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which cover all conditions included in the given system (see Remark 3). The solution $W^1 X_1 = \sum_1 k \in 1, 2, \ldots, R$ is the don't care condition of $I^2$ but is less simple.

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References


Singular Cases

When the discriminant of $W$ has $|D|=0$ there is no solution of the form (5). But here is a solution of the form

$X_k = W^p(x_k)$

$k = 1, 2, \ldots, R$

$s = 1, 2, \ldots, R$

$W(x_k)$ (6)

where $f = 1$ (or an equivalent condition between the independent variables) represents restriction of the domain of definition of the functions $W(x_k)$. The restriction rules out all columns in $D$ which do not have any nonzero elements. Thanks to this restriction the functions $W(x_k)$ in (6) can take any values. This fact can be used to make the final results algebraically simpler.

Example 3: The system given by the single equation

$X_1 + X_2 = 1$

has the discriminant

$D = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$

with zero value. There is no solution of the form (5). To find it for the form (6) we rule out the column $q = 1$ containing only zeros. The condition $x_1 + x_2 = 1$ represents the desired restriction. The maps for $W^1, W^2$ become

$W^1: X_1 = x_1, x_2 = 1$

$W^2: X_1 = x_1, x_2 = 1$

where the shaded area can be used to simplify the results (don't care conditions). Here we have a case with no simplification possible.

There are two solutions,

$W^1: X_1 = x_1, x_2 = 1$

$W^2: X_1 = x_1, x_2 = 1$

The Number of Equivalence Classes of Boolean Functions Under Groups Containing Negative

Introduction

The purpose of this communication is to point out a new derivation of the number of self-complementary symmetry types and to discuss a conjecture of Elspas.

In the synthesis of digital systems, it is sometimes convenient to have both a single and its negation available. This idea suggests a modification of the conventional notion of equivalence under a group. (See Harrison for a discussion of equivalence classes induced by a group.) If $\Phi$ is a permutation group on the domain of Boolean functions, then one usually says that a function $f$ is equivalent to a function $g$ if there exists an $a \in \Phi$ such that $f(a) = g(a)$ for every $a \in \Phi$. The concept of equivalence can be broadened if one allows the condition to be that $f(a) = g(a)$ for every $a \in \Phi$. Under the proposed definition of equivalence, the Boolean functions are again resolved into equivalence classes, and the number of equivalence classes is given by

$\sum_{a \in \Phi} [\sum_{b \in \Phi} \phi(b)]$

For any Boolean function $f$, the number of equivalence classes is

$\sum_{a \in \Phi} [\sum_{b \in \Phi} \phi(b)]$

for any Boolean function $f$. Note that $\Phi$ is really a permutation group on the range $\{0, 1\}$ of the Boolean functions. The situation can be summed up by saying that we are asking for the number of classes of functions under a group $g$ on the domain $\Phi$ on the range.

de Bruijn's Theorem

A recent theorem of de Bruijn states which indicates an immediate solution to the problem of counting the number of classes. Let $f$ be the class of functions from $D$ into $R$, and suppose that has $a$ elements and $R$ elements. Let $\Phi$ be a permutation group of order $a$ and $g$ acting on $D$ while $\Phi$ denotes a group of order $h$ and degree $r$ on $R$. Two functions $f$ and $g$ are said to be equivalent if and only if there exists $a \in \Phi$ such that $f(a) = g(a)$ for every $a \in D$. Since this is a genuine equivalence relation, the family of functions is decomposed into equivalence classes, and we desire the number of these classes. The pertinent theorem of de Bruijn is given in terms of cycle index polynomials. The cycle index polynomial is a multivariate generating function for the cycle structure of $\Phi$ acting on $D$. Let $s_1, \ldots, s_k$ be the number of permutations of $\Phi$ having cycles of length $k$ for $k = 1, 2, \ldots, r$. Naturally

$\sum_{i=1}^r i^r s_i = z$

(1)

Then define the cycle index of $\Phi$ acting on $D$

$Z(\Phi) =\sum_{i=1}^r s_i f_i$

(2)

where the sum is taken over all partitions of $r$, i.e. the non-negative integer solutions of (1).

We can now state de Bruijn's theorem. Theorem 1: Let

$h_i = \exp \left\{ \sum_{k \in \Phi} i^r s_i \right\}

(3)

then the number of equivalence classes is given by

$\sum_{h_i} h_i$

(4)

3. The order of a group is the number of elements in the group. The degree of a permutation group is the cardinality of the object set.
The groups to be considered in the present discussion are listed below.

1) $G_s^*$ is the group of all $2^s$ complementations of variables. This group was first studied as a group on Boolean functions by Ashenhurst. 7

2) $G_s$ denotes the symmetric group on the $n$ variables, i.e., the group of all permutations of input letters. The order of $G_s$ is $n!$.

3) $G_s^*$ is the smallest group containing $G_s^*$ and $G_s$, and is very well known.

4) $G_s^*(Z_2)$ is the general linear group on the variables; the group has been studied by Slepian, 8 Harrison, 9 and Lechner. 10

Proof
Since two domain elements are symmetric, every term of the cycle contains a factor $\alpha^k$ for $k \geq 2$.

Theorem 6: The number of classes of functions with $G_s^*(Z_2)$ on the domain and $R_s$ on the range is exactly one half the number of classes with just $G_s^*$ on the domain.

