 1 + \log[1 + \log(1+x)]$$

$$E_{-3}(x) = e^{cx} = 1 + c_1 x + c_2 \frac{x^2}{2!} + \ldots = 1 + \log\{1 + \log[1 + \log(1+x)]\}.$$

We shall have $a_k = h^{(k)}$. Hence $a_1 = h_1 = 1$, $a_2 = h(h-1) = h_2 - h_1 = -1$, $a_3 = h(h-1)(h-2) = h_3 - 3h_2 + 2h_1 = 2$. $a_4 = h(h-1)(h-2)(h-3) = -3!$ Similarly $a_5 = 4!$, $a_6 = -5!$, etc.

Hence $E_{-1}(x) = 1 + x - \dfrac{1!x^2}{2!} + \dfrac{2!x^3}{3!} - \dfrac{3!x^4}{4!} + \ldots$

To calculate the coefficients of $E_{-2}(x) = 1 + \log[1 + \log(1+x)] = e^{bx}$ we again use formula (25) according to which $b_k = a^{(k)}$ and we have

$b_1 = a_1 = 1$, $b_2 = a^{(2)} = a(a-1) = a_2 - 0_1 = -2$, $b_3 = a_3 - 3a_2 + 2a_1 = 7$, $B_4 = (a-1)(a-2)(a-3) = -35$.

In a similar way we find that the coefficients of $E_{-3}(x) = e^{ex}$ are

$$c_1 = 1,\ c_2 = -3,\ c_3 = 15,\ c_4 = -105, \text{ etc.}$$

It is noteworthy that the Stirling Numbers of the first kind play in the same part in descending from $E_n(x)$ to $E_{(n-1)}x$ as the Stirling Numbers of the second kind in ascending from $E_n(x)$ to $E_{n+1}x$.

The results of §§ 8 and 9 are embodied in the following table.

Coefficients of $E_n(x)$

n	x^0	x	$\frac{x^2}{2!}$	$\frac{x^3}{3!}$	$\frac{x^4}{4!}$	$\frac{x^5}{5!}$	$\frac{x^6}{6!}$	$\frac{x^7}{7!}$
6	1	1	6	51	561	7556	120196	2201856
5	1	1	5	35	315	3455	44590	660665
4	1	1	4	22	154	1304	12915	140115
3	1	1	3	12	60	358	2471	19302
2	1	1	2	5	15	52	203	877
1	1	1	1	1	1	1	1	1
0	1	1	0	0	0	0	0	0
−1	1	1	−1	2!	−3!	4!	−5!	6!
−2	1	1	−2	7	−35	228	−1834	17382
−3	1	1	−3	15	−105	967	−10472	137137
−4	1	1	−4	26	−234	2696	−37919	
−5	1	1	−5	40	−440	6170	−105315	
−6	1	1	−6	57	−741	12244	−245755	

Certain properties of these numbers will be discussed in a subsequent article.

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