TABLES OF SOLID PARTITIONS

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ABSTRACT

The recurrence relations for solid partitions are deduced. A table of partitions is also constructed for values of $n \leqslant 50$.

Gupta (1958) has recently published his tables of partitions for $p(n)$, the number of partitions of n into positive integers. A generalization of the *numerical* partition function has been taken up by several authors. Here, we shall be concerned with the generalization given by Nanda (1951)* from the analogy between the partition theory and thermodynamics of multi-dimensional oscillator assemblies.

The numerical partition function $p(n)$ represents the number of accessible wave functions of a linear oscillator assembly with an indefinite number of systems obeying Bose-Einstein statistics. In the case of two-dimensional oscillator assembly, the state of energy r is r-fold degenerate and therefore in the corresponding partition problem the summand r has to be regarded as capable of occurring in r different ways. Such partitions have been tabulated by Gupta (1951) for $n \leqslant 50$. It is interesting to note that the generating function for these partitions is the same as for MacMahon's *plane partitions* (1916). In this note we consider the partitions corresponding to MacMahon's solid partitions which are generated by

$$\sum_n p^{(3)}(n)x^n = \frac{1}{(1-x)(1-x^2)^3\ldots(1-x^2)^{\frac{r(r+1)}{2}}}. \quad \ldots \quad \ldots \quad (1)$$

Here, following the generalization of Gupta and Nanda, the summand r is regarded as capable of occurring in $r(r+1)/2$ different ways. The difference in the meaning of the term part magnitude here and in MacMahon's work should be noted.

From the generating function (1) we get the recurrence relation

$$np^{(3)}(n) = \tfrac{1}{2}\sum_{m=1}^{n} \{\sigma^2(m)+\sigma^3(m)\}\, p^{(3)}(n-m), \quad \ldots \quad \ldots \quad (2)$$

where $\sigma^s(m)$ denotes the sum of the sth powers of divisors of m. Another recurrence relation was used to check the values obtained from (2). For this we break the partitions into classes such that those having the same integer as the smallest summand are put in the same class. If $p^{(3)}(n, m)$ denotes the number of partitions of n in which m is the smallest summand then the generating function for this is

$$\frac{\frac{m(m+1)}{2}x^m.}{(1-x^m)^{m(m+1)/2}\ldots(1-x^r)^{r(r+1)/2}}. \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

* See also Gupta (1951).

From this we get

$$p^{(3)}(n, m) = \sum_{r=1}^{\infty} (-1)^{r+1} \binom{m(m+1)/2}{r} V^{(3)}(n-rm, m), \qquad \ldots \quad (4)$$

where

$$V^{(3)}(n, m) = \sum_{t=m}^{n} p^3(n, t).$$

We observe that

$$p^{(3)}(n, m) = 0 \qquad \text{for } \frac{n}{2} < m < n$$

and

$$= \frac{n(n+1)}{2} \text{ for } m = n,$$

also

$$\sum_{m=1}^{n} p^{(3)}(n, m) = p^{(3)}(n+1, 1) \equiv p^{(3)}(n), \qquad \ldots \quad \ldots \quad (5)$$

a relation which enables us to get the values of $p^{(3)}(n)$ from the table for $p^{(3)}(n, m)$.

In the following tables values for $p^{(3)}(n, m)$ are given for $n \leqslant 50$. Relation (5) enables us to find the number of solid partitions from these tables. The values calculated by Nanda (1953) for $n \leqslant 25$ have also been included for the sake of completeness.

TABLE 1

Solid Partitions for $n \leqslant 50$

$\overrightarrow{m}$ / $\downarrow n$	1	2	3	4	5	6	7	n
1	1							
2	1	3						
3	4	—	6					
4	10	6	—	10				
5	26	18	—	—	15			
6	59	40	21	—	—	21		
7	141	81	60	—	—	—	28	
8	310	201	90	55	—	—	—	36
9	692	414	182	150	—	—	—	45
10	1483	916	378	210	120	—	—	55
11	3162	1899	861	280	315	—	—	66
12	6583	3973	1737	580	420	231	—	78
13	13602	8059	3458	1275	540	588	—	91
14	27613	16402	6717	2905	675	756	406	105
15	55579	32561	13377	5350	1505	945	1008	120

