Dear Dr. Sloane,

Greetings again! I'm now back on this side of the Atlantic (Dept. of Math, U of Mt.), and as addicted as ever to your Handbook. If you're still collecting items for a second edition, I point out the Hardy & Wright reference to pentagonal numbers on p. 284 of their book (see note, bottom of p. 17 in magazine — read pp. 13–14 for pertinent background). Only if we don't call them pent. nos. by name, just use them by formula. And the use is in partition theory, not prime distribution. I have yet to find a reference on Euler's Theorem, but it would certainly belong as a red. in 2nd Ed. of your book if you can find one! All best wishes, S.E.
STAR-WATCHER'S GUIDE — JANUARY TO MARCH

Two years ago (see News, Oct. 27, 1975) Charles Kowal made observational news when he spotted a 15th-magnitude moon of Jupiter; the year before that he had found the 14th. This year (see November 1977 Sci. Am. or Nov. 12 '77 Science News) observer Kowal made a discovery of far greater interest to those trying to reconstruct the physical history of the solar system: a new object comparable in size (about 300 mi. or 500 km, in diameter) to the largest asteroids, but much farther away — at or around the orbital distance of Uranus, not Jupiter! First sighted on October 15th with the 18-in. Schmidt telescope on Palomar Mountain, its exact orbit is still being determined (tentatively described as nearly circular, with an inclination of 3 to 5 degrees). At 12th to 13th magnitude, it is beyond the reach of most amateur observers, but should show up on many older photographic plates to help determine its orbit. The size-estimate above is for an object with a medium-bright surface like our Moon. If later found to have a darker surface like a carbonaceous chondrite, it would be larger; if icy and more reflective, then smaller.

Also of interest to planetologists is a recent report (same issue of Sci. News) that Neptune is much warmer, relative to the heat it receives from the Sun, than was formerly believed, emitting about 3.5 times as much heat as it takes in. Uranus, on the other hand, seems to have little, if any, internal source of heat.

Meanwhile, a little closer to home, the orbiting Viking probe of Mars has carried out successfully what will probably remain a unique task in its mission: taking temperature-measurements of the midnight side. This is difficult, because it must be entirely within the planet’s shadow when exposure to direct sunlight would burn out its sensors, and it is only entirely in the planet’s shadow twice in a Martian year (1.86 Earth-years). As would be expected (contrary to a reporter Jonathan Eberhart’s statements on p. 329 of the Nov. 12 Sci. News), the warmest region is the eastern rim, since "twilight" means what is coming from the daylit side into darkness. The "cusp" is, as on Earth, re- tained heat relatively well through the night, while the plains regions, where the great dunes of sand begin, cool off relatively rapidly. Olympus Mons and three other volcanic mountains show up as isolated cold-spots.

An early issue of Science News (Sept. 28) reported that scientists speaking at a 3-day symposium held in Boston the middle of September seemed about equally divided as to the results of the biological probes on the Martian surface in July and September of 1976. The reactions observed were simply too unlike anything known by Earth standards of organic and inorganic chemistry to be unambiguously interpreted without further and finer observations. The sensors on board should have been able to detect as few as 1,000,000 b. C. cells per gram of soil, yet Earth-samples from the Moon have been found which contain as few as 100,000.

Earth-bound amateurs can do their own Mars-watching during January as the two planets closest approach on It is the brilliant west at sunset, night, as it descends through the constellation bright white object in Jupiter, seen rising late Regulus in Leo (far north of the two), in the west before Venus is lost from evening star in the West, together with Mercury, in aid March (cont. on 12th).

MATHMATICAL - PHYSICAL CORRESPONDENCE

Christmas 1977

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Year’s subscription, payable in dollars or pounds by check to editor, is 5x single issue price + postage.
The same process may be repeated for $n = 1 = 2$, yielding $n = 21 = 121 = 131 = 141 = 151$; we take any $p = 3$, $n = 21$, so $n = 31$. The stereographic projection of the resulting quadrilateral, its nine lines dividing it into five parts, landing corner in common, divided by each of the two sides meeting in that corner, find the centroids of the remaining quadrilaterals whose construction we suppose here for simplicity sake, and find these again cyclically, forming a similar irregular pentagon, reflected and reduced in size by $1/3 = 1/2$. As the center of the smaller circle via at point $V$, $1/2$-way down the radius of the pentagon for $n = 2$, and $1/3$-way down the radius of the pentagon for $n = 3$, so it is $1/2$-way down that of the 21-gon ("elevenkilaekagon") for $n = 4$. Since the diameter of the smaller circle equals the radius of the larger for $n = 4$, we are not led in this case to any (finite) related polygon, as we should have circumscribed the square about the circle of the pentagon, then we could say that $n = 3, 4, 5$, the related circumscribed polygon has $3, 4, 5$ sides, shrinking to an inscribed square and triangle, respectively, as $n$ grows to $8$ and $16$ - a $p$-gon with $p = 360/2$ arcsos($2/3$) for $n > 8$, but $360/2$ arcsos($n/2$) for $n < 8$.

For $n = 2$ and $h$, there is essentially only one possibility, up to rotation and reflection. For $n = 3$, however, there are two: the one we have given and its double (mod 11) $[2, 5, 11]$. Analogous to the expressions we gave in the last issue for sums and products of taxicab functions of $A_{10}$, $B_{10}$, we find for $A_{11}$ $B_{11}$ $31$ $31$ $31$ $31$ $31$ $31$ $31$ involving $V$, we find for $A_{11} B_{11}$ $31$ $31$ $31$ $31$ $31$ $31$ $31$ sum of sine of cosines $V_1 + V_2 + V_3 + V_4 + V_5 + V_6$, sum of tangents $\sqrt{2}$ product of tangents $[41, 21, 11, 21, 11, 21]$, etc., while $A_{11} B_{11} = 2125$ yield the coordinates of each of these (e.g., sum of tangents $\sqrt{2}$ $\times$ $\sqrt{2} = 2125$, etc.), but not others, in particular not sum of sines or cosines. Similar expressions involving $V_1$ may be found for $A_{12}$ $B_{12}$ $11$ $11$, but not involving $V_1$ for $1/3 = 1/2$, why not? The conical

Dear Friends of the Math-Phys. Correspondence!

The slight one-month delay which most of you experienced in receiving the November issue was caused by the new four-month delay which surface-mail subscribers in Britain and continental Europe will have experienced due to the U.S. coast coast dockworkers' strike, which ended only shortly before Christmas. They will have to wait two issues at the same time, with my apologies, and hope that they will find them worth waiting for.

This issue begins with a series of three recent scientific news-items reviewed by David Black, to which I have added a fourth item without comment. The first is adapted, with the kind approval of author Black and the Royal Astronomical Society, to a forthcoming issue of the Newsletter of the Astronomical Society in America.

I am grateful to Donald Campbell of the Edinburgh Rudolf Steiner School for the selection of the five first three quotations that follow on p. 5, to which I have added four more in a sort of historical memoir. The full text of the von Weizsäcker quotation may be found in Vol. 29 No. 1 of Main Currents in Modern Thought (Sept.-Oct. 1972), reprinted in their Retrospective Issue, Vol. 32 No. 2-3.

There then follows the promised longer article by Louis Locher-Erbant on a very beautiful and little-known theorem due to Euler relating prime distribution to pentagonal numbers. Locher-Erbant was professor of mathematics at the Technische Hochschule in Winterthur for many years, author of a well-received textbook on the calculus as well as numerous articles in the Swiss journal Elemente der Mathematik. He was also the director of the Math.-Astron. Section of the Schweizerische Gesellschaft from 1962 until his death in 1968, and is from the Sternekleiner published by that institution which presents the present article, intended for a non-technical readership, is translated. Research results reported in the article have been brought up to date, and technical notes appended. Thanks also to Professor Lecher's daughter, Frau A. Whitt, for granting permission to make and print the translation, and to Dr. Georg Unger, present director of the Section, for his active support. Thanks also to Prof. R. Stark for the helpful reference to the chapter on partition theory in the book by Hardy and Wright (which has no index).

Istrigued by Thébault's results in the Am. Math. Monthly relating the geometries of the regular heptagon and square, I pursued them further, finding first a trivial generalization to families of concentric 5-cycles in every polygon with odd number of sides (not reported here) before hitting upon what I believe to be a non-trivial generalization to those special odd polygons with $n^2$-odd sides. Ideas then associated with finite projective planes yield results on regular polygons, in particular relating the ratios of the 13- and hexagon.

Finally, there is a poem by John Neame, playfully protesting overly dogmatic statements on the uniqueness of snowflakes. The punctuation is a bit licentious, but you should be able to puzzle it out. Enjoy!

Brian H. Goodlad (Dept. of Biology, Univ. of California, Santa Cruz) writes us that the unidentified "plant genetic material" on p. 19 of issue 20, by scale-considerations (100,000X magnification $\times 10$ microns actual size), is more likely a small molecule than a virus, and in any case some kind of osmotically. Lawrence Edwards points out that the RNA helices are special cases of path-curves too.

We have also received word that George Adams' Universal Forests in Mechanics, investigating the deep projective polarities of kinesis and dynamics, will be released and may be ordered from the Rudolf Steiner Press, 35 Park Rd., London N.W. 875 (22, 98) or St. George Books Service, 8765, Spring Valley, N.Y. 10977. A biography of Oliver Whitcher and collection of Adams' essays was released earlier by Eury Golden Ltd.

With all good wishes for the New Year,

Stephen Eichardt

27 December 1977

Missoula, Montana
One of the great events in the evolution of human consciousness was the gradual shift from Moon-centeredness to Sun-centeredness. The majority of records show that early man thought his spiritual life to be centered on the Moon. Examples of the external evidence for this are the Moon-based calculation of the Jewish calendar and the twenty-eight-foldodd of the Indian and Chinese civilizations. The shift to Sun-centeredness in external, religious matters may be seen as having taken place roughly from the time of the New Kingdom of Egypt to the early Christian era (Julian Calendar, the Manichaean impulse). In an external way, the change of focus finally occurred with the victory of the Copernican model of the universe, which placed the Sun in the center. Rudolf Steiner spoke about aspects of this transition on many occasions.

Because of humanity's long Moon-centered history, one might suspect that relics from that time still persist, hidden somewhere in our being. Although each of us might still possess his Moon orientation to some extent, it should become more evident in a person who was denied his ordinary connection with the Sun. Just such evidence was recently discovered by a group of researchers at Stanford University, and was reported in the October 28 issue of Science.

The person who led researchers to their discovery was denied his connection with the outward Sun through having been born blind. For the last several years, he had experienced periodic inability to conform to societal norms in waking and sleeping. Treatment by hypnosis and drugs did not help.

After 26 days of hospital study, the researchers concluded that he had circadian rhythms of 25.5 hours, "indistinguishable from the period of the lunar day." Furthermore, for the period of the study, "there was a remarkable coincidence between his sleep onset and a local low tide." Attempts to force his body functions back to a 24 hour day rhythm failed.

