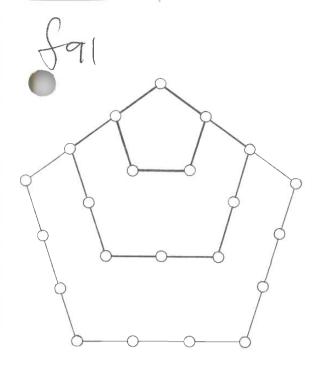
Cans 16.i. B 326 Stephen Eberhart 10 806 Stoddard St. Missoula, Mont. U.S.A. 59801 6 Jan: 78 Dear Dr. Sloame, Greefings again! I'm now back on this side of the Atlantic (Depit of Math, UgMt.), and as addicted as ever to your Hand-book. If you're still col-lecting items for a second Edition, I point out the Hardy of unight reference to pentorgonal numbers on p. 284 g. them No. The book (see notes 6. Hom & p. 17 in magazine - read pp. 13-14 for pertinent background). Only 118 w don't call them pont, nos, by names just use show by formula,
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## MATHEMATICAL - PHYSICAL CORRESPONDENCE

Number 22 Christmas 1977

## Contents:

Letter from the Editor p.	1
Science News Items - Autumn 1977, by David Black p.	
Thoughts on Nature and the Nature of Thought p.	5
The Sequence of Natural Numbers as Art-Work of the Spirit, by Louis Locher-Ernst p.	6
On the Geometry of Regular (n <sup>2</sup> +n+1)-gons, by Stephen Eberhart p. 1	18
New Problems p. 2	1
Hexagoogolplex, by John Sprague p. 2	21
Star-Watcher's Guide - January to March p. 2	22

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Dear Friends of the Math .- Phys. Correspondence:

The slight one-month delay which most of you experienced in receiving the Michaelmas issue was slight indeed by comparison with the four-month delay which surface-mail subscribers in Britain and continental Europe will have experienced due to the U.S. east coast dockworkers' strike, which ended only shortly before Christmas. They will now get two issues at the same time, with my apologies, and hopes that they will find them worth waiting for!

This issue begins with a series of three recent science news-items reviewed by David Black, to which I have added a fourth item without comment. The first is adapted, with the kind approval of author Black and editor O'Neil, from a forthcoming issue of the Newsletter of the Anthroposophical Society in America.

I am grateful to Donald Campbell of the Edinburgh Rudolf Steiner School for the selection of the first three quotations that follow on p. 5, to which I have added four more in a sort of historical menorah. The full text of the von Weizsäcker quotation may be found in Vol. 29 No. 1 of Main Currents in Modern Thought (Sept.-Oct. 1972), reprinted in their Retrospective Issue, Vol. 32 Nos. 2-5.

There then follows the promised longer article by Louis Locher-Ernst on a very beautiful and little-known theorem due to Euler relating prime distribution to pentagonal numbers. Locher-Ernst was professor of mathematics at the Technische Hochschule in Winterthur for many years, author of a well-received text-book on the calculus as well as numerous articles in the Swiss journal Elemente der Mathematik. He was also the director of the Math.-Astron. Section of the Goetheanum in Dornach from 1937 until his death in 1962, and it is from the Sternkalender published by that institution which the present article, intended for a non-technical readership, is translated. Research-results reported in the article have been brought up to date, and technical notes appended. Our thanks to Professor Locher's daughter, Frau A. Wältli, for granting permission to make and print the translation, and to Dr. Georg Unger, present director of the Section, for his active support. Thanks also to Prof. H. Stark for the helpful reference to the chapter on partition theory in the book by Hardy and Wright (which has no index).

Intrigued by Thébault's results in the Am. Math. Monthly relating the geometries of the regular heptagon and square, I pursued them further, finding first a trivial generalization to families of concentric 6-cycles in every polygon with odd number of sides (not reported here) before hitting upon what I believe to be a non-trivial generalization to those special odd polygons with n²+n+1 sides. Ideas normally associated with finite projective planes yield results on regular polygons, in particular relating the geometries of the 13- and hexagon.

Finally there is a poem by John Sprague, playfully protesting overly dogmatic statements on the uniqueness of snowflakes. The punctuation is a bit licentious, but you should be able to puzzle it out. Enjoy!

Brian Goodwin (Dep't of Biology, Univ. of Sussex) writes us that the unidentified "plant genetic material" on p. 19 of issue 20, by scale-considerations (160,000/ $\overline{2}$  magnification  $\Rightarrow$  10 microns actual size), is more likely a cell-nucleus than a chromosome, in any case some kind of organelle. Lawrence Edwards points out that the DNA helices are special cases of path-curve spirals, too.

We have also received word that George Adams' <u>Universal Forces in Mechanics</u>, investigating the deep projective polarities of kinematics and dynamics, has been released and may be ordered from the Rudolf Steiner Press, 35 Park Rd., London NWI 6XT (£2.95) or St. George Book Service, Box 225, Spring Valley, N.Y. 10977. A biography by Olive Whicher and collection of Adams' essays was released earlier by Henry Goulden Ltd.

With all good wishes for the New Year, Stephen Eberhart

### SCIENCE NEWS ITEMS - AUTUMN 1977

by David Black New Haven, Conn.

One of the great events in the evolution of human consciousness was the gradual shift from Moon-centeredness to Sun-centeredness. The majority of records show that early man thought his spiritual life to be centered on the Moon. Examples of the external evidence for this are the Moon-based calculation of the Jewish calendar and the twenty-eight-fold zodiacs of the Indian and Chinese civilizations. The shift to Sun-centeredness in internal, religious matters may be seen as having taken place roughly from the time of New Kingdom Egypt (Akhenaton) to the early Christian Era (Julian Apostate, the Manichaean impulse). In an external way, the change of focus finally occurred with the victory of the Copernican model of the universe, which placed the Sun in the center. Rudolf Steiner spoke about aspects of this transition on many occasions.

Because of humanity's long Moon-centered history, one might suspect that relics from that time still persist, hidden somewhere in our being. Although each of us might still possess his Moon orientation to some extent, it should become more evident in a person who was denied his ordinary connection with the Sun. Just such evidence was recently discovered by a group of researchers at Stanford University, and was reported in the October 28 issue of Science.

The person who led researchers to their discovery was denied his connection with the outward Sun through having been born blind. For the last several years, he had experienced periodical inability to conform to societal norms in waking and sleeping. Treatment by hypnosis and drugs did not help.

After 26 days of hospital study, the researchers concluded that he had circadian rhythms of 2h.9 hours, "indistinguishable from the period of the lunar day." Furthermore, for the period of the study, "there was a remarkable coincidence between his sleep onset and a local low tide." Attempts to force his body functions back to a 2h hour day rhythm failed.

There was also a Stanford survey of 50 people, all blind to varying extents; 38 of them complained of significant sleep-wake disorder. Other experiments have removed normal time clues from ordinary people, and discovered that they tent to resort to circadian rhythms of around 25 hours.

There is no doubt that there is a powerful stream in science which encourages and demands objectivity, respect for truth, and submission of the personality to higher goals. The question is the extent to which that positive stream is able to find expression in the actual conduct of science. The real test comes when discoveries are made of processes which put tremendous power in the hands of whoever controls them. One hopes for altruistic motives espectially when the new discoveries hold potential for inflicting unprecedented harm on all of humanity.

Such a discovery was made by a team led by Herbert Boyer of the University of California at San Francisco [Science News 112(20): 310 (November 12, 1977)]. The discovery "not only eclipses all previous gene-engineering research, but may mark the beginning of a new era in the biological sciences as well." What are the circumstances of this beginning? Two aspects of the circumstances will be described here: the nature of the new discovery, and the conduct of the scientists with regard to their achievement.

What was discovered was a method of persuading a colony of bacteria to produce a human brain hormone. The structure of the hormone was originally deciphered from a five milligram quantity of it, which had been extracted from 500,000 sheep brains. The bacteria produced the same amount in relatively short order. The method of persuasion involved constructing from scratch a gene which codes for the hormone, and splicing the gene into a virus or bacterial plasmid. The gene could then be added to the ones already possessed by the bacteria. "The bacteria heeded the new 'work orders' and ... like bustling factories 'merrily engaged' in producing the hormone."