TABLE I—*contd.*

now read across!
(cont) 2043
4
5

m \ n	16	17	18	19	20
1	1 10445	2 17554	4 24148	8 20294	15 72647
2	64520	1 25986	2 44448	4 69195	8 95077
3	25877	49949	95085	1 80254	3 38003
4	9985	17965	33665	62895	1 17287
5	3510	7995	14505	24405	40755
6	1155	1386	3409	8379	19047
7	1260	1540	1848	2184	2548
8	666	1620	1980	2376	2808
9	—	—	1035	2475	2970
10	—	—	—	—	1540
n	136	153	171	190	210

cont 2043
2044
2045

m \ n	21	22	23	24	25
1	29 92892	56 52954	106 05608	197 65082	366 09945
2	16 92143	31 79406	59 29721	109 93373	202 50589
3	6 31124	11 68226	21 51409	39 34674	71 59108
4	2 14610	3 89805	7 00720	12 59890	22 50405
5	70455	1 26605	2 32605	4 16700	7 34633
6	34083	56007	84819	1 34358	2 22334
7	7000	17976	40726	71974	1 16004
8	3276	3780	4320	13332	35478
9	3510	4095	4725	5400	6120
10	3630	4290	5005	5775	6600
11	—	2211	5148	6006	6930
12	—	—	—	3081	7098
n	231	253	276	300	325

cont

m \ n	26	27	28	29	30
1	674 05569	1234 12204	2247 28451	4071 19735	7338 78402
2	370 96872	675 68512	1224 37970	2207 21343	3959 98810
3	129 48649	233 07439	417 30764	743 74385	1319 27754
4	40 08717	70 92366	124 97237	219 04825	382 53450
5	12 71130	21 88860	37 60325	64 75130	111 24795
6	3 93393	7 10850	12 58432	21 46389	35 77350
7	1 72816	2 47730	3 72211	6 04492	10 53584
8	80046	1 39896	2 21868	3 25962	4 60494
9	6885	23910	65475	1 47060	2 54475
10	7480	8415	9405	10450	40810
11	7920	8976	10098	11286	12540
12	8190	9360	10608	11934	13338
13	4186	9555	10920	12376	13923
14	—	—	5565	12600	14280
15	—	—	—	—	7260
n	351	378	406	435	465

TABLE I—*contd.*

m \ n	31	32	33	34	35
1	13166 31730	23513 22765	41807 14647	74018 98452	1 30514 76707
2	7071 22884	12570 92767	22251 13351	39223 06946	68862 36977
3	2330 11531	4097 67123	7177 09981	12520 67535	21760 54195
4	665 11772	1152 30973	1988 29023	3418 74534	5856 58726
5	190 60313	324 85934	551 14688	930 82880	1567 37305
6	58 77319	96 60427	159 64935	266 05663	443 43334
7	18 92170	33 12904	55 84950	90 39548	143 12018
8	6 25464	9 13059	14 56146	25 27608	45 13050
9	3 98115	5 77980	8 06490	10 83645	14 24070
10	1 14345	2 55640	4 38460	6 77985	9 74215
11	13860	15246	66814	1 90674	4 24347
12	14820	16380	18018	19734	21528
13	15561	17290	19110	21021	23023
14	16065	17955	19950	22050	24255
15	16320	18360	20520	22800	25200
16	—	9316	20808	23256	25840
17	—	—	—	11781	26163
n	496	528	561	595	630