There was also a Stanford survey of 50 people, all blind to varying extents; 36 of them complained of significant sleep-wake disorder. Other experiments have removed normal time clues from ordinary people, and discovered that they tend to revert to circadian rhythms of around 25 hours.

There is no doubt that there is a powerful strain in science which encourages and demands objectivity, respect for truth, and submission of the personality to higher goals. The question is the extent to which that positive strain is able to find expression in the actual conduct of science. The real test comes when discoveries are made of processes which put tremendous power in the hands of whoever controls them. One hopes for realistic notions especially when the new discoveries hold potential for inflicting unprecedented harm on all of humanity. Only a discovery was made by a team led by Herbert Boyer of the University of California at San Francisco (Science News 122[26]: 170 (November 12, 1987)). The discovery "not only symbolizes all previous genetic-engineering research, but may mark the beginning of a new era in the biological sciences as well." What are the circumstances of this beginning? Two aspects of the circumstances will be described here: the nature of the new discovery, and the conduct of the scientists with regard to their achievement.

and reflection, fourteen positions in all, six having any given corner in common. If we select twelve six having corner 7 in common and delete the two sides of each that meet in that corner, the six remaining sides are precisely the heptagon chords which make up Deser's twisted hexagon, with concyclic quadrilaterals.
What was discovered was a method of persuading a colony of bacteria to produce a human brain hormone. The structure of the hormone was originally deciphered from a five milligram quantity of it, which had been extracted from 560,000 sheep brains. The bacteria were grown in a large volume of yeast extract, with the same amount in relatively short order. The method of persuasion involved constructing a molecule of glucose which codes for the hormone, and then using the gene into a virus or bacterial plasmid. The virus could then be added to the ones already possessed by the bacteria.

"The bacteria needed the new 'work orders' and... like bustling factors, they were engaged in producing hormones."

While Philip Handler, president of the National Academy of Sciences, was bailing out the experiment before a Senate subcommittee as "a scientific triumph of the first order," 10 triumph in the field of economics was being prepared behind the scenes. Two years ago, Boyer founded a company called Genentech to construct synthetic gene sequences that would be used to produce valuable "medicinal" drugs. Genentech paid for Boyer's research through a contract with UCSF. UCSF is applying for patents to protect Boyer's new techniques. UCSF's contract obligates them to license the technology to the company that would pay UCSF royalties on the profits. Meanwhile, the researchers are refusing to discuss anything about their work, including its purely "scientific" aspects. Handler's disclosure came as a surprise to the research team. In the light of the circumstances, it is ironic that Handler made his announcement to the Senate in order to bolster his testimony that recombinant DNA research was not only completely safe, but also highly desirable.

The mood of the materialistic natural philosophy of the Nineteenth Century could be found in all branches of science, but it had its bulwark in physics. The physicists were achieving momentous discovery after momentous discovery. Many of them felt that the time was near when they would put themselves out of work, there being nothing left to discover.

Their unibent spirits and strong momentum brought them not only to their goal, but past it, into a region in which few of their old dogmas were pertinent. First came radioactivity, followed in rapid succession by relativity theory, quantum mechanics, and the unraveling of particle physics. The new discoveries severely tested the materialist faith of the physicists. Reluctant to forsake their faith, they have since made their faith in the old dogmas and redefine beyond recognition the terms of others. Regardless of these efforts of physiologists to preserve its facade, the Spirit of Materialism saw that its days in physics were drawing to a close. But the work had borne fruit: the other sciences, even the humanities, worked to model themselves on the pattern of physics, as it was in the Eighteenth Century.

A typical example of the way the spirit of the old mechanism is trying to establish itself in the humanities appeared in a recent issue of a new journal of the arts which seems to be mostly devoted to mechanism. In the quote is somewhat lengthy, it is the beginning of an entire article, but it gives a good feeling for what this trend is about.

"Abstract: The basic arguments of this paper are that art is not intrinsically mysterious and that there is no reason why art should not serve various functions for computers as well as for human beings. Asking what problems computers might be able to solve is an exciting function of the functions of art for humans from a new perspective. The author suggests that what is needed is a new way of thinking about the nature of art.
works must develop compilers in their brains to decode then (music, however, is a special case to be in machine code in certain extraterrestrial contexts, but that is, already decided). One function of art is then to provide observers with practice in constructing de-coding compilers. Other functions of art are also suggested. It is further argued that a complete model would be paid to semantic features of representational visual art and that from this point of view much artworks can be regarded as a program that incorporates a reduction to a process of recognizing reality from a model and (2) inferring an underlying general theoretical construct that it exemplifies." [Apter, M. J., "Can Computers be Appreciated by Artists?" Leonardo 10(1): 17 -20 (Winter 1977).]

Just how fast that picture of mechanistic physics has been left behind by modern discoveries was made clear yet again in a recent review article on particle physics [Weyl, F. 'Schwarzschild: Fundamental Particles with Charm', Scientific American 237(1): 56 -82 (October 1977)]. Most of the experiments in particle physics are made by observing the collissions between two streams of particles which are circulated in opposite directions at speeds approaching that of light in a specially constructed accelerator. If what happened in those collisions (according to the physicists' mathematical formalism) conformed to the view which most believed even in "science" wave of the mechanistic universe, the outcome of a collision must be governed by a strict causality. If one knows the position and velocity of billiard balls, one can ideally predict the outcome of their collision. It is well known that an element of chance has been introduced into the mechanics of the atom, but even so, the probability can be calculated and probabilistic predictions made. Here, we meet something that goes well beyond all of that. "The annihilation of an electron and a positron [has as its] immediate product a photon, a quantum of electromagnetic energy. The photon decays so quickly that it can never be detected, even in principle (it is called a virtual particle), but it nonetheless determines the properties of all subsequent streams of particle... [The laws can be] seen to be a complete freedom for the creation of any particle, so long as it is accompanied by its own antiparticle." Here, the actors all disappear behind the curtain of brief but decisive moment and change into costumes. When the curtain rises, it rises on a new play. The curtain must sit in the audience along with the rest of us and watch what ensues, for the outcome of this new play is largely in the hands of the actors, and they decide what it is to be - behind the "curtain".

Science News 112(11): 196 (September 13, 1977) reports that Chicago's Argonne National Laboratory "has produced the world's most efficient facility for producing beams of polarized protons - that is, protons with their spin all oriented more or less in the same direction. Normally the protons in an accelerator's beam have their spins oriented randomly. To polarize them takes special arrangements, but to separate the protons in spin is a difficult job. For the most part, experiments of this kind have to stop work very early in their studies, because the spins of the protons are going in different directions.

"This polarized proton beam was struck against a liquid hydrogen target. In the collisions between the beam protons and those in the target, the effects of spin were most pronounced when the bouncing protons came off at a large angle to its original direction. Runs were made at energies of 11.75 to 11.6 billion eV. The combination of high energy and high scattering angle indicates that something rather deep inside the proton is responsible for the observed effect. That is, rather simply, that protons bounce off each other and that their spins are parallel.... When the spins are perpendicular, the protons appear to move right through each other.... The physicists and also the material of matter will have fun with that."
[3] For a number of form $n = a^2 + b^2$ (a, b both integral integers) we have
$(1 + 1/2 + \ldots + 1/2^n)/(1 + 1/2 + \ldots + 1/2^{n-1}) \geq 2$
with equality holding if and only if $a = b = 1$, i.e. $n = 6$.

[4] Because of the divergence of the harmonic series, the number of primes $p, q, r, \ldots$ increases and the values of $a, b, c, \ldots$ grow larger, the left side of [3] exceeds every finite bound.

[5] The product $(1 + 1/2 + \ldots + 1/p)/(1 + 1/2 + \ldots + 1/q)$ tends to $\frac{2}{\pi}$ as $p, q \rightarrow \infty$. As this always less than 2 for $p$ and $q$ odd primes, the indicated result follows.

[6] Let the product of all natural numbers from 1 to 10,001 be denoted by $P$. Then among the ten thousand consecutive numbers $P + 2, P + 3, P + 4, \ldots, P + 10,001$ there is not a single prime number since the first is divisible by 2, the second by 3, the third by 5, and so on.

[7] Unfortunately, there seems to be no literature on the pentagonal triangles which is accessible to a wide circle. For further information on the distribution of prime numbers, one good English work is Hardy and Wright's "Introduction to the Theory of Numbers," and an advanced one Ingham's "The Distribution of Prime Numbers," or Trout's excellent little book on Primes in German.

Translator's Notes:

(1) Such primes of form $1 + 2 + 2^2 + \ldots + 2^{n-1} - 1$ are called Bernoulli primes $p = 2^n - 1$ after the French Minorite Father who successfully found the largest known prime number (the first twelve of them) in the year 1666, by what means it is not known. A necessary but not sufficient condition for $2^n - 1$ to be prime is that $n = 2^k - 1$. His largest correct guess was $2^{37} - 1 = 170,111,236,640,671,971,635,476,274,906,984,736$, whose size, because we can note it so compactly when it is "4,756,411" in the familiar way, we may not appreciate. If we were to write it in longhand, it would be a formidable task that would require many years - and that is only $2^{37}$ with 59 decimal digits. $M_{127}$ (discovered in 1968 by Robinson) has 357 decimal digits, and $M_{57}$ (discovered in 1971 by Tuckerman) 2,089 digits, the ratio of binary exponent to number of decimal digits approximating log 10/log 2.

(2) On p. 11 Locher originally wrote that only "somewhat over 200 prime numbers were known" and entered (in a footnote) the prime 1919. On the bottom of page 12, Locher gives (as corrected in a footnote to the 1959 supplement) Lehmer's revised values for the first 200 primes (known by the year 1929). On the bottom of page 14, Locher gives (as corrected in a footnote to the 1955 Supplement) Lehmer's revised values for the first 2000 primes (known by the year 1955). Since the earlier value of Bernoulli was not very familiar, this is a useful task.

(3) Daudt, in his Elements, gave a simple proof of this over 2000 years ago, using a construction similar to that used in note [6] above: Assume that there exists a least, largest prime number, $p$. Then since number differing by 1 can have no factors other than 1 in common, we have that $p + 1$ cannot be divisible by 2 or 3 or 5 or any other prime.
The Sequence of Natural Numbers
As Art-work of the Spirit

by Louis Locher-Ernst

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1. Geometrical forms such as a triangle, circle, or lens, etc., speak to us directly because we can make them our own without difficulty. Conversely, we feel a sense of satisfaction when we are able to take some initial chaotic state of affairs and bring order into it with the help of suitable geometrical forms. We also feel capable of changing one form into other forms. Through this, the realm of forms gains life. By seeking out the simplest basic forms, the characteristic basic gestures of the form-world, we can even make ourselves an alphabet. One can lift oneself to a language of forms whose syllables connote the experiences of the characteristic basic gestures.