While Philip Handler, president of the National Academy of Sciences, was hailing the experiment before a Senate subcommittee as "a scientific triumph of the first order," a triumph in the field of economics was being prepared behind the scenes. Two years ago, Boyer founded a company called Genentech to construct synthetic gene sequences that would be used to produce valuable "medicinal" drugs. Genentech paid for Boyer's research through a contract with UCSF. UCSF is applying for patents to protect Boyer's new techniques. UCSF's contract obligates them to license the patents to Genentech, which would pay UCSF royalties on the profits. Meanwhile, the researchers are refusing to discuss anything about their work, including its purely "scientific" aspects. Handler's disclosure came as a surprise to the research team. In the light of the circumstances, it is ironic that Handler made his announcement to the Senate in order to bolster his testimony that recombinant DNA research was not only completely safe, but also highly desireable. -----

The mood of the materialistic natural philosophy of the Nineteenth Century could be found in all the branches of science, but it had its bulwark in physics. The physicists were achieving momentous discovery after momentous discovery. Many of them felt that the day was drawing near when they would put themselves out of work, there being nothing left to discover.

Their ebullient spirits and strong momentum brought them not only to their goal, but past it, into a region in which few of their old dogmas were pertinent. First came radioactivity, followed in rapid succession by relativity theory, quantum mechanics, and the byzantine marvel of particle physics. The new discoveries severely tested the materialistic faith of the physicists. Reluctant to confront their faith directly, this century has seen the physicists quietly drop some of their old dogmas and redefine beyond recognition the terms of others. Regardless of the efforts of physicists to preserve its home, the Spirit of Materialism saw that its days in physics were drawing to a close. But the work had borne fruit: the other sciences, even the humanities, worked to make themselves more "scientific". That is, they tried to model themselves on the pattern of physics, as it was in the Nineteenth Century.

A typical example of the way the spirit of the old mechanism is trying to establish itself in the humanities appeared in a recent issue of a new journal of the arts which seems to be mostly devoted to mechanism in art. The quote is somewhat lengthy, since it is the summary of an entire article, but it gives a good feeling for what this trend is about.

"Abstract: The basic arguments of this paper are that art is not intrinsically mysterious and that there is no reason why art should not serve various functions for computers as well as for human beings. Asking what such functions might be for computers leads to an examination of the functions of art for humans from a new perspective. The author suggests that artworks are like computer programs and observers of art-

works must develop compilers in their brains to decode them (music, however, may be said to be in machine code in certain of its aspects, that
is, already decoded). One function of art is then to provide observers
with practice in constructing de-coding compilers. Other functions of
art are also suggested. It is further argued that more attention should
be paid to semantic features of representational visual art and that from
this point of view such artworks can be regarded as a program that incorporates a model. Compiling here involves two processes: (1) reconstructing reality from a model and (2) inferring an underlying general theoretical construct that it exemplifies." [Apter, M. J.: "Can Computers be
Programmed to Appreciate Art?" Leonardo 10(1): 17-21 (Winter 1977)]

Just how far that picture of mechanistic physics has been left behind by modern discoveries was made clear yet again in a recent review article on particle physics [Roy F. Schwitters: "Fundamental Particles with Charm", Scientific American 237(h): 56 - 82 (October 1977)]. Most of the experiments in particle physics are made by observing the collisions between two streams of particles which are circulated in opposite directions at speeds approaching that of light in a specially constructed accelerator. If what happened in those collisions (according to the physicists' explanations) conformed to the view which most believers in "science" have of the mechanistic universe, the outcome of a collision must be governed by a strict causality. If one knows the position and velocity of billiard balls, one can ideally predict the outcome of their collision. It is well known that an element of chance has been introduced into the mechanics of the atom, but even so, the probability can be calculated and probabilistic predictions made. Here, we meet something that goes well beyond all of that. "The annihilation of an electron and a positron ... [has as its] immediate product ... a photon, a quantum of electromagnetic energy. The photon decays so quickly it can never be detected, even in principle (it is called a virtual particle), but it nonetheless determines the properties of all subsequent states of the system .... [The laws can be] seen to confer almost complete freedom for the creation of any particle, so long as it is accompanied by its own antiparticle." Here, the actors all disappear behind the curtain for a brief but decisive moment and change their makeup and costumes. When the curtain rises, it rises on a new play. Physics must sit in the audience along with the rest of us and watch what ensues, for the choice of the play is largely in the hands of the actors, and they decide what it is to be - behind the "curtain".

Science News 112(13): 196 (September 2h, 1977) reports that Chicago's Argonne National Laboratory "has produced the world's most energetic facility for producing beams of polarized protons — that is, protons with their spins all oriented more or less in the same direction. Ordinarily the protons in an accelerator's beam have their spins oriented randomly. To polarize them takes special arrangements, but to separate the effects of spin from other factors in a collision experimenters must know which way the spins in the proton beam are going.

"The polarized proton beam was struck against a liquid hydrogen target. In the collisions between the beam protons and those in the target, the effects of spin were most pronounced when the bouncing proton came off at a large angle to its original direction. Runs were made at energies of 11.75 [to] 13.1 billion eV. The combination of high energy and high scattering angle indicates that something rather deep inside the target proton is responsible for the observed effect. That is, rather simply, that protons bounce well off each other when their spins are parallel... When the spins are antiparallel..., the protons ... appear to pass right through each other ...! The physicists and philosophers [asking] about the materiality of matter will have fun with that."



## THOUGHTS ON NATURE AND THE NATURE OF THOUGHT

Plato: "For God, desiring that as far as possible all things might be good and none evil and having received all that is visible not in a state of rest but moving without harmony or measure, brought it from its disorder into order, thinking that this was in all ways better. Now it is a law that what is most perfect can do only that which is most beautiful. Therefore he took thought and perceived that of all things which are visible nothing that is without reason will ever be more beautiful than that which has reason, and that without soul reason cannot dwell in anything. Because then he argued thus, in forming the universe he created reason in soul and soul in body, that he might make a work that was by nature most beautiful and perfect. In this way then we should affirm according to the probable account that this universe is a living creature in very truth possessing soul and reason by the providence of God." [Timaeus]

Aristotle: "The origin of heaven and of the natural world is an eternal entity which moves without being moved and is substance and actuality; it moves like an object of thought or desire or of love, whereas other things move by being moved.... It is a life such as the best that we live, though we live it for a short time only, a life of eternal active thinking, the source of the highest pleasure. Now this highest form of pure thinking must take as its object the highest of all objects of thought. So God thinks himself, since he is the highest of all things, and his thinking is a thinking of thinking." [Metaphysics]

Lucretius: "The atoms which form the basis of the universe did not arrange themselves consciously or by design or determine their own movements; over an infinite period of time every possible combination of atoms was produced, by collision or the force of their own gravity, until at last those combinations were produced which are the basis of the earth, the sea, the heaven, and of life." [De Rerum Naturae]

 $\frac{\mathrm{John}}{\mathrm{with}}$  the Evangelist: "In the beginning was the Word, and the Word was with  $\frac{\mathrm{God}}{\mathrm{god}}$ , and the Word was  $\mathrm{God}$ . He was in the beginning with  $\mathrm{God}$ ; all things were made through him, and without him was not anything made that was made. In him was life, and the life was the light of men. The light shines in the darkness, and the darkness has not overcome it."  $\frac{\mathrm{Gospel}}{\mathrm{Gospel}}$ 

Michael Faraday: "Particles are nothing other than force centres. It is force, or forces, of which matter is constituted. As such they are in any case materially penetrable, probably to their centre... Matter always fills out all of space, as far as gravity reaches. Each and every atom (inasmuch as such exist) extends thus through the entire solar system, yet with an ever conserved centre of force." [Cited in Ernst Lehrs' Mensch und Materie (my back-translation - Editor)]

Rudolf Steiner: "Whoever tries to work out for himself a view of the relation of man to the world becomes aware of the fact that he creates this relation, at least in part, by forming mental pictures about the things and events in the world.... He begins to say to himself 'It is impossible for me to have a relationship to any thing or event unless a mental picture appears in me.'... One only avoids the confusion... if one notices that, inside everything we can experience..., there is something which cannot suffer the fate of having a mental picture interpose itself between the process and the person observing it. This something is thinking." [Philosophy of Freedom]

 $\underline{\text{C. F. von}}$  Weizsäcker: "For the formulation that seems indispensible to me, if we are to clarify the extent to which we can believe in the mathematical laws of Nature, must contain the proposition that these laws are preconditions for the possibility of experience." [Platonic Nat. Science]

## THE SEQUENCE OF NATURAL NUMBERS AS ART-WORK OF THE SPIRIT

### by Louis Locher-Ernst

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1. Geometrical forms such as a triangle, circle, or lemniscate, etc., speak to us directly because we can make them our own without difficulty. Conversely, we feel a sense of satisfaction when we are able to take some initially chaotic state of affairs and bring order into it with the help of suitable geometrical forms. We also feel capable of changing one form into other forms. Through this, the realm of forms gains life. By seeking out the simplest basic forms, the characteristic basic gestures of the form-world, we can even make ourselves an alphabet. One can lift oneself to a language of forms whose syllables connote the experiences of these characteristic basic gestures.

