Amicable Pairs and Aliquot Sequences

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October 31, 2013

If n is a positive integer, let s(n) denote the sum of all positive divisors of n that are strictly less than n. Then n is said to be **perfect** or **1-sociable** if s(n) = n. We mentioned perfect numbers in [1], asking whether infinitely many exist, but did not report their reciprocal sum [2]

$$\frac{1}{6} + \frac{1}{28} + \frac{1}{496} + \frac{1}{8128} + \frac{1}{33550336} + \frac{1}{8589869056} + \dots = 0.2045201428\dots$$

This constant can, in fact, be rigorously calculated to 149 digits (and probably much higher accuracy if needed).

Define $s^k(n)$ to be the k^{th} iterate of s with starting value n. The integer n is amicable or **2-sociable** if $s^2(n) = n$ but $s(n) \neq n$. Such phrasing is based on older terminology [3]: two distinct integers m, n are said to form an "amicable pair" if s(m) = n and s(n) = m. The (infinite?) sequence of amicable numbers possesses zero asymptotic density [4] and, further, has reciprocal sum [5, 6]

$$\frac{1}{240} + \frac{1}{284} + \frac{1}{1184} + \frac{1}{1210} + \frac{1}{2620} + \frac{1}{2924} + \frac{1}{5020} + \frac{1}{5020} + \frac{1}{6232} + \frac{1}{6368} + \dots = 0.01198415\dots$$

In contrast with the preceding, *none* of the digits are provably correct. The best rigorous upper bound for this constant is almost 10^9 ; deeper understanding of the behavior of amicable numbers will be required to improve upon this poor estimate.

Fix $k \ge 3$. An integer n is k-sociable if $s^k(n) = n$ but $s^{\ell}(n) \ne n$ for all $1 \le \ell < k$. No examples of 3-sociable numbers are known [7, 8]; the first example for $4 \le k < 28$ is the 5-cycle {12496, 14288, 15472, 14536, 14264} and the next example is the 4-cycle {1264460, 1547860, 1727636, 1305184}. Let S_k denote the sequence of all k-sociable numbers and S be the union of S_k over all k. It is conjectured that the (infinite?) sequence S possesses zero asymptotic density and progress toward confirming this appears in [9]. No one is ready to compute the reciprocal sum of S; a proof of convergence would seem to be faraway.

As an aside, we mention the sequence of **prime-indexed primes**, which is clearly infinite and has reciprocal sum [10]

1	1	1	1	1	1	1	1	1	_ 1	1	1	1	1	- 1.0432015	
$\overline{3}$	$\overline{5}^{+}$	11	17	$\overline{31}$	$\overline{41}$	$\overline{59}$	$\overline{67}$	$\overline{83}$	109	127	157	179	$-\frac{191}{191}$	$\cdots = 1.0452015$	-1.0452015

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Again, this is conjectural only. The best rigorous lower/upper bounds for this constant are 1.04299 and 1.04365 [2] Such bounds are tighter than those (1.83408 and 2.34676) for the reciprocal sum of twin primes [11].

Our main interest is in the "aliquot sequence" $\{s^k(n)\}_{k=1}^{\infty}$, where we assume WLOG that n is even. For example, if n = 12, the sequence $\{16, 15, 9, 4, 3, 1\}$ is finite (terminates at 1). From earlier, we know that infinite cyclic behavior is possible. Does an infinite *unbounded* aliquot sequence exist? On the one hand, starting with n = 276, extensive computation has yielded 1769 terms with no end in sight [12, 13, 14, 15, 16]; probabilistic arguments in [17, 18], based on the arithmetic mean of s(2n)/(2n), also support a belief that most sequences grow without bound.

On the other hand, the geometric mean of s(2n)/(2n):

$$\sqrt[N]{\prod_{n=1}^{N} \frac{s(2n)}{2n}} = \exp\left(\frac{1}{N} \sum_{n=1}^{N} \ln\left(\frac{s(2n)}{2n}\right)\right)$$

(which seems a more appropriate tool than a simple average) predicts the opposite. Bosma & Kane [19] proved that

$$\begin{array}{ll} \lambda & = & \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \ln\left(\frac{s(2n)}{2n}\right) = 2\alpha(2) + \sum_{p \ge 3} \alpha(p) - \sum_{j \ge 1} \left((2\beta_j(2) - 1) \prod_{p \ge 3} \beta_j(p) \right) \frac{1}{j} \\ & = & -0.0332594808... < 0 \end{array}$$

which implies that the geometric mean $\mu = \exp(\lambda) = 0.9672875344... < 1$. The indicated numerical estimates are due to Sebah [20]. Sums and products over p are restricted to primes; further,

$$\begin{split} \alpha(p) &= \left(1 - \frac{1}{p}\right) \sum_{m=1}^{\infty} \frac{1}{p^m} \ln\left(\frac{p^{m+1} - 1}{p^m(p-1)}\right), \\ \beta_j(p) &= \left(1 - \frac{1}{p}\right) \sum_{m=0}^{\infty} \frac{1}{p^m} \left(\frac{p^{m+1} - 1}{p^m(p-1)}\right)^{-j}. \end{split}$$

The fact that $\mu < 1$ suggests that aliquot sequences tend to decrease ultimately, evidence in favor of the Catalan-Dickson conjecture. It would be good to compute other related constants, appearing in [21], to similar levels of precision.

From [1, 22], the probability that s(n) exceeds n, for arbitrary n, is

$$\lim_{n \to \infty} \frac{1}{n} \cdot \left| \left\{ i \le n : \frac{s(i)}{i} > 1 \right\} \right| = 0.2476...$$

(what was called A(2)). The fact that these odds are significantly less than 1/2 again suggests that unboundedness is a rare event, if it occurs at all.

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