How different the realm of numbers appears to us! We shall consider here only the natural numbers 1, 2, 3, 4, 5, 6, ... Whereas the forms, by virtue of the fact that we can change them into one another, reveal to us a mysterious connection with light and color, the realm of numbers shows itself to us as dark, offering no ready access to experience. Forms permit us to slip into them with our experiencing capacity. It is easy to steep oneself in a 5- or 6-colored star and arrive soon at various combinations. Numbers, on the other hand, have something rigid and unchanging about them that seems unapproachable. Just try to get into the number 9; it forbids us to come too near. Whereas, when we occupy ourselves with forms, those speak to us, numbers become testifiers and draw away from our experimental grasp.

Into this realm of the sequence of numbers 1, 2, 3, ... we wish now to dare a few steps. The first thing we find when we test our awareness of what is going on here, is that we scarcely know what else to report than just 1, 2, 3, ... In so doing, we have the conviction that we could extend the sequence as far as please. To carry this out, we need some sort of processing in a rational process. In the circular order, the culture of the number 10 serves as a basis for this, with the rhythmicity given by its successive powers $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, etc. In our first school-years we become accustomed to arranging every natural number to fit into this system.

As ingenious as this arrangement is, the accustomedness to it, if one thinks of furthering, can lead to a misunderstanding which we wish to clear out of the way. It will suffice to explain it by a few simple examples. If one considers the nine times table:

<table>
<thead>
<tr>
<th>1×9</th>
<th>2×9</th>
<th>3×9</th>
<th>4×9</th>
<th>5×9</th>
<th>6×9</th>
<th>7×9</th>
<th>8×9</th>
<th>9×9</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>18</td>
<td>27</td>
<td>36</td>
<td>45</td>
<td>54</td>
<td>63</td>
<td>72</td>
<td>81</td>
</tr>
</tbody>
</table>

then one notices that disregarding the parenthesized case — the sum of the digits is always nine: $1 + 8 = 9$, $2 + 7 = 9$, ..., $7 + 2 = 9$. One might easily suppose that in this is expressed some particularly remarkable property of the number nine.

Another example: Take an arbitrary three-digit number, exchange the first and third digit with one another, and see the difference between the one and the new number. Then take the resulting number, interchange first and third digits again, and add that difference to this new number. The result in every case is either 0 or, in some special case, one, e.g.:

<table>
<thead>
<tr>
<th>392</th>
<th>659</th>
<th>503</th>
<th>396</th>
<th>100</th>
<th>909</th>
</tr>
</thead>
<tbody>
<tr>
<td>567</td>
<td>521</td>
<td>182</td>
<td>699</td>
<td>001</td>
<td>909</td>
</tr>
<tr>
<td>899</td>
<td>1087</td>
<td>936</td>
<td>1099</td>
<td>999</td>
<td>1089</td>
</tr>
</tbody>
</table>

The following comments contain some supplementary indications.

[1] To find the digits of the number $n$, written with help of base $b$, one forms the chain $n = a_0 \times b^0 + a_1 \times b^1 + a_2 \times b^2 + \ldots + a_b \times b^b$, where the $a$s are numbers, with $a_0 = b$ and $a_b = 1$, up to $b + a_b = a_b + b = b + 1$, then $a_0 = b + 1$, and so on. One then sees that e.g. $b = 10 = 2 + 8 = 2^1 + 2^3$, $b + 1 = 11 = 2 + 9 = 2^1 + 2^3 + 2^1$, etc. When one forms the sum $s = 1 + b + b^2 + \ldots + b^b$, one then sees that e.g. $b = 10 = 2 + 8 = 2^1 + 2^3$, $b + 1 = 11 = 2 + 9 = 2^1 + 2^3 + 2^1$, etc.

If the number $n$ has the natural representation

$$n = p_1 \times 2^{q_1} \cdots (\text{where } p_i \times 2^{q_i} \cdots \text{ are primes})$$

then its divisor-sum has the value

$$s = (1 + \frac{1}{2^1} + \frac{1}{2^{q_1}}) \cdots (1 + \frac{1}{2^{q_1}} + \frac{1}{2^{q_2}} + \frac{1}{2^{q_3}} + \frac{1}{2^{q_4}} + \frac{1}{2^{q_5}} + \frac{1}{2^{q_6}} + \cdots + \frac{1}{2^{q_b}} + 1) \times (1 + \frac{1}{2^1} + \frac{1}{2^{q_1}} + \frac{1}{2^{q_2}} + \frac{1}{2^{q_3}} + \frac{1}{2^{q_4}} + \frac{1}{2^{q_5}} + \cdots + \frac{1}{2^{q_b}} + 1)$$

as may be seen immediately by multiplying out. One can also write

$$s = (1 + \frac{1}{2^1} + \frac{1}{2^{q_1}} + \frac{1}{2^{q_2}} + \frac{1}{2^{q_3}} + \frac{1}{2^{q_4}} + \frac{1}{2^{q_5}} + \cdots + \frac{1}{2^{q_b}} + 1) \times (1 + \frac{1}{2^1} + \frac{1}{2^{q_1}} + \frac{1}{2^{q_2}} + \frac{1}{2^{q_3}} + \frac{1}{2^{q_4}} + \frac{1}{2^{q_5}} + \cdots + \frac{1}{2^{q_b}} + 1)$$

The number $n$ is thus abundant if and only if

$$s = (1 + \frac{1}{2^1} + \frac{1}{2^{q_1}} + \frac{1}{2^{q_2}} + \frac{1}{2^{q_3}} + \frac{1}{2^{q_4}} + \frac{1}{2^{q_5}} + \cdots + \frac{1}{2^{q_b}} + 1) \times (1 + \frac{1}{2^1} + \frac{1}{2^{q_1}} + \frac{1}{2^{q_2}} + \frac{1}{2^{q_3}} + \frac{1}{2^{q_4}} + \frac{1}{2^{q_5}} + \cdots + \frac{1}{2^{q_b}} + 1) \geq 2n.$$

For the sake of convenience, we shall use the following definitions:

- An abundant number is a number $n$ such that $s > 2n$.
- A perfect number is a number $n$ such that $s = 2n$.
- A deficient number is a number $n$ such that $s < 2n$.

Thus, for a number to be abundant, it must have a divisor-sum that is at least twice the number itself.

6. The matters set forth here, to which many others could be added, gain a heightened significance if one is prepared to enter into a battle going on beneath the wrestling for understanding of the newer findings of physics and lending today toward a certain climax, although it made its first appearance historically in the time of the Scholastics. Do the concepts grasped by thinking indicate real entities, and if they are only as mirror-images in conscious understanding, or are they mere nomes, abstractions formed from the experiences of the senses? Without being able to go into particulars here, but only mentioned that a number of results in the mathematics of our century have made it particularly clear that certain concepts are merely nominalistic, while others possess realistic significance. For example in set theory we must distinguish clearly between the "collection" (Gesamtheit) as mere noun and the "set" (Menge) as entity if we are to overcome the so-called antinomies, as discovered by P. Finzler. Another important result is the theorem of Skolem (1929) from which it follows that the sequence of natural numbers may not be characterized by any finite axiomatization, but any such finite axiomatization would not permit interpretations that are inequivalent to the natural numbers.

The concept "natural number" is of nominalistic nature, a mere abstraction, while individual numbers, as E. Steiner pointed out on occasion, represent entities. One may also consider the thought how completely numbers place themselves at our disposal, remaining all of their own, existing ever serving beings.

In any case one will no longer harbor the commonplace view that the sequence of natural numbers is something hand, even to be disdainful, after gaining insight into its wonderful weaving. This may vanish a feeling. Even if that is felt to be hard to grasp, even if it only shimmers through as though glimmering from a great distance, we have been permitted to take a look into the workshop of the Cherubim, from whence the number-baggage take their origin.

Let us consider, too, what great significance it could have if this feeling were to be wakened in young persons. Whoever has once been permitted to take such a look will never shun the number-sequence by slavish labor. It represents a most wonderful kind of art-work of the spirit. To be sure, it is an art-work that springs from worlds of necessity, but whoever wishes to be freely creative must, above all, learn to fit in readily with necessities.
Let us now think of some number \( n \) as given, and find by how much it exceeds all lesser pentagonal numbers. E.g. \( n = 30 \) yields the excesses:
\[
\begin{align*}
\text{n - 1} & = 29, \\
\text{n - 2} & = 28, \\
\text{n - 5} & = 25, \\
\text{n - 7} & = 23, \\
\text{n - 12} & = 15, \\
\text{n - 22} & = 8, \\
\text{n - 26} & = 2.
\end{align*}
\]
We take these excesses and check their divisor-sums. Either by calculation or from the list:
\[
\begin{align*}
\varphi(29) & = 28, \\
\varphi(28) & = 21, \\
\varphi(25) & = 20, \\
\varphi(23) & = 22, \\
\varphi(15) & = 14, \\
\varphi(8) & = 4, \\
\varphi(2) & = 1.
\end{align*}
\]
Now we add the first two of these divisor-sums, subtract the next two, etc.
\[
\begin{align*}
30 - 56 & = 31 - 2h + 39 + 2h - 15 - 7 = 72. \\
\text{The result is the divisor-sum of 30.}
\end{align*}
\]
For \( n = 38 \) the excesses are:
\[
\begin{align*}
\text{n - 2} & = 36, \\
\text{n - 5} & = 31, \\
\text{n - 7} & = 28, \\
\text{n - 12} & = 16, \\
\text{n - 22} & = 2, \\
\text{n - 26} & = 2.
\end{align*}
\]
We add and subtracting alternate pairs in this series yields:
\[
\begin{align*}
\text{\varphi(36) + \varphi(31) + \varphi(28) + \varphi(16) + \varphi(2) + \varphi(2) = 350.}
\end{align*}
\]
For \( n = 16 \) the excesses are:
\[
\begin{align*}
\text{n - 2} & = 14, \\
\text{n - 3} & = 13, \\
\text{n - 4} & = 12, \\
\text{n - 7} & = 9, \\
\text{n - 12} & = 4, \\
\text{n - 22} & = 2.
\end{align*}
\]
We add and subtracting alternate pairs in this series yields:
\[
\begin{align*}
\text{\varphi(14) + \varphi(13) + \varphi(12) + \varphi(9) + \varphi(4) + \varphi(2) = 268.}
\end{align*}
\]
And we find our divisor-sums:
\[
\begin{align*}
\varphi(a) & = \varphi(b - 2) + \varphi(b - 5) + \varphi(b - 7) + \varphi(b - 12) + \varphi(b - 15) + \ldots
\end{align*}
\]
This altogether remarkable state of affairs holds true for every number \( n \) which is not a pentagonal number. A simple convention makes it possible to encapsulate the pentagonal numbers as well. If \( n \) is such a number, then there occurs in the end an excess of zero. As divisor-sums of zero, we must take now the pentagonal number \( a \) itself.