How different the realm of numbers appears to us! We shall consider here only the natural numbers 1, 2, 3, 1, 5, 6, .... Whereas the forms, by virtue of the fact that we can change them into one another, reveal to us a mysterious connection with light and color, the realm of numbers shows itself to us as dark, offering no ready access to experience.

Forms permit us to slip into them with our experiencing capacity. It is easy to steep oneself in a 5- or 6-pointed star and arrive soon at various sensations. Numbers, on the other hand, have something rigid and unchanging about them that seems unapproachable. Just try to get into the number 9; it forbids us come too near. Whereas, when we occupy ourselves with forms, these speak to us, numbers become taciturn and draw away from our experiential grasp.

Into this realm of the sequence of numbers 1, 2, 3, ... we wish now to dare a few steps. The first thing we find when we test our awareness of what is going on here, is that we scarcely know what else to report than just 1, 2, 3, .... In so doing, we have the conviction that we could extend the sequence as far as we please. To carry this out, we need some sort of parceling in a rhythmic process. In the circles of present-day culture the number 10 serves as a basis for this, with the rhythmization given by its successive powers  $10^{\circ}$  = 1,  $10^{\circ}$  = 10,  $10^{\circ}$  =  $10 \cdot 10$  = 1000, etc. In our first school-years we became accustomed to arranging every natural number to fit into this system.

As ingenious as this arrangement is, the accustomedness to it, if one thinks no further, can lead to a misunderstanding which we wish to clear out of the way. It will suffice to explain it by a few simple examples.

If one considers the nine times table

$$1.9 = 9$$
,  $2.9 = 18$ ,  $3.9 = 27$ , .....,  $9.9 = 81$ ,  $10.9 = 90$ ,  $(11.9 = 99)$ ,  $12.9 = 36$ , .....,  $19.9 = 171$ ,

then one notices that — disregarding the parenthesized case — the sum of the digits is always nine: 1+8=9, 2+7=9, ..., 1+7+1=9. One might easily suppose that in this is expressed some particularly remarkable property of the number nine.

Another example: Take an arbitrary three-digit number, exchange the first and third digit with one another, and find the difference between the given and the new number. Then take the resulting number, interchange first and third digits again, and add that difference to this new number. The result in every case is either 1089 or, in one special case, zero. E.g.: 752 105 511 396 100 099

Here, too, one is surprised and supposes this to be some particularly remarkable property of the number 1089. It would be easy to give a whole list of similar examples.

But such phenomena have nothing at all to do with individual properties of the numbers in question, depending instead on the choice of base 10. As a first requirement for concentrating on individual properties, we must free ourselves from being tied to the parceling-basis, although we shall naturally continue to write all numbers in the base 10 system. But first we wish to show how a given number can be written in other systems. If b is the basis, then instead of the powers

$$10^1 = 10$$
,  $10^2 = 100$ ,  $10^3 = 1000$ .

we use the powers

$$b^1 = b$$
,  $b^2$ ,  $b^3$ ,  $b^{\perp}$ , .....

to arrange our work. If for example n=3h3(10) is to be expressed in the system with base b=6, one carries out the following divisions:

$$343 = 6.57 + 1$$
,  $57 = 6.9 + 3$ ,  $9 = 6.1 + 3$ ,  $1 = 6.0 + 1$ .

Then we have n = 1331(6). The remainder from the first division represents the first digit, counting from the right. And in fact we have  $1 + 3 \cdot 6 + 3 \cdot 6^2 + 1 \cdot 6^3 = 313(10)$ .

In order to express 3h3(10) in the systems with base b=3 or b=7 [1] one carries out the following divisions:

$$3h3 = 3 \cdot 11h + 1$$
 $11h = 3 \cdot 38 + 0$ 
 $3h3 = 7 \cdot h9 + 0$ 
 $38 = 3 \cdot 12 + 2$ 
 $12 = 3 \cdot h + 0$ 
 $1 = 3 \cdot 1 + 1$ 
 $1 = 3 \cdot 0 + 1$ 

Then we have 343(10) = 110,201(3) as well as 1000(7). And in fact  $1 + 0.03 + 2.03^2 + 0.03^3 + 1.03^4 + 1.03^5 = 343(10) = 1.07^3$ .

2. The digits which any number displays depend on the choice of parceling. Now we shall show a property of the number 28 which is independent of this means of representation. Altogether, the divisors of 28 are 1, 2, 4, 7, 14, 28. Their sum is 56. The sum of the divisors which are smaller than the number itself, i.e. the sum of 1, 2, 4, 7 and 14, is 28. In order to be able to express ourselves more succinctly, let us call the sum of all the divisors of a number, excepting the number itself, the content of the number. Let the sum of all the divisors, including the number itself, be designated henceforth as s. The number 28 has the property that its content is equal to the number itself; the value of s is therefore 28 + 28 = 56, twice that of the number.

The content of a number and also its divisor-sum s have evidently nothing to do with parceling. On pages 8 and 9 there is given a list of the numbers from 1 to 360 with their corresponding divisor-sums. For example, to the right of 30 there stands the number 72. And in fact, all of the divisors of 30 - namely 1, 2, 3, 5, 6, 10, 15 and 30 - yield 72 as their sum. The content of 30 is thus \$\mathbb{12}\$. [2]

Now the content of a number can be either smaller or larger than the number itself, or it can be equal to it. The number 15 with content 1+3+5=9 belongs to the former class, the number 20 with content 1+2+1+5+10=22 to the second class, while 28 with content 28 is a member of the third. If one adds to the content the number itself, then the three classes are characterized by the fact that the sum s of the divisors of the number n is either less than or greater than its double 2n, or else it is equal to that double value. The sole exception to this is the number one whose content coincides with s.

## THE SUM OF ALL DIVISORS OF NUMBERS FROM 1 TO 180

1	1	37 <sup>P</sup>	38	73 <sup>p</sup>	74	109 <sup>p</sup>	110	145	180
2P	3	38	60	74	114	110	216	146	222
3 P	4	39	56	75	124	111	152	147	228
h	7	a40	90	76	140	a112	248	148	266
5 <sup>p</sup>	6	41P	42	77	96	113P	114	149P	150
6	12	42ª	96	78ª	168	114	240	150ª	372
7 <sup>P</sup>	8	43P	44	79 <sup>p</sup>	80	115	144	151P	152
8	15	lili	84	a 80	186	116	210	152	300
9	13	45	78	81	121	117	182	153	234
10	18	46	72	82	126	118	180	154	288
11 <sup>p</sup>	12	47P	118	83P	84	119	144	155	192
128	28	48ª	124	812	224	120ª	360	156ª	392
13 <sup>p</sup>	14	49	57	85	108	121	133	157 <sup>p</sup>	158
14	24	50	93	86	132	122	186	158	240
15	24	51	72	87	120	123	168	159	216
16	31	52	98	a88	180	124	224	a160	378
17 <sup>P</sup>	18	53 <sup>P</sup>	54	89 <sup>D</sup>	90	125	156	161	192
18ª	39	54ª	120	90ª	234	126ª	312	162ª	363
19 <sup>p</sup>	20	55	72	91	112	127P	128	163 <sup>P</sup>	164
<sup>a</sup> 20	42	a <sub>56</sub>	120	92	168	128	255	164	294
21	32	57	80	93	128	129	176	165	288
22	36	58	90	94	144	130	252	166	252
23 <sup>P</sup>	24	59 <sup>P</sup>	60	95	120	131 <sup>p</sup>	132	167 <sup>P</sup>	168
2hª	60	60ª	168	96 <sup>a</sup>	252	132ª	336	168ª	480
25	31	61 <sup>p</sup>	62	97 <sup>p</sup>	98	133	160	169	183
26	42	62	96	98	171	134	204	170	324
27	40	63	104	99	156	135	240	171	260
28	56	64	127	a100	217	136	270	172	308
29 <sup>p</sup>	30	65	84	101 <sup>p</sup>	102	137P	138	173 <sup>p</sup>	174
30ª	72	66 <sup>a</sup>	144	102ª	216	138ª	288	174ª	360
31 <sup>p</sup>	32	67P	68	103P	104	139 <sup>p</sup>	140	175	248
32	63	68	126	a 10h	210	a140	336	a <sub>176</sub>	372
33	48	69	96	105	192	141	192	177	240
34	54	a70	144	106	162	142	216	178	270
35	48	71 <sup>p</sup>	72	107 <sup>p</sup>	108	143	168	179 <sup>p</sup>	180
36 <sup>a</sup>	91	72ª	195	108ª	280	14hª	<b>LO</b> 3	180ª	546

