# Amicable Pairs and Aliquot Sequences 

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If $n$ is a positive integer, let $s(n)$ denote the sum of all positive divisors of $n$ that are strictly less than $n$. Then $n$ is said to be perfect or 1 -sociable if $s(n)=n$. We mentioned perfect numbers in [1], asking whether infinitely many exist, but did not report their reciprocal sum [2]

$$
\frac{1}{6}+\frac{1}{28}+\frac{1}{496}+\frac{1}{8128}+\frac{1}{33550336}+\frac{1}{8589869056}+\cdots=0.2045201428 \ldots
$$

This constant can, in fact, be rigorously calculated to 149 digits (and probably much higher accuracy if needed).

Define $s^{k}(n)$ to be the $k^{\text {th }}$ iterate of $s$ with starting value $n$. The integer $n$ is amicable or 2-sociable if $s^{2}(n)=n$ but $s(n) \neq n$. Such phrasing is based on older terminology [3]: two distinct integers $m, n$ are said to form an "amicable pair" if $s(m)=n$ and $s(n)=m$. The (infinite?) sequence of amicable numbers possesses zero asymptotic density [4] and, further, has reciprocal sum [5, 6]
$\frac{1}{240}+\frac{1}{284}+\frac{1}{1184}+\frac{1}{1210}+\frac{1}{2620}+\frac{1}{2924}+\frac{1}{5020}+\frac{1}{5020}+\frac{1}{6232}+\frac{1}{6368}+\cdots=0.01198415 \ldots$.
In contrast with the preceding, none of the digits are provably correct. The best rigorous upper bound for this constant is almost $10^{9}$; deeper understanding of the behavior of amicable numbers will be required to improve upon this poor estimate.

Fix $k \geq 3$. An integer $n$ is $k$-sociable if $s^{k}(n)=n$ but $s^{\ell}(n) \neq n$ for all $1 \leq \ell<k$. No examples of 3 -sociable numbers are known [7, 8]; the first example for $4 \leq k<28$ is the 5 -cycle $\{12496,14288,15472,14536,14264\}$ and the next example is the 4 -cycle $\{1264460,1547860,1727636,1305184\}$. Let $S_{k}$ denote the sequence of all $k$-sociable numbers and $S$ be the union of $S_{k}$ over all $k$. It is conjectured that the (infinite?) sequence $S$ possesses zero asymptotic density and progress toward confirming this appears in [9]. No one is ready to compute the reciprocal sum of $S$; a proof of convergence would seem to be faraway.

As an aside, we mention the sequence of prime-indexed primes, which is clearly infinite and has reciprocal sum [10]
$\frac{1}{3}+\frac{1}{5}+\frac{1}{11}+\frac{1}{17}+\frac{1}{31}+\frac{1}{41}+\frac{1}{59}+\frac{1}{67}+\frac{1}{83}+\frac{1}{109}+\frac{1}{127}+\frac{1}{157}+\frac{1}{179}+\frac{1}{191}+\cdots=1.0432015 \ldots$.

[^0]Again, this is conjectural only. The best rigorous lower/upper bounds for this constant are 1.04299 and 1.04365 [2] Such bounds are tighter than those (1.83408 and 2.34676 ) for the reciprocal sum of twin primes [11].

Our main interest is in the "aliquot sequence" $\left\{s^{k}(n)\right)_{k=1}^{\infty}$, where we assume WLOG that $n$ is even. For example, if $n=12$, the sequence $\{16,15,9,4,3,1\}$ is finite (terminates at 1). From earlier, we know that infinite cyclic behavior is possible. Does an infinite unbounded aliquot sequence exist? On the one hand, starting with $n=276$, extensive computation has yielded 1769 terms with no end in sight $[12,13,14,15,16]$; probabilistic arguments in $[17,18]$, based on the arithmetic mean of $s(2 n) /(2 n)$, also support a belief that most sequences grow without bound.

On the other hand, the geometric mean of $s(2 n) /(2 n)$ :

$$
\sqrt[N]{\prod_{n=1}^{N} \frac{s(2 n)}{2 n}}=\exp \left(\frac{1}{N} \sum_{n=1}^{N} \ln \left(\frac{s(2 n)}{2 n}\right)\right)
$$

(which seems a more appropriate tool than a simple average) predicts the opposite. Bosma \& Kane [19] proved that

$$
\begin{aligned}
\lambda & =\lim _{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N} \ln \left(\frac{s(2 n)}{2 n}\right)=2 \alpha(2)+\sum_{p \geq 3} \alpha(p)-\sum_{j \geq 1}\left(\left(2 \beta_{j}(2)-1\right) \prod_{p \geq 3} \beta_{j}(p)\right) \frac{1}{j} \\
& =-0.0332594808 \ldots<0
\end{aligned}
$$

which implies that the geometric mean $\mu=\exp (\lambda)=0.9672875344 \ldots<1$. The indicated numerical estimates are due to Sebah [20]. Sums and products over $p$ are restricted to primes; further,

$$
\begin{aligned}
& \alpha(p)=\left(1-\frac{1}{p}\right) \sum_{m=1}^{\infty} \frac{1}{p^{m}} \ln \left(\frac{p^{m+1}-1}{p^{m}(p-1)}\right), \\
& \beta_{j}(p)=\left(1-\frac{1}{p}\right) \sum_{m=0}^{\infty} \frac{1}{p^{m}}\left(\frac{p^{m+1}-1}{p^{m}(p-1)}\right)^{-j} .
\end{aligned}
$$

The fact that $\mu<1$ suggests that aliquot sequences tend to decrease ultimately, evidence in favor of the Catalan-Dickson conjecture. It would be good to compute other related constants, appearing in [21], to similar levels of precision.

From $[1,22]$, the probability that $s(n)$ exceeds $n$, for arbitrary $n$, is

$$
\lim _{n \rightarrow \infty} \frac{1}{n} \cdot\left|\left\{i \leq n: \frac{s(i)}{i}>1\right\}\right|=0.2476 \ldots
$$

(what was called $A(2)$ ). The fact that these odds are significantly less than $1 / 2$ again suggests that unboundedness is a rare event, if it occurs at all.

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