The law extended in this fashion is then valid for every natural number.

For example \( n = 28 \) Excesses 12, 15, 8, 6, 5, 4, 2, 0, with divisor-sums 12, 15, 8, 6, 5, 4, 2, 1, yields indeed \( 2\cdot12 + 15 + 5 + 2 + 1 = 52 \).

The law discovered under the same pentagonal theorem by L. Euler (between 1732 and 1750), unfortunately it has attracted very little attention, even among mathematicians. There are even many today who know nothing of it, although it must be counted as one of the most beautiful discoveries of modern times. Its proof requires considerable means.

To the right of the equals sign in \([\text{?}]\) there occur divisor-sums of numbers which are all smaller than \( n \). The structures of these numbers will be more than that of the number itself, but their content is the divisor-sum of \( n \).

Now we shall show a property of the number 28 which is independant of this means of representation. Altogether, the divisor-sums of 28 are 1, 2, 4, 7, 11, 28. Their sum is 56. The sum of the divisors which are smaller than the number itself, i.e. the sum of 1, 2, 4, 7, 11, and 28, is 56. It is possible to express this more concisely, let us call the sum of all the divisors of a number, excepting the number itself, i.e. the content of the number, let the sum of all the divisors, including the number itself, be designated henceforth as \( s \). The number 28 has the property that its content is equal to the number itself, the value of \( s \) is therefore 28 + 28 = 56, twice that of the number.

The content of a number and also its divisor-sum \( s \) have evidently nothing to do with parceling. On pages 8 and 9 there is a table of the numbers from 1 to 30, with their corresponding divisor-sums. For example, to the right of 30 there stands the number 72. And in fact, all of the divisors of 30 namely 1, 2, 3, 5, 6, 10, 15, and 30 yield 72 as their sum. The content of 30 is 56. Now the content of a number can be either smaller or larger than the number itself, or it can be equal to it. The number 15 with content 1 + 3 + 5 = 9 belongs to the former class, the number 20 with content 1 + 2 + 4 + 5 + 10 = 22 to the second class, while 28 with content 28 is a member of the third. If one adds to the content the number itself, then the three classes are characterized by the fact that the sum \( s \) of the divisors of the number \( n \) is either less than or greater than its double \( 2s \), or else it is equal to that double value. The sole exception to this is the number one whose content coincides with \( s \).
5. According to the theorem just described, we know the approximate number of primes occurring in the sequence 1, 2, 3, 4, 5, 6 to n, becoming more accurate as we increase the length of the partial segment of the sequence of all natural numbers, i.e., with increasing n, forming the true value of the ratio, tending toward 1:1. But it is precisely the departure from this rule which merits our real interest. Only in these do we find expression of the individual structures of the numbers. Now, it is in fact possible to give an exact law determining the distribution of primes as well, albeit in a form which is scarcely usable for direct computation. A distinguished role is played herein by the so-called pentagonal numbers. There are two kinds of pentagonal numbers, which may be determined as follows:

\[ C^2 = 0 \]

\[ 1^2 = 1 \]
\[ 1 + 2^2 = 5 \]
\[ 1 + 2 + 3^2 = 12 \]
\[ 1 + 2 + 3 + 4^2 = 22 \]
\[ 1 + 2 + 3 + 4 + 5^2 = 35 \]
\[ 1 + 2 + 3 + 4 + 5 + 6^2 = 51 \]
\[ 1 + 2 + 3 + 4 + 5 + 6 + 7^2 = 77 \]

The following figures show how these numbers arise geometrically:

They may also be represented by means of regular pentagons, and what is more, and particularly important, but would lead too far here, they may be characterized by a peculiar, purely arithmetical property:

If one orders them in the sequence

...77, 57, 40, 26, 15, 7, 2, 1, 2, 3, 5, 12, 22, 35, 51, 70, ...

and forms in each case the difference of two successive numbers, subtracting from each the preceding number, then one obtains the sequence

... -20, -17, -13, -9, -5, -2, 3, 12, 22, 35, 51, 70, ...

This displays throughout a constant step of 3 from term to term.

We now mark off these pentagonal numbers in the sequence of natural numbers:

\[ a \quad A \quad b \quad 5 \quad b \quad 7 \quad b \quad 9 \quad b \quad 10 \quad b \quad 11 \quad b \quad 12 \]
\[ 13 \quad 15 \quad 16 \quad 17 \quad 18 \quad 19 \quad 20 \quad 21 \quad 22 \quad 23 \quad 24 \]
\[ 25 \quad 26 \quad 27 \quad 28 \quad 29 \quad 30 \quad 31 \quad 32 \quad 33 \quad 34 \quad 35 \]
\[ 37 \quad 38 \quad 39 \quad 40 \quad 42 \quad 43 \quad 44 \quad 45 \quad 46 \quad 47 \quad 48 \]
\[ 49 \quad 50 \quad 52 \quad 53 \quad 54 \quad 56 \quad 57 \quad 58 \quad 59 \quad 60 \]
\[ 61 \quad 62 \quad 63 \quad 64 \quad 65 \quad 66 \quad 67 \quad 68 \quad 69 \quad 70 \quad 71 \quad 72 \]
One can easily specify ten thousand successive natural numbers, or a single one of which is prime. Instead of ten thousand, one could pick any other number; that is, there are prime number gaps exceeding any arbitrary length. (6)*

It sometimes happens that both neighbors of a normal abundant number are prime numbers; for example, 11 and 13, 29 and 31 are such twins. It is not known to this day whether there are only finitely many such pairs of twins. As far as direct inspection permits us to tell, they seem to keep reappearing, e.g., 1,001,001,001,001,001,001,001,001,001.

Between any number n and its double 2n there is at least one prime number. It can be shown by elementary but fairly complicated means that from n = 720 on there are always at least 100 prime numbers between n and 2n.

When we ask about the distribution of primes within the natural sequence, we must always keep in mind that we are dealing with the kind of distribution in relation to the size of number, although size alone has little to do with individual structure. The following example of three nearly like-size numbers is quite instructive:

\[ 730,273 = 73 \times 10,001, \]
\[ 730,277 = 71 \times 1,001, \]
\[ 730,279 = 73 \times 1,001. \]

Even closely neighboring numbers can display totally different structures. Despite this circumstance, we are able to give a rule about the average distribution. Let the number of primes occurring in the sequence from 1 to n be called P(n), e.g., P(100) = 25, P(200) = 36, P(300) = 62.

The order of magnitude of P(n) is then given by

\[ P(n) \approx \frac{n}{\log n} \]

This approximation becomes ever better as n increases. Exactly expressed, we have the remarkable statement: The value of \( \frac{1}{\log n} \) P(n) (\( n = 2, 3, 4, 5, \ldots \))

tends toward 1 as n becomes arbitrarily large.

This rule for the average distribution, which was first conjectured - in somewhat sharper form (\#6) - by the not yet 20-year-old G. H. Hardy, was not able to be verified until 1896, and then only by application of the most difficult means. To make the matter more readily understandable, we have given it here in an elementary form. A few years ago a somewhat simpler proof was found, but one which still requires a rather extensive mathematical training to understand.

To give the reader an idea of the approximation, we list together in the following table the exact number P(n) of primes up to n = 1,000, up to 10,000, and even on, with the values of the sum 1/2 + 1/3 + 1/4 + 1/5 + \ldots + 1/n and the product indicated above, tending ever closer to 1.

<table>
<thead>
<tr>
<th>n</th>
<th>P(n)</th>
<th>1/2 + 1/3 + \ldots + 1/n</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,000</td>
<td>168</td>
<td>1.089...</td>
</tr>
<tr>
<td>10,000</td>
<td>1,229</td>
<td>1.080...</td>
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<tr>
<td>100,000</td>
<td>9,592</td>
<td>1.063...</td>
</tr>
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<td>1,000,000</td>
<td>78,198</td>
<td>1.051...</td>
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<td>5,000,000</td>
<td>566,597</td>
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<td>10,000,000</td>
<td>3,762,155</td>
<td>1.024...</td>
</tr>
<tr>
<td>100,000,000</td>
<td>31,337,088</td>
<td>1.015...</td>
</tr>
<tr>
<td>500,000,000</td>
<td>251,357,088</td>
<td>1.007...</td>
</tr>
</tbody>
</table>

(*) \#6

The sum of all divisors of numbers from 181 to 360:

\[ 181^p \]
\[ 182^p \]
\[ 183^p \]
\[ 184^a \]
\[ 185^b \]
\[ 186^a \]
\[ 187^b \]
\[ 188^a \]
\[ 189^b \]
\[ 190^a \]
\[ 191^p \]
\[ 192^a \]
\[ 193^b \]
\[ 194^b \]
\[ 195^a \]
\[ 196^a \]
\[ 197^p \]
\[ 198^a \]
\[ 199^b \]
\[ 200^a \]
\[ 201^p \]
\[ 202^p \]
\[ 203^a \]
\[ 204^a \]
\[ 205^b \]
\[ 206^a \]
\[ 207^a \]
\[ 208^a \]
\[ 209^a \]
\[ 210^a \]
\[ 211^a \]
\[ 212^a \]
\[ 213^a \]
\[ 214^a \]
\[ 215^a \]
\[ 216^a \]
We call numbers of the first kind poor or deficient, those of the second kind rich or abundant. Those of the third kind have been called since antiquity perfect, and in the list from 2 to 360 one finds only two perfect numbers, namely 6 and 28.

It is not difficult to prove [3] that the prime numbers of the form \( n = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \) will be seen, is the first abundant number. The next abundant ones are 18, 20, 24, 30, 36, 60, 20, 106, 122, 156.

We call all the normal abundant numbers. Some of them are indicated in the list by an "n" placed to the right. Besides these normal ones there are other abundant numbers, for example 100 and 306; they are indicated by an "m" placed to the left.

Our list displays no odd abundant number. Such numbers do exist, however, and are called odd abundant numbers. Among the first abundant numbers we note 12, 14, 18, 20, 24, 30, 36, 60, 70, 72, 80, 90, 100, 120, 140, 150, 160, 180, 200, 210, 220, 240, 260, 300, 310, 330, 350, 390, 400, 420, 440, 450, 500, etc. It is abundant.

Every multiple of an abnormal number is again abundant.

There are numbers whose abundance extends in every way, for example, 100, 120, etc. is abundant.

The least content which a number can have is one. The poorest, or most deficient, numbers have this property; they are the prime numbers.

For a reason, the first divisible number is not counted among the primes. Every prime number has the only content of the prime number. The divisor-sum \( a + b = 12 \) with divisor-sum 12.

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For reasons, the first divisible number is not counted among the primes. Every prime number has the only content of the prime number. The divisor-sum \( a + b = 12 \) with divisor-sum 12.
We call numbers of the first kind poor or deficient, those of the second kind rich or abundant. Those of the third kind have been called since antiquity perfect numbers. The list of 2 to 360 one finds only two perfect numbers, namely 6 and 28.