## THE SUM OF ALL DIVISORS OF NUMBERS FROM 181 TO 360

181 <sup>p</sup>	182	217	256	253	288	289	307	325	434
182	336	218	330	254	384	290	540	326	492
183	248	219	296	255	432	291	392	327	440
184	360	<sup>a</sup> 220	504	256	511	292	518	328	630
185	228	221	252	257 <sup>p</sup>	258	293 <sup>p</sup>	294	329	38և
186 <sup>a</sup>	384	222 <sup>a</sup>	456	258 <sup>a</sup>	528	294ª	684	330 <sup>a</sup>	864
187	216	223 <sup>p</sup>	224	259	30h	295	360	331 <sup>P</sup>	332
188	336	a 224	504	<sup>a</sup> 260	588	296	570	332	588
189	320	225	403	261	390	297	480	333	494
190	360	226	342	262	396	298	L50	33h	504
191 <sup>p</sup>	192	227 <sup>p</sup>	228	263 <sup>p</sup>	264	299	336	335	408
192 <sup>a</sup>	508	228 <sup>a</sup>	560	264ª	720	300 <sup>a</sup>	868	336ª	992
193 <sup>p</sup>	19և	229 <sup>p</sup>	230	265	324	301	352	337 <sup>p</sup>	338
19և	294	230	432	266	480	302	456	338	549
195	336	231	384	267	360	303	408	339	456
<b>a</b> 196	399	232	450	268	476	a304	620	a 31,0	756
197 <sup>P</sup>	198	233 <sup>p</sup>	234	269 <sup>p</sup>	270	305	372	341	384
198 <sup>a</sup>	468	234 <sup>a</sup>	546	270ª	720	306ª	702	342ª	780
199 <sup>‡</sup>	200	235	288	2 <b>71</b> P	272	307 <sup>p</sup>	308	343	400
<sup>a</sup> 200	465	236	420	<b>a</b> 2 <b>7</b> 2	558	<sup>a</sup> 308	672	344	660
201	272	237	320	273	448	309	h16	345	576
202	306	238	432	274	414	310	576	346	522
203	2110	239 <sup>p</sup>	240	275	372	311 <sup>p</sup>	312	347P	348
204ª	504	240 <sup>a</sup>	744	276ª	672	312 <sup>a</sup>	840	348ª	840
205	252	241 <sup>p</sup>	242	277 <sup>p</sup>	278	313 <sup>p</sup>	314	349 <sup>p</sup>	350
206	312	242	399	278	420	314	474	<sup>a</sup> 350	744
207	312	243	36L	279	416	315	624	351	560
<sup>a</sup> 208	434	2114	113 la	<sup>a</sup> 280	720	316	560	a 352	756
209	270	245	342	281 <sup>p</sup>	282	317 <sup>p</sup>	318	353 <sup>p</sup>	354
210 <sup>a</sup>	576	21:6ª	504	282ª	576	318 <sup>a</sup>	648	354ª	720
211 <sup>p</sup>	212	247	280	283 <sup>P</sup>	284	319	360	355	432
212	378	248	480	284	50h	<sup>a</sup> 320	762	356	630
213	288	249	336	285	480	321	432	357	576
214	324	250	468	286	50և	322	576	358	540
215	264	251 <sup>p</sup>	252	287	336	323	360	359 <sup>p</sup>	360
216 <sup>a</sup>	600	252 <sup>a</sup>	728	288 <sup>a</sup>	819	324ª	847	360 <sup>a</sup>	1170

We call numbers of the first kind poor or deficient, those of the second kind rich or abundant. Those of the third kind have been called since antiquity perfect. In the list from 2 to 360 one finds only two perfect numbers, namely 6 and 28.

One might expect that in the sequence of numbers 1, 2, 3, 4, 5, 6, ... some sort of simple perpetual law would hold for all the contents, and also for the divisor-sums s. The matter is, however, extraordinarily

complicated, as the following considerations will show.

12, it will be seen, is the first abundant number. The next abundant ones are 18, 20, 24, 30, 36, 40. It is not difficult to prove [3] that all of the multiples of 6, with the exception of the perfect number 6, are abundant. These multiples on (n = 2, 3, h, · · · ) we shall call the normal abundant numbers. They are indicated in the list by an "a" placed to the right. Besides these normal ones there are other abundant numbers, for example 40 and 308; they are indicated by an "a" placed to the left. Our list displays no odd abundant number. Such numbers do exist, however, beyond 360, e.g. 1575 = 9.25.7 with divisor-sum 322h.

It is not difficult to give examples of entire classes of abundant numbers which do not belong to the normal ones. We mention in this re-

gard only the following facts:

Every number of the form  $4.5^n$  (such as 4.5 = 20, 4.25 = 100, 4.125

= 500, etc.) is abundant. -

Every multiple of an abundant number is again abundant. -There are numbers whose abundancy exceeds any given bound, i.e. numbers for which s is greater than the tenfold or hundredfold etc. value of the number itself. [h] -

The least content which a number can have is one. The poorest, or most deficient, numbers have this property; they are the prime numbers. For good reasons, the number one is not counted among them. Every prime

number p has only the content 1 and the divisor-sum s = 1 + p.

In the list, prime numbers are indicated by a letter "p". It will be noticed at once that they are always neighbors of normal abundant numbers. In fact, every prime number is either of the form 6n+1 or 6n-1, since the other possible forms 6n + 2, 6n - 2, 6n + 3, 6n - 3 all represent numbers which are either divisible by 2 or by 3. But not every neighbor of a normal abundant number is a prime.

Every number may be represented as product of these most deficient numbers, the representation being unique up to order of factors. For every number n there is thus a natural representation n = paqbrc..., where p, q, r, ... are prime numbers. For example, 360 = 2.2.2.3.3.5 =

23.32.5

The normal deficient numbers are the prime numbers and all their powers, that is e.g. 7,  $7^2 = 49$ ,  $7^3 = 343$ , etc. But besides these there are entire classes of further deficient numbers. It is not hard to show:

All numbers consisting of two odd primes are deficient, for example all numbers of the form  $3^a \cdot 5^b$ , or  $5^a \cdot 7^b \cdot [5]$  —

All numbers of the form 5a.7b.11c are deficient. - Numbers of the form 3a.5b.7c are only deficient for a few small values of a, b, c, as for example 3.5.7 = 105 and 9.5.7 = 315, while 9.25.7 = 1575 is abundant. -

Already in this division into rich and poor, we see an extraordinary diversity.

It is naturally with special interest that one turns to the perfect numbers. These are extremely rare. It is not known to this day, despite laborious investigations, whether there is an unlimited or only a finite number of them. Twelve of them were known by the year 1950; at the time of this writing (October 1958), their number had been brought up to 17. The structure of an even perfect number is easy to describe. We have:

Whenever  $1+2+2^2+\cdots+2^m=p$  is prime, then  $n=2^m\cdot p$  is a perfect number, and conversely every even perfect number has this structure. The difficulty lies in knowing when the sum in question is prime. For example, m=1 yields the prime number 1+2=3 and  $n=2^1\cdot 3=6$  is perfect. m=2 also yields a prime number  $1+2+2^2=7$ , leading to the perfect number  $n=2^2\cdot 7=28$ . However, m=3 yields  $1+2+2^2+2^3=15$ . But m=1 yields  $1+2+2^2+2^3+2^{11}=31$ , so that  $n=2^{11}\cdot 31=196$  represents the third perfect number. The 17 values of m=1 leading to perfect numbers known as of 1958 are m=1, 2, 1, 6, 12, 16, 18, 30, 88, 106, 126, 520, 606, 1278, 2202 and 2280 [to which seven more have since been added: 3216, 1252, 1122, 9688, 9910, 11212 and 19936 — see translator's note (\*)].

Not a single odd perfect number is known. However, despite great efforts made, it is also not known whether none actually exists. On the other hand, it has been determined that there is surely none less than 1h0,000,000,000,000 and that a possibly existent odd perfect number would have to possess at least six different prime factors.

Thus here, too, we see that the properties of the sequence of natural numbers are far more complicated than one might at first be in-

clined to suppose. -

From the list, we see that the four numbers 5h and 56 and 87 and 95 share the same divisor-sum s = 120. We call such numbers related. Two related numbers can display a particularly remarkable link with one another. The seven numbers 20h, 220, 22h, 2h6, 28h, 286 and 33h with divisor-sum s = 50h are related. Among them, the two numbers 220 and 28h have the following property: The content of 220 is 28h and that of 28h is 220. For in fact:

1+2+4+5+10+11+20+22+44+55+110 = 284,1+2+4+71+142 = 220.

The content of one number thus yields the other. There is a story, handed down by tradition, that Pythagoras, in answer to a question on the nature of friendship, said that friends were to one another as 220 and 28h. This is still worth pondering today.