Proof
Every linear transformation leaves the origin invariant so every term of the cycle index contains a factor $x^k$ for $k \geq 2$.

The calculated numerical results follow in Table 1.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\theta_s R_s(Z_2)$</th>
<th>$\theta_s R_s^*(Z_2)$</th>
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<tbody>
<tr>
<td>1</td>
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<tr>
<td>10</td>
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<td>2</td>
</tr>
</tbody>
</table>

It is interesting to note the ease with which these results were obtained and to compare these methods with those of Elspas' 8 and Ninomiya. 9 Both these men computed the same results but with considerable effort and only for the group $G_s$.

Self-Complementary Classes

We shall use the results of the previous section to obtain some results concerning groups without negation on the range. Suppose $G$ is a group on the domain and we require the number of classes of $G$ closed under complementation of the functions. Clearly only classes of neutral functions can have this property. Neutral functions are those functions with as many ones as zeros in their graphs, i.e., functions of weight $2^{n-1}$.

Theorem 7: The number of classes of functions under $G$ which are equivalent to their complements, i.e., self-complementary, is

$$Z_G(0, 2, 0, 2, \cdots, 0, 2).$$

Proof
Let $T_n$ be the number of classes of functions under $G$ alone. Note that $Z_G(2, \cdots, 2) = T_n$. Let $N_n$ be the number of self-complementary classes. Then

$$\frac{1}{2}Z_G(2, \cdots, 2) + Z_G(0, 2, \cdots, 0, 2) = \frac{1}{2}(T_n - N_n) + N_n$$

which implies

$$N_n = Z_G(0, 2, \cdots, 0, 2).$$

Theorem 8: The number of self-complementary classes under $G_s^*$ is

$$(2^n - 1)2^{n-1}.$$
Correspondence

neutral Boolean functions. The second part of the theorem follows from noting that the largest term in $Z(G(0, 2, \ldots, 0, 2))$ is $2^{n-1}/g$. An upper bound is certainly

$$\frac{1}{g} \sum_{x \in G} 2^{x-1} = 2^{n-1}.$$ 

The ratio we desire can now be computed, but it is convenient to have an estimate for the binomial coefficient. Using Stirling's formula, we get

$$\sqrt{\frac{2}{\pi}} \frac{1}{2^{2n-1}} \cdot \frac{1}{2^{n-1}}.$$ 

Finally, the pertinent theorem can be proven.

**Theorem 12:** Let $G$ be any group defined on the domain of the Boolean functions. If the order of $G$ does not exceed $2^{n-1}$ for any $n > 0$, then the number of self-complementary classes of functions tends to zero with the number of classes of neutral functions for increasing $n$.

**Proof**

Compute the ratio for the upper bound. As an immediate corollary to Theorem 12, we note that self-complementary classes are rather rare for the five groups that we have discussed.

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It is well known that the number of threshold functions of $n$ arguments is less than

$$B_n = 2 \sum_{i=1}^{n} \left( \frac{n-1}{i} \right).$$

Using this bound, Cameron shows that the number of $n$-argument functions realizable by a network of at most $k$ threshold gates is asymptotically less than $2^n$. As Cameron pointed out, this implies that at least one switching function of $n$ arguments (and probably most of them) requires more than $2n^{0.5}$ threshold gates for realization. The purpose of this communication is to generalize these results to functions incompletely specified; an improvement in the asymptotic bound $2n^{0.5}$ will also be obtained. Applications in character recognition and self-organizing systems are discussed.

The basic bound $B_n$ is derived from the following basic lemma (see Cameron for a good discussion of its proof; the proof in Winder is virtually identical, but less well explained):

**Lemma:** If $m$ hyperplanes are passed through the origin of an $(n+1)$-dimensional Euclidean space, the space is divided into a number of regions—at most

$$B_n = 2 \sum_{i=0}^{n} \left( \frac{n-1}{i} \right).$$

The bound $B_n$ is then obtained by considering an $(n+1)$-dimensional "realization space"—the space consisting of points $a = (a_0, a_1, \ldots, a_n)$, each of which represents the realization of some threshold function (a bias and $n$ weights). (We assume $a \geq 1$ logic.) By taking all possible choices of sign, we consider $2^n$ hyperplanes,

$$a_0 \pm a_1 \pm a_2 \pm \cdots \pm a_n = 0.$$ 

Two points in the realization space represent the same function if and only if they are not separated by any of these hyperplanes. Thus the regions, with boundaries defined by these hyperplanes, correspond one-to-one with threshold functions. Thus setting $m = 2^n$ in the lemma gives $B_n$.

Now, suppose we select exactly $m$ out of the $2^n$ possible input combinations. How many switching functions are no two of which agree in value on all $m$ of these points, can be realized by a single threshold gate? Clearly, by the same argument, there are at most $B_m$. (Because the $m$ points correspond to $m$ of the hyperplanes, and again, we're asking how many regions the realization space is divided into by $m$ hyperplanes.)

**Bounds on Threshold Gate Realizability**

**Summary**

A well-known bound estimates the number of switching functions realizable by a single threshold gate. In this communication this bound is generalized to apply to incompletely specified functions. Application is made to prove analytically an experimental result of Koford: the number of patterns discriminated by a threshold gate is twice the number of inputs (roughly). Also, Cameron's lower bound on the number of threshold gates needed in a network to realize an arbitrary function is improved. Finally, a lower bound on the number of gates needed in a two-level network is found; it is substantially lower than Koford's experimental results.

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