Suppose that in the set of numbers 1, 2, 3, 4, 5, 6, ... some sort of simple perpetual law would hold for the divisors of each number, also for the divisor-sum $s$. The matter is, however, extraordinarily complicated, as the following considerations will show. We call the perfect numbers known as of 1956 are 1, 2, 6, 12, 16, 18, 28, 36, 42. If we let $s = 28$, we find that $s = 1 + 2 + 4 + 7 + 14 + 28 = 56$.

It is not difficult to give examples of entire classes of abundant numbers which do not belong to the normal ones. We mention in this regard only the following facts:

Every number of the form $15^k$ (such as $15 = 20, 125 = 100, 125 = 500$, etc.) is abundant.

Every multiple of an abundant number is again abundant.

There are numbers whose abundance exceeds any given bound, i.e., numbers for which $s$ is greater than the tenfold or hundredfold etc. value of the number itself. [A]

The least abundant which a number can have is one. The poorest, or most deficient, numbers have this property; they are the prime numbers.

For any number less than 1, a prime number is not counted among the contents, and every prime number $p$ has only the content 1 and the divisor-sum $s = p + 1$.

In the list, prime numbers are indicated by a letter "p". It will be noticed at once that they are always neighbors of odd numbers. In fact, every prime number is either of the form $6n + 1$ or $6n - 1$, since the other possible forms $6n + 2, 6n - 2, 6n + 3, 6n - 3$ all represent numbers which are either divisible by 2 or 3. But not every neighbor of a normal abundant number is a prime.

It may be represented as a product of most abundant numbers, the representation being unique up to order of factors. For every number $n$ there is thus a natural representation $n = p_1^q_1 \cdot \ldots \cdot p_r^q_r$, where $p_1, q_1, \ldots, p_r, q_r$ are prime numbers. For example, 360 = $2^3 \cdot 3^2 \cdot 5$.

The normal deficient numbers of the prine numbers and all their powers are $1, 7, 25, 49, 121, 169, \ldots$. But beside these there are entire classes of further deficient numbers. It is not known whether all numbers consisting of two odd primes are deficient, for example, all numbers of the form $10n + 1$, or $10n - 1$.

We call numbers of the form $5n \pm 1$ deficient. It is not known whether all numbers of this kind are deficient. This is not the case for $n = 5, 10, 15, \ldots$.

In the list of numbers known as of 1956 are 1, 2, 6, 12, 16, 18, 28, 36, 42. If we let $s = 28$, we find that $s = 1 + 2 + 4 + 7 + 14 + 28 = 56$.

This is still worth pondering today.

The six smallest pairs of friendly or amicable numbers are

1. 220, 284
2. 1184, 1210
3. 2620, 2924
4. 5020, 5564
5. 6232, 6368

Some 100 pairs are known. It is not known whether any general law holds for their structure, lending special importance to certain primes, let it be expressly observed. (#)

The obvious question, in what manner the primes, the poorest numbers, are distributed in the sequence of integers, led to investigations of the deepest and most difficult kind. Indeed, this question poses a strange riddle to our thinking. If we count 1, 2, 3, etc., to 18, may successive 18s, this period may be subdivided into various shorter periods, e.g., twice nine days, or three times six days. If we take, however, the past 18 days, such a division into sub-periods of equal length is not possible. It becomes 12 days and 22 days, while 23 days can serve only as a whole period-unit. It would seem as though something so conceptually simple and perfectly clear as prime numbers should pose no difficulty to our thinking in trying to grasp the law of their appearance within the progressive sequence. To be sure, it is easy to see that the sequence of primes never comes to an end. (wee) In order to convey some impression, we may mention the following data.

Between 1 and 100 there are 25 prime numbers, between 100 and 200 there are 22 of them, between 200 and 300 there are 16, between 36100 and 26000 only 1, but between 299000 and 300000 again 9 of them.
One can easily specify ten thousand successive natural numbers, or a single one of which is prime. Instead of ten thousand, one could pick any other number; that is, there are prime number gaps exceeding any arbitrary length. [6] —

It sometimes happens that both neighbors of a normal abundant number are prime numbers; for example, 11 and 13, 29 and 31 are such twins. It is not known to this day whether there are only finitely many such pairs of twins. As far as direct inspection permits us to tell, they seem to keep reappearance, e.g., 1,000,000,000,000,000,007, 1,000,000,000,000,000,009, 1,000,000,000,000,000,011 and 1,000,000,000,000,000,013.

Between any number n and its double 2n there is at least one prime number. It can be shown by elementary but fairly complicated means that from n = 726 on there are always at least 100 prime numbers between n and 2n.

When we ask about the distribution of primes within the natural sequence, we must always bear in mind that we are dealing with the kind of distribution in relation to the size of number, although size alone has little to do with individual structure. The following example of three nearly like-size numbers is quite instructive:

370,273 = 137-79-109,
370,277 = 17-21-39,17,
370,279 = 7-13-331.

Even closely neighboring numbers can display totally different structures. Despite this circumstance, we are able to give a rule about the average distribution. Let the number of primes occurring in the sequence from 1 to n be called P(n), e.g., P(100) = 25, P(200) = 36, P(300) = 62.

The order of magnitude of P(n) is then given by

P(n) = n (1/2/3/4/5/6/7/8/9/10/11/12/13/14/15/16/17/18/19/20/21/22/23/24/25/26/27/28/29/30)

This approximation becomes ever better as n increases. Exactly expressed, we have the remarkable statement: The value of

\[ \frac{1}{2/3/4/5/6/7/8/9/10/11/12/13/14/15/16/17/18/19/20/21/22/23/24/25/26/27/28/29/30} \]

tends toward 1 as n becomes arbitrarily large.

Our rule for the average distribution, which was first conjectured — in somewhat sharper form (see next), by the not yet 20-year-old G. F. Gauss, was not able to be verified until 1896, and then only by application of the most difficult means. To make the matter more readily understandable, we have given it here as an elementary form. A few years ago a somewhat simpler proof was found, but one which still requires a rather extensive mathematical training to understand.

To give the reader an idea of the approximation, we list together in the following table the exact number P(n) of primes up to n = 1,000, up to 30,000, and up on, with the values of the sum 1/2 + 1/3 + 1/4 + 1/5 + 1/n and the product indicated above, tending ever closer to 1.

<table>
<thead>
<tr>
<th>n</th>
<th>P(n)</th>
<th>sum</th>
<th>product</th>
</tr>
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<tbody>
<tr>
<td>1,000</td>
<td>168</td>
<td>1.089</td>
<td>1.089</td>
</tr>
<tr>
<td>10,000</td>
<td>1,229</td>
<td>1.080</td>
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<td>1,000,000</td>
<td>78,198</td>
<td>1.051</td>
<td>1.051</td>
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<td>1.043</td>
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<tr>
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<td>5,763,185</td>
<td>1.036</td>
<td>1.036</td>
</tr>
<tr>
<td>1,000,000,000</td>
<td>50,847,531</td>
<td>1.032</td>
<td>1.032</td>
</tr>
</tbody>
</table>

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The sum of all divisors of numbers from 181 to 360:

The sum of all divisors of numbers from 1 to 180

<table>
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5. According to the theorem just described, we know the approximate number of primes occurring in the sequence 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, ..., 100. ...
Let us now think of some number $n$ as given, and find by how much it exceeds all lesser pentagonal numbers. E.g. $n = 30$ yields the excesses $n - 1 = 29$, $n - 2 = 28$, $n - 5 = 25$, $n - 7 = 23$, $n - 12 = 18$, $n - 15 = 15$, $n - 22 = 8$, $n - 26 = 4$.

We take these excesses and check their divisor-sums $s$. Either by calculation or from the list:

- $s_{29} = 30$, $s_{28} = 56$, $s_{25} = 51$, $s_{23} = 24$, $s_{18} = 39$, $s_{15} = 23$, $s_{15} = 7$.

Now we add the first two of these divisor-sums, subtract the next two, and add the next two, etc.:

$$30 + 56 = 29 + 39 + 23 + 15 = 72.$$ 

The result is the divisor-sum of 30.

For $n = 38$ the excesses are: 17, 36, 33, 31, 26, 23, 16, 12, 5. The divisor-sums of this zero, we must take now the pentagonal number $n - 1$.

Adding and subtracting alternate pairs in this series yields

$$38 + 91 = 129 + 32 + 62 + 31 + 28 + 15 + 8 + 5.$$ 

In general, if $n$ yields the excesses $n - 2$, $n - 5$, $n - 7$, $n - 12$, $n - 15$, etc., and we call their divisor-sums $s_{n-2}$, $s_{n-5}$, $s_{n-7}$, $s_{n-12}$, $s_{n-15}$, etc., then we have the law

$$s_n = s_{n-1} - s_{n-2} - s_{n-5} - s_{n-7} - s_{n-12} - s_{n-15} - \cdots.$$ 

This altogether remarkable state of affairs holds true for every number $n$ which is not a pentagonal number. A simple convention makes it possible to enunciate the pentagonal numbers as well. If $n$ is such a number, then there occurs in the end an excess of zero. As divisor-sum of this zero, we must take now the pentagonal number $n - 1$.

The law extended in this fashion is then valid for every natural number.

For example $n = 12$: Excesses 15, 10, 7, 5, 0, which yield 12 + 17 - 8 + 6 - 12 = 25 in $s_{12}$. The pentagonal number is 12.

Similarly as discovered under the same pentagonal theorem by L. Ruler (between 1711 and 1750). Unfortunately it has attracted very little attention, even among mathematicians. There are even many today who know nothing of it, although it must be counted as one of the most beautiful discoveries of modern times. Its proof requires considerable means.

To the right of the equals-sign in [9] there occur divisor-sums of numbers which are all smaller than $n$. The structures of the excess-numbers true predetermine the divisor-sum of $n$. The number $n$ is therefore prime if and only if the resulting value of $s_n$ is $n + 1$.

That is to say: The structures of the excess-numbers, also determine whether $n$ is prime or not, steps 3 in the sequence of second differences, play an essential part in this weving. It is by the law which we have given that the strange irregular-appearing behavior of the divisor-sums $s_2$, $s_3$, $s_5$, $s_7$, $s_{12}$, etc., of successive numbers and in particular the appearance of prime numbers in after regularity.

Here, too, one is surprised and supposes this to be some particularly reproducible property of the number 1209. It would be easy to give a whole list of similar examples.