The six smallest pairs of friendly or amicable numbers are

220 1184 2620 6232 10,744 17,296 284 1210 2924 6368 10,856 18,416.

Some hOO pairs are known. But it is not known whether any general law holds for their structure, lending special importance to certain primes, let it be expressly observed. (\*\*)

4. The obvious question, in what manner the primes, the poorest numbers, are distributed in the sequence of natural numbers, has led to investigations of the deepest and most difficult kind. Indeed, this question poses a strange riddle to our thinking. If we count 1, 2, 3, etc. to 18, say 18 successive days, then this period may be subdivided into various shorter periods, e.g. twice nine days, or three times six days. If we take, however, the past 19 days, then such a division into subperiods of equal length is not possible. It becomes so again for 20, 21 and 22 days, while 23 days can serve only as a whole period-unit. It would seem as though something so conceptually simple and perfectly clear as prime numbers should pose no difficulty to our thinking in trying to grasp the law of their appearance within the progressive sequence. To be sure, it is easy to see that the sequence of primes never comes to an end. (\*\*\*) In order to convey some impression, we may mention the following facts.

Between 1 and 100 there are 25 prime numbers, between 100 and 200 there are 21 of them, between 200 and 300 there are 16, between 268300 and 268h00 only 1, but between 299900 and 300000 again 9 of them. -

One can easily specify ten thousand successive natural numbers, no a single one of which is prime. Instead of ten thousand, one could pic any other number; that is, there are prime number gaps exceeding any arbitrary length. [6] -

It sometimes happens that both neighbors of a normal abundant number are prime numbers; for example, 11 and 13, 29 and 31 are such twins. It is not known to this day whether there are only finitely many such pairs of twins. As far as direct inspection permits us to tell, they seem to keep reappearing, e.g. 1,000,000,009,619 and 1,000,000,009,651.

Between any number n and its double 2n there is at least one prime number. It can be shown by elementary but fairly complicated means that from n = 720 on there are always at least 100 prime numbers between n and 2n. -

When we ask about the distribution of primes within the natural sequence, we must always bear in mind that we are dealing with the kind of distribution in relation to the size of number, although size ultimately has little to do with individual structure. The following example of three nearly like-size numbers is quite instructive:

$$370,273 = 13.79.109,$$
  
 $370,277 = 17.23.917,$   
 $370,279 = 7.13^2.313.$ 

Even closely neighboring numbers can display totally different structures. Despite this circumstance, we are able to give a rule about the average distribution. Let the number of primes occurring in the sequence from 1 to n be called P(n). E.g. P(100) = 25, P(200) = 46, P(300) = 62. The order of magnitude of P(n) is then given by  $P(n) \approx n: (\frac{1}{2} + \frac{1}{3} + \frac{1}{h} + \frac{1}{5} + \cdots + \frac{1}{n}) \; .$ 

$$P(n) \approx n: (\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n}).$$

This approximation becomes ever better as n increases. Exactly expressed, we have the remarkable statement: The value of

$$\frac{1}{n} \cdot P(n) \cdot (\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{n})$$

tends toward 1 as n becomes arbitrarily large.

This rule for the average distribution, which was first conjectured - in somewhat sharper form (\*\*\*\*) - by the not yet 20-year-old C. F. Gauss, was not able to be verified until 1896, and then only by application of the most difficult means. To make the matter more readily understandab we have given it here an elementary form. A few years ago a somewhat sim pler proof was found, but one which still requires a rather extensive mathematical training to understand.

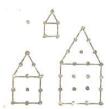
To give the reader an idea of the approximation, we list together in the following table the exact number P(n) of primes up to n = 1000, up to 10,000, and so on, with the values of the sum  $1/2 + 1/3 + 1/4 + 1/5 + \cdots + 1/n$ and the product indicated above, tending ever closer to 1.

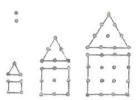
n	P(n)	$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$	$\frac{1}{n} \cdot P(n) \cdot (\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n})$
1,000	168	6.4854	1.089
10,000	1,229	8.7876	1.080 · · · ·
100,000	9,592	11.0901	1.063
1,000,000	78,498	13.3927 · · ·	1.051
10,000,000	664,579	15.6953	1.043
100,000,000	5,761,455	17.9978	1.036
1,000,000,000	50,847,534 (**)	20.3004	1.032

5. According to the theorem just described, we know the approximate number of primes occurring in the sequence 1, 2, 3, h, 5, 6 to n, becoming more accurate as we increase the length of the partial segment of the sequence of all natural numbers, i.e. with increasing n, forming the true value of the ratio, tending toward 1:1. But it is precisely the departures from this rule which merit our real interest. Only in these do we find expression of the individual structures of the numbers. Now, it is in fact possible to give an exact law determining the distribution of primes as well, albeit in a form which is scarcely usable for direct computation. A distinguished role is played herein by the so-called pentagonal numbers. There are two kinds of pentagonal numbers, which may be determined as follows:

,	ac ocamane o	" WO TOTTOMO.	0-	= 0	
12	= 1	1+12		= 2	
1+22	= 5	1+2+22		= 7	
1+2+32	= 12	1+2+3+3	,2	= 15	
1+2+3+42	= 22	1+2+3+1	1 + 42	= 26	
1+2+3+4+52	= 35	1+2+3+1	1+5+52	= 40	
1+2+3+4+5+6	2 = 51	1+2+3+1	1+5+6+62	= 57	
* * * * * * * * * * * * * * *		*******			

The following figures show how these numbers arise geometrically:





They may also be represented by means of regular pentagons, and what is more, and particularly important, but would lead too far here, they may be characterized by a peculiar, purely arithmetical property.

If one orders them in the sequence

..., 77, 57, h0, 26, 15, 7, 2, 0, 1, 5, 12, 22, 35, 51, 70, ... and forms in each case the difference of two successive numbers, subtracting from each the preceding number, then one obtains the sequence ..., -20, -17, -1h, -11, -8, -5, -2, 1, h, 7, 10, 13, 16, 19, .... This displays throughout a constant step of 3 from term to term.

We now mark off these pentagonal numbers in the sequence of natural numbers:

61	62	63	64	65	66	67	68	69	70	71	72
49	50	51	52	53	54	55	56	57	58	59	60
37				41	15	43	44	45	46	47	48
25							32			35	36
13			16					21	-	23	
1	2							9	10	11	12

Let us now think of some number n as given, and find by how much it exceeds all lesser pentagonal numbers. E.g. n=30 yields the excesses

$$n-1=29$$
,  $n-2=28$ ,  $n-5=25$ ,  $n-7=23$ ,  $n-12=18$ ,  $n-15=15$ ,  $n-22=8$ ,  $n-26=4$ .

We take these excesses and check their divisor-sums s . Either by calculation or from the list:

$$s_{29} = 30$$
,  $s_{28} = 56$ ,  $s_{25} = 31$ ,  $s_{23} = 24$ ,  $s_{18} = 39$ ,  $s_{15} = 24$ ,  $s_{8} = 15$ ,  $s_{1} = 7$ .

Now we add the first two of these divisor-sums, subtract the next two, add the next two, etc.:

$$30 + 56 - 31 - 24 + 39 + 24 - 15 - 7 = 72$$
.

The result is the divisor-sum of 30.

For n = 38 the excesses are: 37, 36, 33, 31, 26, 23, 16, 12, 3. The divisor-sums are: 38, 91, 48, 32, 42, 24, 31, 28, 4.

Adding and subtracting alternate pairs in this series yields 38+91-18-32+12+21-31-28+1=60, which is the divisor-sum of 38.

For n = 16: Excesses 15, 1 $\mu$ , 11, 9,  $\mu$ , 1 with divisor-sums  $2\mu$ ,  $2\mu$ , 12, 13, 7 and 1, where  $2\mu + 2\mu - 12 - 13 + 7 + 1 = 31$ , the divisor-sum of 16. In general, if n yields the excesses

n-1, n-2, n-5, n-7, n-12, n-15, etc.,

and we call their divisor-sums

sn-1, sn-2, sn-5, sn-7, sn-12, sn-15, etc.,

then we have the law

$$s_n = s_{n-1} + s_{n-2} - s_{n-5} - s_{n-7} + s_{n-12} + s_{n-15} \cdots$$
 [†]

This altogether remarkable state of affairs holds true for every number n which is not a pentagonal number. A simple convention makes it possible to encorporate the pentagonal numbers as well. If n is such a number, then there occurs in the end an excess of zero. As divisor-sum of this zero, we must take now the pentagonal number n itself. The law extended in this fashion is then valid for every natural number.

For example n = 12: Excesses 11, 10, 7, 5, 0 with divisor-sums 12, 18, 8, 6, 12. This yields indeed  $12 + 18 - 8 - 6 + 12 = 28 = s_{12}$ .

This law was discovered under the name pentagonal theorem by L. Euler (between 17hl and 1750). Unfortunately it has attracted very little attention, even among mathematicians. There are even many today who know nothing of it, although it must be counted as one of the most beautiful discoveries of modern times. Its proof requires considerable means.

To the right of the equals-sign in  $[\dagger]$  there occur divisor-sums of numbers which are all smaller than n. The structures of these excess-numbers thus predetermine the divisor-sum of n. The number n is therefore prime if and only if the resulting value of  $s_n$  is n+1. That is to say: The structures of numbers that are smaller than n, namely its excess-numbers, also determine whether n is prime or not.

We are looking here into the subtle, wonderful weaving of the natural numbers. The pentagonal numbers, of which only 2,5 and 7 are prime as may easily be seen, and whose progression is determined by steps of 3 in the sequence of second differences, play a decisive part in this weaving. It is by the law which we have given that the strange irregular-seeming behavior of the divisor-sums s2, s3, s1, s5, s6, s7 etc. of successive numbers and in particular the appearance of prime numbers is after all regulated. [7]

The matters set forth here, to which many others could be added, gain a heightened significance if one is prepared to enter into a battle going on beneath the wrestling for understanding of the newer findings of physics and tending today toward a certain climax, although it made its first appearance historically in the time of the Scholastics. Do the concepts grasped by thinking indicate real entities, albeit appearing only as mirror-images in conscious understanding, or are they mere nomina, abstractions formed from the experiences of the senses? Without being able to go into particulars here, be it only mentioned that a number of results in the mathematics of our century have made it particularly clear that certain concepts are merely nominalistic. while others possess realistic significance. For example in set theory we must distinguish clearly between the "collection" [Gesamtheit] as more nomen and the "set" [Menge] as entity if we are to overcome the so-called antinomies, as discovered by P. Finsler. Another important result is the theorem of Skolem (1929) from which it follows that the sequence of natural numbers may not be characterized by any finite axiom system, for any such finite system would also permit interpretations that are inequivalent to the natural numbers.

The concept "natural number" is of nominalistic nature, a mere abstraction, while individual numbers, as R. Steiner pointed out on oc-

casion, represent entities.

One may also consider the thought how completely numbers place themselves at our disposal, renouncing all life of their own, remaining

ever serving beings.

In any case one will no longer harbor the commonplace view that the sequence of natural numbers is something banal, even to be disdained, after gaining insight into its wonderful weaving. This insight may waken a feeling. Even if what is felt is hard to grasp, even if it only shimmers through as though gleaming from a great distance, we have been permitted to take a look into the workshop of the Cherubin, from whence the number-beings take their origin.

Let us consider, too, what great significance it could have if this feeling were to be awakened in young persons. Whoever has once been permitted to take such a look will never abuse the number-sequence by slavish labor. It represents a most wonderful kind of art-work of the spirit. To be sure, it is an art-work that springs from worlds of necessity. But whosever wishes to be freely creative must, above all, learn

to fit in rightly with necessities.

The following comments contain some supplementary indications.

[1] To find the digits of the number n, written with help of base b, one forms the chain  $n=bn_1+a_0$ ,  $n_1=bn_2+a_1$ ,  $n_2=bn_3+a_2$ , etc., up to  $b\cdot 0+a_k$  (where all the a's are smaller than b). We obtain thus  $n=a_0+a_1b+a_2b^2+\cdots+a_nb^n$ , and  $a_0,a_1,a_2,\cdots,a_k$  are the digits we seek.

[2] If the number n has the natural representation  $n = p^{a} \cdot q^{b} \cdot r^{c} \cdots \text{ (where p, q, r, \cdots are primes)}$  then its divisor-sum has the value

 $s = (1+p+\cdots+p^{a})(1+q+\cdots+q^{b})(1+r+\cdots+r^{c})\cdots,$ 

as may be seen immediately by multiplying out. One can also write  $s = n(1+1/p+\dots+1/p^a)(1+1/q+\dots+1/q^b)\dots.$ 

The number n is thus abundant if and only if

 $(1+1/p+\cdots+1/p^{a})(1+1/q+\cdots+1/q^{b})\cdots > 2$ , [††]

- [3] For a number of form  $n=2^a\cdot 3^b$  (a,b both positive integers) we have  $(1+1/2+\cdots+1/2^a)(1+1/3+\cdots+1/3^b)\geqslant 2$  with equality holding if and only if a=b=1, i.e. n=6.
- [h] Because of the divergence of the harmonic series, as the number of primes p, q, r, · · · increases and the values of a, b, c, · · · grow larger, the left side of [††] exceeds every finite bound.
- [5] The product  $(1+1/p+\cdots+1/p^a)(1+1/q+\cdots+1/q^b)$  tends to  $\frac{p}{p-1}\cdot\frac{q}{q-1}$  as a and b increase. As this is always less than 2 for p and  $\frac{p}{q}$  odd primes, the indicated result follows.
- [6] Let the product of all natural numbers from 1 to 10,001 be denoted by  $\mathbb{P}$ . Then among the ten thousand consecutive numbers  $\mathbb{P}+2$ ,  $\mathbb{P}+3$ ,  $\mathbb{P}+h$ , ...,  $\mathbb{P}+10,001$  there is not a single prime number since the first is divisible by 2, the second by 3, the third by h, and so on.
- [7] Unfortunately, there seems to be no literature on the pentagonal theorem which is accessible to a wider circle. For further information on the distribution of prime numbers, one good English text at elementary level is Ore's Number Theory and its History, an intermediate one Hardy and Wright's Introduction to the Theory of Numbers, and an advanced one Ingham's tract On The Distribution of Prime Numbers, or Trost's excellent little book on Primzahlen in German.

### Translator's Notes:

- (\*) Such primes of form  $1+2+2^2+\cdots+2^m=2^{m+1}-1$  are called Mersenne primes Mq after the French Minorite Father who successfully found or guessed nine of the first twelve of them in the year 1644, by what means it is not known. A necessary but not sufficient condition for Mq to be prime is that q=m+1 itself be prime. His largest correct guess was  $M_{127}=2^{127}-1=170,141,183,460,469,231,731,687,303,715,884,105,727,$  whose size, because we can notate it so compactly when it is "parcelled" in the familiar way, we may not appreciate. If we were to write it instead as a string of ones to be added, say at typical typewriter-density of ten ones to the inch, then scanning the length of such a number even at the speed of light (186,000 miles per second) would take approximately 45.7 billion billion years and that is only  $M_{127}$  with 39 decimal digits.  $M_{11213}$  (discovered in 1968 by Robinson) has 3375 decimal digits, and  $M_{19937}$  (discovered in 1971 by Tuckerman) 6001 digits, the ratio of binary exponent to number of decimal digits approaching log 10: log 2.
- (\*\*) On p.11 Locher originally wrote that only "somewhat over 200 pairs [were] known" whereas Escott (Scripta Mathematica, 12, 61-72) listed 390 pairs known by the year 1916. On the bottom of p.12, however, Locher gives (as corrected in a footnote to the 1959 Sternkalender) Lehmer's revised value for P(1,000,000,000), which is 56 greater than the earlier value found by Bertelsen cited in Ore as well as Hardy and Wright. (In the same footnote, Locher gives Lehmer's value for P(1010) as 155,052,512.)
- (\*\*\*) Euclid, in his Elements, gave a simple proof of this over 2000 years ago, using a construction similar to that used in note [6] above: Assume that there exists a last, largest prime number, p, and form the product of all primes up to it,  $P = 2 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot p$ . Then since numbers differing by 1 can have no factors greater than 1 in common, we have that P+1 cannot be divisible by 2 or 3 or 5 or any other prime

up to the supposed last, largest one, p; therefore it is either a new, vastly larger prime in its own right, or else composed of one or more new, larger prime factors, either way contradicting the assumption that p was the last, largest prime, hence there is no such last prime. Putting this result together with that in [6], we see that while the sequence of primes never ceases, it does thin out, becoming ever more rarified. (Euclid also knew, and proved, the indicated form of all even perfect numbers:  $2^{m}(2^{m+1}-1)$ , where  $2^{m+1}-1$  is prime.)