But such phenomena have nothing at all to do with individual properties of the numbers in question, depending instead on the choices of base 10. As a first requirement for concentrating on individual properties we must free ourselves from being tied to the parceling-basis, although we shall naturally continue to write all numbers in the base 10 system. But first we wish to show how a given number can be written in other systems if $b$ is the basis, then instead of the powers

$$10^0, 10^1, 10^2, 10^3, \ldots$$

we use the powers

$$b^0, b^1, b^2, b^3, \ldots$$

to arrange our work. If for example $n = 333(10)$ is to be expressed in the system with base $b = 6$, one carries out the following divisions:

$$333 = 6^2 + 1$$
$$57 = 6 + 1$$
$$9 = 6 + 1 + 1$$
$$3 = 6 + 1 + 1$$

Then we have $n = 333(6)$. The remainder from the first division represents the first digit; counting from the right. And in fact we have

$$1 + 3 + 6 + 1 + 6 + 1 = 333(10).$$

In order to express $333(10)$ in the systems with bases $b = 3$ or $b = 7$ [1] one carries out the following divisions:

$$333 = 74(3) + 1$$
$$74 = 3 + 1$$
$$1 = 3 + 1$$

Then we have $333(3) = 112, 201(3)$ as well as $1000(7)$. And in fact

$$1 + 0 + 3 + 2 + 3^2 + 0 + 7 + 1 + 7 = 333(10) = 1^2 + 7^2.$$ 

2. The digits which any number displays depend on the choice of parceling. On pages 8 and 9 there is given a list of the numbers from the number 20 which is independent of this means of representation. Altogether, the divisors of 20 are 1, 2, 4, 5, 10, 20. Their sum is 50. The divisors which are smaller than the number itself, i.e., the sum of 1, 2, 4, 5, 10, 20, is 28. In order to be able to express ourselves more concisely, let us call the sum of all the divisors of a number, excepting the number itself, the content of the number. Let the sum of all the divisors, including the number itself, be designated henceforth as $s$. The number 28 has the property that its content is equal to the number itself; the value of $s$ is therefore 28, $28 = 50$, twice the number.

The content of a number and also its divisor-sum $s$ have evidently nothing to do with parceling. On pages 8 and 9 there is given a list of the numbers from 1 to 30 whose divisor-sums are divisible by 3. For example, to the right of 30 there stands the number 72. And if in all, of the divisors of 30 namely 1, 2, 3, 5, 6, 10, 15 and 30 yields that their sum. The content of 30 is $30^2$ [2].

Now the content of a number can be either smaller or larger than the number itself, or it can be equal to it. The number 15 with content 1 + 3 + 5 = 9 belongs to the former class, the number 20 with content 2 + 4 + 5 + 10 = 22 to the second class, while 28 with content 28 is a number of the third. If one adds to the content the number itself, then the three classes are characterized by the fact that the sum $s$ of the divisors of the number is either less than or greater than its double 2s, or else it is equal to that double value. The sole exception to this is the number $n$ whose content coincides with $n$. 

[7]
The sequence of natural numbers

as art-work of the spirit

by Louis Locher-Ernst

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Geometrical forms such as a triangle, circle, or lemniscate, etc., speak to us directly because we can make them clear to our own without difficulty. Conversely, we feel a sense of satisfaction when we are able to take some initially chaotic state of affairs and bring order into it with the help of suitable geometrical forms. We also feel capable of changing one form into other forms. Through this, the realm of forms gains life. By seeking out the simplest basic forms and bringing the characteristic basic gesture of the form-world, we can even make ourselves an alphabet. One can lift oneself to a language of forms whose syllables connote the experience of the characteristic basic gestures.

How different the realms of numbers appear to us! We shall consider here only the natural numbers 1, 2, 3, 4, 5, 6, ... Whereas the form, by virtue of the fact that we can change them into another, reveals to us a mysterious connection with light and color, the realm of numbers shows itself to us as dark, offering no ready access to experience. Forms permit us to slip into them with our experiencing capacity. It is easy to sweep oneself in a 5- or 6-pointed star and arrive soon at various monstrosities. Numbers, on the other hand, have something rigid and unchanging about them that seems unapproachable. Just try to get into the number 9; it forbids us come too near. Whereas, when we occupy ourselves with forms, those speak to us, numbers become testy and draw away from our experiential grasp.

Into this realm of the sequence of numbers 1, 2, 3, ... we wish now to dare a few steps. The first thing we will do is test our awareness of what is going on here, that we scarcely know what else to report than just 1, 2, 3, ... . In so doing, we have the conviction that we could extend the sequence as far as we please. To carry this out, we need some sort of process in a rhythmic process. In the circular forms of the artistic culture the number 10 serves as a basis for this, with the rhythmization by its successive powers 1\(^1\), 10\(^1\), 100\(^1\), 1000\(^1\), etc. In our first school-years we become accustomed to arranging every natural number to fit into this system. An ingenious as this arrangement is, the accustomedness to it, if one thinks no further, can lead to a misunderstanding which we wish to clear out of the way. It will suffice to explain it by a few simple examples.

If one considers the nine times table,

\[1 \times 1 = 1, 1 \times 2 = 2, 1 \times 3 = 3, ... \]

then one notices that disregarding the parenthesis case — the sum of the digits is always nine: \(1 + 8 = 9, 2 + 7 = 9, ... 7 + 1 = 8\). One might easily suppose that in this is expressed some particularly remarkable property of the number nine.

Another example: Take an arbitrary three-digit number, exchange the first and third digit with one another, and find the difference between the given and the new number. Then take the resulting number, interchange first and third digits again, and add that difference to this new number. In every case is either 0 or, in one special case, zero. E.g.

\[
\begin{align*}
123 & \quad 213 & \quad 900 \\
245 & \quad 542 & \quad 297 \\
367 & \quad 763 & \quad 400
\end{align*}
\]

The matter not forth here, to which many others could be added, gain a heightened significance if one is prepared to enter into a battle going on beneath the wrestling for understanding the newer findings of physics and tending today toward a certain climax, although it made its first appearance historically in the time of the Scholasticism. Do the concepts generated by thinking indite real entities, allying itself only as mirror-images in conscious understanding, or are they mere notions, abstractions formed from the experiences of the senses? Without being able to go into particulars here, but only mentioning that a number of results in the mathematics of our century have made it particularly clear that certain concepts are merely nominalistic, while others possess realistic significance. For example in set theory we must distinguish clearly between the "collection" (Gesamtheit) as mere noun and the "set" (Menge) as entity if we are to overcome the so-called antinomies, as discovered by P. Finisier. Another important result is the theorem of Skolem (1929) from which it follows that the sequence of natural numbers may not be characterized by any finite axioms, whereas such finite statements would also permit interpretations that are inequivalent to the natural numbers.

The concept "natural number" is of nominalistic nature, a mere abstraction, while individual numbers, as R. Steiners pointed out on occasion, represent entities.

One may also consider the thought how completely numbers place themselves at our disposal, removing all life of their own, remaining ever serving beings.

In any case one will no longer harbor the commonplace view that the sequence of natural numbers is meaningless, even to be disdainful, after gaining insight into its wonderful weaving. This insight may awaken a feeling. Even if that feeling is hard to grasp, even if it only shimmers through as though glimmering from a great distance, we have been permitted to take a look into the workshop of the Cherubim, from whence the number-balise take their origin.

Let us consider, too, what great significance it could have if this feeling were to be awakened in young persons. Whoever has once been permitted to take such a look will never shun the number-sequence by slavish labor. It represents a most wonderful kind of art-work of the spirit. To be sure, it is an art-work that springs from worlds of necessity. But whoever wishes to be freely creative must, above all, learn to fit in rightly with necessities.

The following comments contain some supplementary indications.

[1] If one finds the digits of the number \(n\), written with help of base \(b\), one forms the chain \(n = a_0 + a_1 b + a_2 b^2 + \ldots + a_k b^k\), where each of the digits is one of \(\{0, 1, 2, \ldots, b\}\) and \(a_0 a_1 a_2 \ldots a_k\) are the digits we seek.

[2] The number \(n\) has the natural representation

\[n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots (\text{where } p_1, p_2, \ldots \text{ are primes})\]

then its divisor-sum has the value

\[(1 + p_1 + \ldots + p_1^{\alpha_1})(1 + q_2 + \ldots + q_2^{\alpha_2})(1 + r_3 + \ldots + r_3^{\alpha_3})\]

as may be seen immediately by multiplying out. One can also write

\[n = (1 + 1/p_1)(1 + 1/p_2)(1 + 1/q_2)(1 + 1/r_3)\ldots\]

The number \(n\) is therefore abundant if and only if

\[(1 + 1/p_1)(1 + 1/p_2)(1 + 1/q_2)(1 + 1/r_3)\ldots > 2.\]
THOUGHTS ON NATURE AND THE NATURE OF THOUGHT

Plate: "For God, desiring that as far as possible all things might be good and beautiful and all and having received all that is possible in this state of rest but moving without harmony or measure, brought it from its disorder into order, thinking that this was in all ways better. But it is a law that what is most perfect can do nothing that is not most beautiful. Therefore he took thought and perceived that of all things which are visible nothing that is without reason will ever be more beautiful, and that which has reason cannot dwell in anything. Because then he argued thus, in forming the universe he created reason in soul and soul in body, that he might make that world which was by nature beautiful and beautiful in such a way then we should affirm according to the probable account that this universe is a living creature in very truth possessing soul and reason by the providence of God." [Plato]

Aristotle: "The origin of heaven and of the natural world is an eternal entity which moves without being moved and is substance and actuality, inaccessible to a wider circle. For further information on the distribution of prime numbers, one good English book is Hardy and Wright's "Introduction to the Theory of Numbers," and an advanced one Lang's "The Distribution of Prime Numbers," or Trotter's excellent little book on Frobenius in German."

Translator's Notes:

(#) Such primes of form $1 + 2 + 2^2 + \ldots + 2^n = M_n$ are called Mersenne primes $M_n$ after the French Minotaur who successfully found the Mersenne prime of the first twelve of the year 1616, by which means it is not known. A necessary but not sufficient condition for $M_n$ to be prime is that $q = 2^n - 1$ itself be prime. His largest correct guess was $M_{31} = 2^{31} - 1 = 170,141,183,860,595,231,731,687,305,715,881,057,727$, whose size, because we can note it so compactly when it is "uncollected" in the familiar way, we may not appreciate. If we were to write it in base ten, the number of ones to be added, may at typical typewriter-density of ten ones to the inch, then scanning the length of the number at the speed of light (186,000 miles per second) would take approximately one hundred million years. [9] and the is only $M_{23}$ with 99 decimal digits. [10] (discovered in 1868 by Robinson) has 957 decimal digits, and $M_{23}$ is discovered in 1971 by Tuckerman $M_{50}$ digits, the ratio of binary exponent to number of decimal digits approaching $\log \log 2$.

[**] On p.11 Tuckerman originally wrote that only "somehow over 200 pairs [were] known" whereas Scott (Euler's Mathematische Werke, 12, 61-72) listed 1390 pairs known by the year 1910. On the bottom of p.11, however, Locher gives (as corrected in a footnote to the 1959 Siebenkinder) Lehmer's revised values which are $P(3,000,000,000,000)$ which is 55,005,922 digits, that the earlier value found by Bortelsen cited in Ore as well as Hardy and Wright. (in the same footnote, Locher gives Lehmer's value for $P(10^{10})$ as $55,052,512$.)