(\*\*\*\*) Gauss suspected some such relationship already at age 14 when he received his first table of logarithms. The sharper form in which he later stated it was that

$$\frac{1}{n} \cdot P(n) \cdot \log n \longrightarrow 1$$
 as  $n \longrightarrow \infty$ 

where "log" here means natural logarithm to the base of Euler's exponential function  $e^{X},\ e=2.718281828h59\cdots$ . To see the connection with Locher's simplified statement, we need to introduce another constant named by Euler  $\gamma=0.57721566h9\cdots$ , equal to the limit of the first n terms of the harmonic series minus the natural logarithm of n, as n tends to infinity:  $\gamma=\frac{1}{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{2}+\frac{1}{1}+\cdots+\frac{1}{n}-\log n\right),$ 

whence 
$$\log n \approx (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \gamma)$$
; what Locher

has done, therefore, is simply to combine the 1 and the  $-\gamma$  into a single summand 1 - 0.5772...  $\doteq$  0.4228 and then suppress it, since in the limit it becomes arbitrarily small in comparison with log n,

so that 
$$\log n \approx (\frac{1}{2} + \frac{1}{3} + \frac{1}{b} + \frac{1}{5} + \cdots + \frac{1}{n})$$
 as well,

converging to the same limit-values, only slightly more slowly, hence not so "sharply" stated, but avoiding introduction of transcendental functions. Chebyshef proved a weaker statement of this prime number theorem around 1850, asserting only that P(n) [or  $\pi(n)$  as most books call it] was of the same order of magnitude as n divided by logn. Gauss' sharper form was proved independently by Hadamard and de la Vallée Poussin in 1896, about 100 years after Gauss first stated it.

References to the prime number theorem may be found given in comment [7] above. As references to generalized pentagonal numbers 1, 2, 5, 7, 12, 15, ..., Sloane (Handbook of Integer Sequences) cites the American Math. Monthly Vol. 76 (1969) p. 88h and R. Honsberger's Ingenuity in Mathematics (1970) p. 119; Chapt. XVIII of A. H. Beiler's Recreations in the Theory of Numbers generalizes the strictly pentagonal numbers 1, 5, 12, 22, ..., to other polygonal forms. The strictly pentagonal numbers 1, 5, 12, 22, ... (those on the left on p. 13 above — see also the cover for a drawing of 22 as gnomonically nested pentagons) are all of form n(3n-1)/2, n=1, 2, 3, ...; the numbers of the related sequence 2, 7, 15, 26, ... (those on the right on p. 13) are all of form n(3n+1)/2, n=1, 2, 3, ..., the two sequences overlapping in the number 0 when extended to include n=0. These two kinds of pentagonal numbers show up as exponents in a famous identity by Euler, cited in Hardy and Wright (p. 28h): n=1 (1-x<sup>n</sup>) =  $n=-\infty$  (-1)<sup>n</sup>  $n^{\frac{1}{2}(3n+1)}$  = n=1 (-1)<sup>n</sup>  $n^{\frac{1}{2}(3n-1)}$  n=1 n=1 (1-x<sup>n</sup>) n=1 n=1 n=1 (1-x<sup>n</sup>) n=1 n=1

i.e. 
$$(1-x)(1-x^2)(1-x^3)\cdots = 1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-\cdots$$

Hardy and Wright then go on (pp. 285-286) to relate coefficients of terms in the above series to problems in partition theory, which is perhaps the "peculiar, purely arithmetical property" which Locher had in mind on p. 13.

[S. Eberhart]

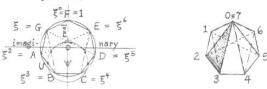
# ON THE GEOMETRY OF REGULAR (n<sup>2</sup>+n+1)-GONS by Stephen Eberhart

Of the three problems posed on p. 22 of the last issue, in connection with the geometry of the regular heptagon, the first two remain unsolved: 1) Given A+B+C =  $180^{\circ}$  and A:B:C = 1:2:h, find (if possible) a quadratic irrational expression (presumably in terms of  $\sqrt{7}$ ) for  $\sin A + \sin B + \sin C$ ; 2) given A+B+C =  $180^{\circ}$  but A:B:C in any arbitrary ratio, offer some insight into the remarkable identity  $\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$ . The third problem however, generalizing Demir's result on pp. 13-1h of that issue, has borne considerable fruit which will be reported here.

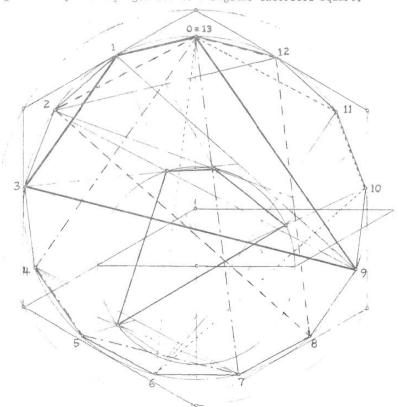
Victor Thébault had posed as a problem (Ell5 $\mu$ ) in the American Math. Monthly: Let ABCDEFG be a regular heptagon inscribed in a unit circle, with U the midpoint of side AB and V the midpoint of the radius perpendicular to side BC; then UV =  $\sqrt{2}/2$ , or half the side-length of a square inscribed in the same circle, one trigonometric proof of which we gave in the last issue. Hüseyin Demir gave a simpler proof and obtained a stronger result: AB and its mirror DC are only two out of six chords of the heptagon whose midpoints lie at that same distance from point V, the other four being BG, GD, and their mirrors CE, EA, forming a twisted hexagon ABGDCE (see upper left illustration on next page). The result follows immediately if we associate points U and V with numbers u and v in the complex plane, thinking of points A to G as powers of  $\xi$ , a primitive 7th root of unity; if we choose v = -1/2 on the real axis, then we have A =  $\xi^2$  and B =  $\xi^3$  (or their conjugates), whence  $u = (\xi^2 + \xi^3)/2$  and the square of the distance in question  $UV^2 = (u - v)(\bar{u} - \bar{v}) = (\xi^2 + \xi^3 + 1)(\xi^5 + \xi^{l_1} + 1)/l_1 = (-1 + 3)/l_1 = 2/l_1$ , whence  $UV = V\bar{V}/2$  as required by Thébault.

The conjugate symmetry makes clear that we could have associated U equally well with  $u = (\xi^2 + \xi^3)/2$  or  $\bar{u} = (\xi^5 + \xi^4)/2$ , but what about the other four chord-midpoints which Demir found to lie at the same distance from V? Once we are given them, we can associate them in a similar way with appropriate complex numbers and carry out a similar product to compute the distance of each one from V as above, but how could we discover them if they were not given? Look at the product: Each factor has three terms, so the product will have nine; one term in each factor is 1, so one term in the product will be 1; if each of the powers of  $\xi$  from  $\xi$  to  $\xi^6$ occurs once, then the other two terms must both be  $\xi 7 = \xi^0 = 1$ , occurring twice; since  $\xi + \xi^2 + \cdots + \xi^{m-1} = -1$  for any  $\xi$  equal to a primitive mth root of unity, the nine terms of the (numerator of) the product will sum to -1+3=2 as above. The question thus boils down to this: What is the necessary condition that each of the powers of  $\xi$  from  $\xi$  to  $\xi^6$  ( $\xi$  to  $\xi^{m-1}$  in general) occur exactly once in the product? Look at the exponents:  $\hat{l} = \xi^{0}$ , and the numbers 5, h, 0 are just the negatives (modulo 7) of the numbers 2, 3, 0. Each term of the product is thus a power of form  $\xi^{\rm d}$  where d is one of the nine possible differences which may be formed by choosing two numbers (with repetition) from the set  $\{2,3,0\}$   $\equiv$   $\{2,3,7\}$ . 2-2=3-3=7-7=0, hence  $\xi^0=1$  will occur three times in any case. Taking differences of distinct numbers of the set {p,q,r} will yield exponents from 1 to 6, hence powers from  $\xi$  to  $\xi^6$ , once apiece if and only if those numbers form a perfect difference set modulo 7. What we seek, then, are the solutions p,q if we fix r=7. Since the properties of perfect difference sets (or p.d.s.'s) are well known, this gives us the key to still greater generalization. In the case of mod 7, there is essentially only one solution  $\{p,q,r\}$ , which may be visualized as a triangle spanning 1, 2, and 4 sides of a heptagon with corners numbered from 1 to 7 (it is the triangle of problem 1 above - see upper right illustration on next page); all others may be derived from it by rotation

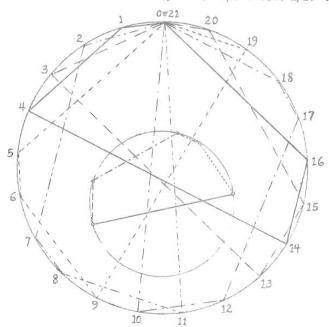
and reflection, fourteen positions in all, six having any given corner in common. If we select those six having corner 7 in common and delete the two sides of each that meet in that corner, the six remaining sides are precisely the heptagon chords which make up Demir's twisted hexagon, with concyclic midpoints.



Perfect difference sets are known to exist mod m whenever m is of form  $n^2+n+1$  and n is either a prime or a power of a prime; m = 7 corresponds to n = 2, so the next case will be m =  $13 = 3^2+3+1$ , from n = 3. The p.d.s.'s will now form quadrangles, such as that given below for  $\{1,3,9,13\}$ . We rotate and reflect this quadrangle to all eight positions having corner 13 in common, delete the two sides of each meeting in that corner, and find the centroids of the remaining triangles to be again concyclic, forming similar quadrangles (one shown below) but reflected and reduced in size by  $\sqrt{3}/3$  or  $1/\sqrt{3}$ , leading to a regular circumscribed hexagon the way the heptagon led to a regular inscribed square.



The same process may be repeated for  $n = h = 2^2$ , yielding m = 21=  $h^2+h+1$ : We take any p.d.s. mod 21, e.g. {1, h, 1h, 16, 21}, draw this as an (irregular) inscribed pentagon, and rotate it to each of the five positions having corner 21 in common, delete from each the two sides meeting in that corner, find the centroids of the remaining quadrangles (whose construction we surpress here for simplicity's sake), and find these again concyclic, forming a similar (irregular) pentagon, reflected and reduced in size by  $1/\sqrt{L}=1/2$ . As the center of the smaller circle was at point V, 1/2-way down the radius of the heptagon for n=2, and 1/3 of the radius of the 13-gon ("treiskaidecagon") for n=3, so it is 1/L-way down that of the 21-gon ("eiskaieikosagon") for n = L. Since the diameter of the smaller circle equals the radius of the larger for n = h, we are not led in this case to any (finite) related polygon. (To be consistent, we should have circumscribed the square about the circle of the heptagon; then we could say that as n = 1, 2, 3, h the related circumscribed polygon has 3, h, 6, oo sides, shrinking to an inscribed square and triangle, respectively, as n grows to 8 and 16 - a p-gon with  $p = 360^{\circ}/2 \arccos(2/\sqrt{n})$  for n > 4, but  $360^{\circ}/2 \arccos(\sqrt{n}/2)$  for n < 4.



For n = 2 and h, there is essentially only one p.d.s., up to rotation and reflection. For n = 3, however, there are two: the one we have seen and its double (mod 13)  $\{2,6,5,13\}$ . Analogous to the expressions we gave in the last issue for sums and products of trig. functions of angles A,B,C obeying A+B+C =  $180^\circ$  and A:B:C = 1:2:h, involving  $\sqrt{7}$ , we find for A:B:C = 1:3:9 the sum of sines to be  $\sqrt{(13+3\sqrt{13})/8}$ , sum of cosines  $(1+\sqrt{13})/h$ , sum of tangents = product of tangents  $-\sqrt{65}-18\sqrt{13}$ , etc., while A:B:C = 2:6:5 yield the conjugates of some of these (e.g. sum of tangents = product of tangents =  $\sqrt{65+18\sqrt{13}}$ ) but not others, in particular not sum of sines or cosines. Similar expressions involving  $\sqrt{21}$  may be found for A:B:C = 1:h:16, but not involving  $\sqrt{31}$  for 1:5:25 — why not? The cen-

troid geometry continues to work nicely for n = 5,7,8,9,1l etc. (but not for n = 6,10,12 as these are not primes or prime powers, so nop.d.s.'s exist), but as yet not a single nice expression involving Vm has been found for sums or products of trig. functions of angles A,B,C with  $A+B+C=180^{\circ}$ ,  $A+B+C=1810^{\circ}$ ,

Finally, we may look at the orthic triangles of such triangles: For n=2 (the "heptagonal triangle" with angles A:B:C = 1:2: $\mu$ ) we saw in the last issue that the orthic triangle was similar to the original one, reduced in size by 1/2. For n=3 we see there are two such triangles (one with A:B:C = 1:3:9 and the other 2:6:5); these turn out to be orthic triangles of one another, so that if one takes orthic triangles twice one obtains a triangle similar to the original one, reduced in size by  $(1/2)^2 = 1/h$  (due to properties of the 9-point circle, which see in the literature). 1:h:16 and 2:8:11 are similarly paired for n=h.

## NEW PROBLEMS

- 1. Among the numbers from 1 to 360, only 220 and 28h have one another as contents in an alternating cycle; most others eventually reduce to 1 acyclically by repeated taking of contents, although a few grow first before they shrink (e.g.  $22h \rightarrow 280 \rightarrow hh0 \rightarrow 6h0 \rightarrow 890 \rightarrow 730 \rightarrow 602 \rightarrow h5h \rightarrow 230 \rightarrow 202 \rightarrow 10h \rightarrow 106 \rightarrow 56 \rightarrow 6h \rightarrow 63 \rightarrow hl \rightarrow 1)$ ; a few others seem to "explode", growing without bound (e.g. 120 and 180), or do these, too, ultimately shrink? Dr. Georg Unger points out that some larger numbers form larger content-cycles (e.g. 12h96, 1h288, 15h72, 1h536, 1h26h is a 5-cycle); can the reader discover a 3- or h-cycle?
- 2. Every triangle ABC possesses two points the Brocard points (see Court's <u>College Geometry</u> in the College Outline Series) M,N such that angles MAB, MBC, MCA, NCB, NBA, NAC are all alike the Brocard angle. For the "heptagonal triangle" with A:B:C = 1:2: $\mu$  this angle is  $\mu$  arccot $\sqrt{7}$ ; what is it for the triangles with A:B:C = 1:3:9 or 1: $\mu$ :16?

#### HEXAGOOGOLPLEX

## by John Sprague

## STAR-WATCHER'S GUIDE - JANUARY TO MARCH

Two years ago (see Time, Oct. 27, 1975) Charles Kowal made observational news when he spotted a lith moon of Jupiter; the year before that he had found the 13th. This year (see November 1977 Sci. Am. or Nov. 12'77 Science News) observer Kowal made a discovery of far greater interest to those trying to reconstruct the physical history of the solar system: a new object comparable in size (about 300 mi. or 500 km. in diameter) to the largest asteroids, but much farther away — at or around the orbital distance of Uranus, not Jupiter! First sighted on October 18th with the 18-in. Schmidt telescope on Palomar Mountain, its exact orbit is still being determined (tentatively described as nearly circular, with an inclination of 3 to 5 degrees). At 18th to 19th magnitude, it is beyond the reach of most amateur observers, but should show up on many older photographic plates to help determine its orbit. The sizeestimate above is for an object with medium-bright surface like our Moon. If later found to have a darker surface like a carbonaceous chondrite, it would be larger; if icy and more reflective, then smaller.

Also of interest to planetologists is a recent report (same issue of <u>Sci</u>, <u>News</u>) that Neptune is much warmer, relative to the heat it receives from the Sun, than was formerly believed, emitting about 3.5 times as much heat as it takes in. Uranus, on the other hand, seems to have

little, if any, internal source of heat.

Meanwhile, a little closer to home, the orbiting Viking probe of Mars has carried out successfully what will probably remain a unique task in its mission: taking temperature-measurements of the midnight side. This is difficult, because it must be entirely within the planet's shadow or else exposure to direct sunlight would burn out its sensors, and it is only entirely in the planet's shadow twice in a Martian year (1.88 Earthyears). As would be expected (contrary to reporter Jonathan Eberhart's statements on p. 329 of the Nov. 12 Sci. News), the warmest region is the eastern rim, since "dawn" — seen from the backside — means what is coming from the daylit side into darkness. The "canyonlands", as on Earth, retained heat relatively well through the night, while the plains regions, where the great dust-storms begin, cool off relatively rapidly. Olympus Mons and three other volcanic mountains show up as isolated cold-spots.

An earlier issue of Science News (Sept. 21) reported that scientists speaking at a 3-day symposium held in Boston the middle of September seemed about equally divided as to the results of the biological probes on the Martian surface in July and September of 1976. The reactions observed were simply too unlike anything known by Earth-standards of organic and inorganic chemistry to be unambiguously interpreted without further and finer observations. The sensors on board should have been able to detect as few as 1,000,000 E. Coli cells per gram of soil, yet Earth-samples from the Mojave Desert have been found which contain as few as 100,000.

Earth-bound amateurs can do their own Mars-watching during January as the two planets closest approach on 
It is the brilliant the East at sunset, night, as it descritioning the constellar bright white object 

Tanuary February February culminating at mides its retrograde lation of Cancer. The procedure of the constellar in the cons

through the constelbright white object mini is Jupiter, seen rising later Regulus in Leo (Sanorth of the two). in the East before Venus is lost from pass one another, with the 22nd of that month. red object, rising in culminating at midbes its retrograde loop lation of Cancer. The rising earlier in Gewhile Saturn is to be near the 1st mag. starturn is the one further Mercury may be seen low sunrise in mid Jan., but view, reappearing as

evening star in the West, together with Mercury, in mid March (conj. on 12th).