[***] Doklad, in his Elements, gave a simple proof of this over 2000 years ago, using a construction similar to that used in note [4] above: Assume that there exists a last, largest prime number, $p$. Then since number 1 can have no factors greater than 1 in common, we have that $P - 1$ cannot be divisible by 2 or 3 or 5 or any other prime

[1] For a number of form $n = a^2 + b^3$ (a, b both positive integers) we have $1 + 1/2 + \ldots + 1/n^2 > 2$ with equality holding if and only if $a = b = 1$, i.e. $n = 6$.

[2] Because of the divergence of the harmonic series, the number of primes $p, q, r, \ldots$ increases and the values of $a, b, c, \ldots$ grow larger, the left side of [1] exceeds every finite bound.

[3] The product $(1 + 1/n^2 + \ldots + 1/n^k)/(1 - 1/q - 1/r - \ldots)$ tends to $\frac{p}{q}$ as $n$ increases. As this is always less than 2 for $p$ and $q > 1$ odd primes, the indicated result follows.

[4] Let the product of all natural numbers from 1 to 10,001 be denoted by $P$. Then among the ten thousand consecutive numbers $P + 2, P + 3, P + 5, \ldots, P + 10,001$, there is not a single prime number since the first is divisible by 2, the second by 3, the third by 5, and so on.

[5] Unfortunately, there seems to be no literature on the pentagonal theorem which is accessible to a wide circle. For further information on the distribution of prime numbers, one good English book is Hardy and Wright's "Introduction to the Theory of Numbers," and an advanced one Lang's "The Distribution of Prime Numbers," or Trotter's excellent little book on Frobenius in German.

[6] The atomic world form the basis of the universe did not arrange themselves consciously or by design or determine their own movements; over an infinite period of time every possible combination of atoms was produced, by collision or the force of their own gravity, until at last those combinations which produced the race of living beings, the earth, the sea, the heavens, and of life. [De natura rerum]

John the Evangelist: "In the beginning was the Word, and the Word was with God, and the Word was God. He was in the beginning with God. All things were made through him, and without him was not anything made that was made. In him was life, and the life was the light of men. The light shines in darkness, and the darkness comprehends not it. [John 1:1-5]

Michael Faraday: "Particles are moving other than force centres. It is in force, or forces, of which matter is constituted, moved in any case materially penetrable, probably to their centre. Matter at rest fills out all of space, as far as gravity reaches. Each and every atom (immovable as such exist) extends thus through the entire solar system, yet with an ever concerned centre of force." [Cited in P. von Bortelsen, "Physik und Materie" (my back-translation — Editor)]

Rudolf Steiner: "Man has a view of the relation of man to the world becomes aware of the fact that he creates this relation, at least in part, by forming mental pictures about the things of the world and the mind in the world, which is 55,005,922 digits, that the earlier value found by Bortelsen cited in Ore as well as Hardy and Wright. (in the same footnote, Locher gives Lehmer's value for $P(10^{10})$ as $55,052,512$.)

C. P. von Weizsäcker: "For the formulation that seems indispensable to us, if we are to clarify the extent to which we can believe in the mathematical laws of nature, must contain the proposition that these laws are for the convenience of the principle of experience." [Philosophy of Physics]
works must develop compilers in their brains to decode then (music, however, tends to be in machine code in certain contexts). To the extent that is already decided. One function of art is then to provide observers with practice in constructing de-coding compilers. Other functions of art are still open to debate. It is further argued that most music must be paid to semantic features of representational visual art and that from this point of view such artworks can be regarded as a program that incorporates the visual art. I believe these two procedures involving reality from a model and (2) inferring an underlying general- 

Just how fast that picture of mechanistic physics has been left behind by modern discoveries was made clear yet again in a recent review article on particle physics [Roy F. Schwitters; "Fundamental Particles with Charm", Scientific American 217(1): 56-82 (October 1977)]. Most of the experiments in particle physics are made by observing the collis-
tions between two streams of particles which are circulated in opposite directions at speeds approaching that of light in a specially construc-
ted accelerator. If what happened in those collisions (according to the physics of quantum mechanics) is translated to the view which most think is true in "science" and of the mechanistic universe, the outcome of a collision must be governed by a strict causality. If one knows the position and velocity of all the particles, one can predict in principle the outcome of their collision. It is well known that an element of chance has been intro-
duced into the mechanics of the atom, but even so, the probability can be calculated and probabilistic predictions made. Here, we meet some-
thing that goes well beyond all of that. "The annihilation of an elec-
tron and a positron ... has as its immediate product ... a photon, a quantum of electromagnetic energy. The photon decays so quickly it can never be detected, even in principle (it is called a virtual parti-
cle), but it nonetheless determines the properties of all subsequent states ... [The laws can be] seen to be complete freedom for the creation of any particle, so long as it is accom-
pained by its own antiparticle." Here, the actors all disappear behind the curtains of their brief but decisive moment and change their costumes. When the curtain rises, it rises on a new play. The curtain must sit in the audience along with the rest of us and watch what ensues, for the script is so written and read in the hands of the actors, and they decide what it is to be - behind the "curtain."

Science News 112(13): 196 (September 11, 1977) reports that Chi-
icago's Argonne National Laboratory "has produced the world's most ener-
gy efficient facility for producing beams of polarized protons - that is, protons with their spin all oriented more or less in the same direction. Nor-
キャリレーのプロトトンなどのアセサイドのヘビーの碇点の中に、プロトノンとその他の一部の粒子が乱れた。乱れは粒子を単純にそれぞれの粒子が乱れた。乱れは粒子を単純にそれぞれの粒子が乱れた。乱れは粒子を単純にそれぞれの粒子が乱れた。乱れは粒子を単純にそれぞれの粒子が乱れた。乱れは粒子を単純にそれぞれの粒子が乱れた。乱れは
nup to the supposed last, largest one; p; therefore it is either a new, vastly larger prime in its own right, or else composed of one or more new, larger primes, all of which factors, either way, contradict the assumption that p was the last, largest prime, hence there is no such last prime. Putting this result together with that in [6], we see that while the se-
quence of primes never ceases, it does thin out, becoming ever more ir-
perfect. (Euclid also knew, and proved, the indicated form of all even
perfect numbers: 2n(2n-1) - 1, where 2n-1 is prime.)

(1.7) Gauss suspected some such relationship already at age 12 when he received his first table of logarithms. The sharper form in which he later stated it was that

\[ \frac{1}{2} \log n \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty \]

where "log" here means natural logarithms to the base of Euler's exponen-
tial function e, e = 2.718281828459 \ldots". To see the connection with
Locher's simplified statement, we need to introduce another constant named by Euler, the Euler-Mascheroni constant γ, \gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right),

which

\[ \log n \approx \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) = \gamma \]

what Locher has done, therefore, is simply to combine the 1 and the γ into a single summand \(1 - 0.5772 \ldots \) \approx 0.2238 and then suppress it, since in the limit it becomes arbitrarily small in comparison with \log n, so that

\[ \log n = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \]

as well, converging to the same limit values, only slightly more slowly, hence not as "sharply" stated, but avoiding introduction of transcendental functions. Chebyshev proved a weaker statement of this prime number theorem around 1850, asserting only that \(\pi(x)\) (or \(\pi(n)\) as most books call it) was of the same order of magnitude as a divided by \log n. Gauss' sharper form was proved independently by Hadamard and de la Vallee Poussin in 1896, about 100 years after Gauss first stated it.

(1) For references to the prime number theorem may be found given in con-
nect with the study of the Riemann zeta-function - as references to the play in large collections of the hands of the actors, and they decide what it is to be - behind the "curtain."

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What was discovered was a method of persuading a colony of bacteria to produce a human brain hormone. The structure of the hormone was originally deciphered from a five milligram quantity of it, which had been extracted from 500,000 sheep brains. The bacteria proved to have the same amount in relatively short order. The method of persuasion involved constructing from scratch a gene which codes for the hormone, and then getting the gene into a virus or bacterial plasmid. It could then be added to the ones already possessed by the bacteria.

"The bacteria needed the new 'work orders' and ... like bustling factors in producing 'hormones',"...

While Phillip Handler, president of the National Academy of Sciences, was hailing the experiment before a Senate subcommittee as "a scientific triumph of the first order," a triumph in the field of economics was being prepared behind the scenes. Two years ago, Boyer founded a company called Genentech to construct synthetic gene sequences that would be used to produce valuable "medicinal" drugs. Genentech paid for Boyer's research through a contract with UCSF. UCSF is applying for patents to protect Boyer's new techniques. UCSF's contract obligates them to license the technology and pay UCSF royalties on the profits. Meanwhile, the researchers are refusing to discuss anything about their work, including its purely "scientific" aspects. Handler's disclosure came as a surprise to the research team. In the light of the circumstances, it is ironic that Handler made his announcement to the Senate in order to bolster his testimony that recombinant DNA research was not only completely safe, but also highly desirable.

The mood of the materialistic natural philosophy of the Nineteenth Century could be found in all the branches of science, but it had its bulwark in physics. The physicists were achieving momentous discoveries after momentous discovery. "Many of the forces that determine the work night when they would put themselves out of work, there being nothing left to discover."

Their ebullient spirits and strong momentum brought them not only to their goal, but past it, into a region in which few of their old dogmas were pertinent. First came radioactivity, followed in rapid succession by relativity theory, quantum mechanics, and the navels of particle physics. The new discoveries severely tested the natural philosophy of the physicists. Reluctant to renounce their faith directly, this century has seen the physicists quixotically adopt some of their old dogmas and redefine beyond recognition the terms of others. Regardless of the efforts of physicists to preserve its hue, the Spirit of Materialism saw that its days in physics were drawing to a close.

But the work had borne fruit: the other sciences, even the humanities, worked to model themselves on the pattern of physics, as it was in the Nineteenth Century.

A typical example of the way the spirit of the old mechanization in trying to establish itself in the humanities appeared in a recent issue of a new journal of the arts which seems to be mostly devoted to mechanism in art. The quote is somewhat lengthy, so it is the summary of an entire article, but it gives a good feeling for what this trend is about.

"Abstract: The basic arguments of this paper are that art is not intrinsically mysterious and that there is no reason why art should not serve as a source of new functions for computers as well as for human beings. Asking what features of art might be developed into the new functions for computers is an exploration of the functions of art for humans from a new perspective. The author suggests that artworks are like computer programs and observers of art..."
One of the great events in the evolution of human consciousness was the gradual shift from Moon-centered to Sun-centeredness. The majority of records show that early man thought his spiritual life to be centered on the Moon. Examples of the external evidence for this are the Moon-based calculation of the Jewish calendar and the twenty-eight-fold medicine of the Indian and Chinese civilizations. The shift to Sun-centeredness in internal, religious matters may be seen as having taken place roughly from the time of New Kingdom Egypt (Amenhotep) to the early Christian Era (Julian Apollonius, the Manichaeans impetus). In an external way, the change of focus finally occurred with the victory of the Copernican model of the universe, which placed the Sun in the center. Rudolf Steiner spoke about aspects of this transition on many occasions.

Because of humanity's long Moon-centered history, one might suspect that relics from that time still persist, hidden somewhere in our being. Although each of us might still possess his Moon orientation to some extent, it should become more evident in a person who was denied his ordinary connection with the Sun. Just such evidence was recently discovered by a group of researchers at Stanford University, and was reported in the October 28 issue of Science.

The person who led researchers to their discovery was denied his connection with the outward Sun through having been born blind. For the last several years, he had experienced periodic inability to conform to societal norms in waking and sleeping. Treatment by hypnosis and drugs did not help.

After 26 days of hospital study, the researchers concluded that he had circadian rhythms of 28.8 hours, "indistinguishable from the period of the lunar day." Furthermore, for the period of the study, "there was a remarkable coincidence between his sleep onset and a local low tide." Attempts to force his body function back to a 24 hour day rhythm failed.

There was also a Stanford survey of 50 people, all blind to varying extents; 38 of them complained of significant sleep-wake disorder. Other experiments have removed normal time clues from ordinary people, and discovered that they tend to resort to circadian rhythms of around 25 hours.

There is no doubt that there is a powerful stream in science which encourages and demands objectivity, respect for truth, and submission of the personality to higher goals. The question is the extent to which that positive stress is able to find expression in the actual conduct of science. The real test comes when discoveries are made of processes which put tremendous power in the hands of whoever controls them. One hopes that realistic notions especially when the new discoveries hold potential for inflicting unprecedented harm on all of humanity. Only a discovery was made by a team led by Herbert Boyer of the University of California at San Francisco (Science News 112[26]:150 (November 18, 1977)). The discovery "not only opens all previous gene-engineering research, but may mark the beginning of a new era in the biological sciences as well." What are the circumstances of this beginning? Two aspects of the circumstances will be described here: the nature of the new discovery, and the conduct of the scientists with regard to their achievement.
Dear Friends of the *Math.-Phys. Correspondence*

The slight one-month delay which most of you experienced in receiv- ing the Michaelmas issue was caused purely by a clerical error. The four-month delay which surface-mail subscribers in Britain and continental Europe will have experienced due to the U.S. coast coast dockworkers' strike, was caused only slightly delayed by my failure to send out the two issues at the same time, with my apologies, and hope that they will find them worth waiting for.

This issue begins with a series of three recent science news-teams reviewed by David Black, to which I have added a fourth item without comment. The first is adapted, with the kind approval of author Black and with his permission, from the forthcoming issue of the *Newsletter* of the Authors' Society in America.

I am grateful to Donald Campbell of the Edinburgh Rudolf Steiner School for the selection of the first three quotes that follow on p. 5, to which I have added more four in a sort of historical morsel. The full text of the von Zeitschneider quotation may be found in Vol. 29 No. 1 of Modern Occult in Modern Thought (Sept.-Oct. 1972), reprinted in their Retrospective Issue, Vol. 32 Nos. 2-5.

There then follows the promised longer article by Louis Loeger-Brant on a very beautiful and little-known theorem due to Euler relating prize distribution to pentagonal numbers. Loeger-Brant was professor of mathematics at the Technische Hochschule in Winterthur for many years, author of a well-received textbook on the calculus as well as numerous articles in the Swiss journal *Elemente der Mathematik*. He was also the director of the Math.-Astron. Section of the Schweizerische for 37 years until his death in 1975, and it is from the *Mathematischen* published by that institution which the present article, intended for a non-technical readership, is translated. Research results reported in the article have been brought up to date, and technical notes appended. Our thanks to Professor Loeger’s daughter, Frau A. Whitt, for granting permission to make and print the translation, and to Dr. Georg Lang, present director of the Section, for his active support. Thanks also to Prof. E. Stark for the helpful reference to the chapter on partition theory in the book by Hardy and Wright (which has no index).

Irriguingly, Tschau’s results in the Am. Math. Monthly relating the geometries of the regular heptagon and square, I presume them further, to finding first a trivial generalization to families of concentric 6-cycles in every polygon with odd number of sides (not reported here) before hitting upon what I believe to be a non-trivial generalization to those special odd polygons with n=6 sides. Ideas normally associated with finite projective planes yield results on regular polygons, in particular relating the geometries of the 13- and hexagon.

Finally, there is a long by John Sprague, playfully protesting overly dogmatic statements on the uniqueness of snowflakes. The punctuation is a bit licentious, but you should be able to puzzle it out. Enjoy!

Brian Smallwood (Dept. of Biology, Univ. of California, Berkeley) uses writing the unidentified "Plant genetic material" on p. 19 of issue 20, by scale considerations (100,000/75 magnification, 10 microns actual size), is more likely a co-emulsion than a true virus, in any case some kind of organelle. Lawrence Edwards points out that the DNA helices are special cases of path-curves (spiral), too.

We have also received word that George Adams’ *Universal Forces in Mechanics*, investigating the deep projective polarities of kinematics and dynamics, has been released and may be ordered from the Rudolph Steiner Press, 27 Park Rd., London NW 3, 672 (63,92) or St. George Books Inc, Box 825, Spring Valley, N.Y. 10977. A biography of Oliver Heaviside and collection of Adams’ essays was released earlier by Brain Goulson, Ltd.

With all good wishes for the New Year,

*Stephen Nicholls*
trigometric geometry continues to work nicely for $n = 5, 7, 8, 9, 11$ etc. but not for $n = 6, 10, 12$ as these are not prime or prime powers, so no
p.n.a.'s exist, but as yet not a simple nice expression involving $n$ has been found for sums or products of trig functions of angles $A, B, C$
with $A + B + C = 180^\circ$, $A : B : C = \text{integer}$, $180^\circ/n = m$, or for any other $A : B : C
\equiv \text{integer}$ (mod $n$), when $m < n$. Do they not exist? Is $n = 4$ a "watershed" case for the trigonometry as well?

Finally, we may look at the orthic triangles of such triangles:
For $n = 2$ (the "heptagonal triangle" with angles $A : B : C = 120^\circ$) we saw in the last issue that the orthic triangle was similar to the original one, reduced in size by $1/2$. For $n = 3$ we see there are two such tri-
angles (one with $A : B : C = 121.9^\circ$ and the other $216.5^\circ$); these turn out to be orthic triangles of one another, so that if one takes orthic triangles
twice one obtains a triangle similar to the original one, reduced in size by $(1/2)^2 = 1/4$ (due to properties of the 5-point circle, which see in
the literature). $121.9^\circ$ and $216.5^\circ$ are similarly paired for $n = 4$

NEW PROBLEMS

1. Among the numbers from 1 to 160, only 220 and 284 have the same other
as contents in an alternating cycle; most others eventually reduce to 1
since cyclically by repeated taking of contents, although a few grow first be-
fore they shrink (e.g. 220 $\Rightarrow 220 \Rightarrow 110 \Rightarrow 55 \Rightarrow 50 \Rightarrow 25 \Rightarrow 12 \Rightarrow 6 \Rightarrow 3 \Rightarrow 1$); a few others seem to "ex-
plode" growing without bound (e.g. 120 and 100), or do these, too, ul-
timately shrink? Dr. Georg Unger points out that some larger numbers
form larger content-cycles (e.g. 12196, 12258, 15728, 15956, 16268 is a
5-cycle); can the reader discover a 3- or 4-cycle?

2. Every triangle ABC possesses two points - the Brocard points (see
Court's College Geometry in the College Outline Series) - $X_n$, such that
angles $XAB, XBC, XCA, XCB, XAC, XCA$ are all alike - the Brocard angle.
For the "heptagonal triangle" with $A : B : C = 120^\circ$, this angle is arcocos $\frac{\pi}{7}$;
what is it for the triangles with $A : B : C = 131.9^\circ$ or $114.46^\circ$

HEXAGONALPLEX

by John Sprague
Two years ago (see *Sci. News*, Oct. 27, 1975) Charles Kowal made observational news when he spotted a 13th moon of Jupiter; the year before that he had found the 14th. This year (see *Nov. 1977 Sci. News*) observer Kowal made a discovery of far greater interest to those trying to reconstrcut the physical history of the solar system: a new object comparable in size (about 300 mi. or 500 km. in diameter) to the largest asteroids, but much farther away — at or around the orbital distance of Uranus, not Jupiter. First sighted on October 15th with the 18-in. Schmidt telescope on Palomar Mountain, its exact orbit is still being determined (tentatively described as nearly circular, with an inclination of 3 to 5 degrees). At 18th to 19th magnitude, it is beyond the reach of most amateur observers, but should show up on any older photographic plates to help determine its orbit. The size estimate above is for an object with medium-bright surface like our Moon, if later found to have a darker surface like a carbonaceous chondrite, it would be larger, if icy and more reflective, then smaller.

Also of interest to planetary astronomers is a recent report (same issue of *Sci. News*) that Neptune is much warmer, relative to the heat it receives from the Sun, than was formerly believed, emitting about 3.5 times as much heat as it takes in. Uranus, on the other hand, seems to have little, if any, internal source of heat.

Meanwhile, a little closer to home, the orbiting Viking probe of Mars has carried out successfully what will probably remain a unique task in its mission: taking temperature-measurements of the midnight side. This is difficult, because it must be entirely within the planet's shadow unless exposure to direct sunlight would burn out its sensors, and it is only entirely in the planet's shadow twice in a Martian year (1.86 Earth-years). As would be expected (contrary to a newspaper article in *The New York Times*), the warmest region is the eastern rim, since "dawn" — seen from the backside means what is coming from the daylight side into darkness. The "canyonlands," as on Earth, retained heat relatively well through the night, while the plains regions, where the great dust-storms begin, cool off relatively rapidly. Olympus Mons and three other volcanic mountains show up as isolated cold-spots.

An earlier issue of *Sci. News* (Sept. 26) reported that scientists in a 3-day symposium held in Boston in mid-September agreed about an improved technique to measure emissions from the Martian surface in July and October of 1976. The results observed were not too unlike anything known by Earth-standard of organic and inorganic chemistry to be unambiguously interpreted without further and finer observations. The sensors on board should have been able to detect as few as 1,000,000 g. Cell walls per gram of soil, yet Earth-samples from the Moon and Mars have been found which contain as few as 100,000. Earth-bound amateurs can do their own Mars-watching during January as the two planets closest approach. It is the brilliant red planet at sunset, as it descends through the constellation of Leo. A binocular should reveal the blue-white object near the pole, seen rising later. The process is reversed the next night.