m \ n	36	37	38	39	40
1	2 29223 01583	4 01050 25130	6 99091 06888	12 14270 77241	21 01799 91927
2	1 20433 12447	2 09836 14248	3 64288 27758	6 30209 54451	10 86553 18673
3	37679 43120	65013 52674	1 11789 94054	1 91584 25146	3 27272 95697
4	9999 65454	17015 81818	28864 06281	48808 07408	82287 23944
5	2631 21893	4406 28107	7357 89262	12252 32276	20341 64689
6	737 24329	1217 47822	1999 33874	3266 57002	5322 45490
7	224 58198	354 70869	567 11207	919 76668	1499 06834
8	78 50700	130 97838	209 66490	323 29836	489 32616
9	20 22345	32 03565	55 40085	98 78625	170 81970
10	13 45025	17 90415	23 31175	29 67305	41 47000
11	7 22073	11 05302	15 74034	21 53217	28 42851
12	1 05560	3 05721	6 77391	11 44494	17 36514
13	25116	27300	29575	1 61707	4 73928
14	26565	28980	31500	34125	36855
15	27720	30360	33120	36000	39000
16	28560	31416	34408	37536	40800
17	29070	32130	35343	38709	42228
18	14706	32490	35910	39501	43263
19	—	—	18145	39900	43895
20	—	—	—	—	22150
n	666	703	741	780	820

TABLE I—*contd.*

m \ n	41	42	43	44
1	36 25831 31144	62 34574 40421	106 86374 19584	182 60674 05332
2	18 67173 23660	31 98382 30139	54 61682 64991	92 98484 90731
3	5 57316 88350	9 46173 21140	16 01609 89378	27 03293 59133
4	1 38321 59690	2 31858 34659	3 87568 41193	6 46119 73327
5	33678 16571	55605 76837	91579 35714	1 50452 99787
6	8655 59282	14060 38105	22803 40790	36914 57742
7	2436 58107	3928 87271	6279 99862	9965 14960
8	738 08937	1126 63089	1751 59866	2776 49796
9	282 78765	448 29315	684 36915	1011 26385
10	65 28610	113 05965	201 33575	346 83385
11	36 71646	46 39602	57 79455	79 55706
12	24 53451	33 29235	43 63866	55 96032
13	10 45681	17 55481	26 42913	37 07977
14	39690	2 41115	7 13475	15 67965
15	42120	45360	48720	52200
16	44200	47736	51408	55216
17	45900	49725	53703	57834
18	47196	51300	55575	60021
19	48070	52440	57000	61750
20	48510	53130	57960	63000
21	—	26796	58443	63756
22	—	—	—	32131
n	861	903	946	990

m \ n	45	46	47
1	311 10232 35156	528 47525 27400	895 18870 53300
2	157 84199 44639	267 17306 90660	450 97722 28585
3	45 50050 36262	76 37587 52539	127 86310 43784
4	10 74329 38259	17 81801 54117	29 47818 44719
5	2 46589 81544	4 03202 49180	6 57772 47496
6	59615 84132	96041 94327	1 54332 98748
7	15744 03177	24833 76070	39189 58855
8	4441 38255	7102 72497	11255 19240
9	1472 75334	2147 46585	3184 27755
10	570 19050	896 77720	1357 03150
11	125 20926	217 14121	386 99991
12	70 25733	86 96727	106 09014
13	49 95809	65 06409	82 90828
14	26 17125	39 13035	54 55695
15	3 51040	10 46880	22 92060
16	59160	63240	67456
17	62118	66555	71145
18	64638	69426	74385
19	66690	71820	77140
20	68250	73710	79380
21	69300	75075	81081
22	69828	75900	82225
23	—	38226	82800
n	1035	1081	1128

TABLE I—*concld.*

m \ n	48	49	50	51
1	1512 18472 89960	2547 56885 16119	4280 62165 96263	7174 23226 21601
2	759 17160 67571	1274 61228 18996	2134 50759 30601	
3	213 50675 53976	355 61826 58018	590 86557 60149	
4	48 65118 92764	80 10499 01953	131 59112 10238	
5	10 70639 35732	17 38833 82508	28 17992 46378	
6	2 47444 70079	3 94890 59867	6 32194 51760	
7	61881 51466	97713 48353	1 54102 95229	
8	17635 00290	27327 38637	42031 36392	
9	4833 88245	7515 09915	11832 44235	
10	1987 62025	2831 79435	3990 58946	
11	664 33917	1086 17377	1696 29669	
12	144 75474	827 81447	396 67173	
13	103 49066	127 38453	154 58989	
14	73 04010	94 57980	119 83755	
15	38 05740	56 55240	78 40560	
16	5 00344	15 01644	32 76172	
17	75888	80784	85833	
18	79515	84816	90288	
19	82650	88350	94240	
20	85260	91350	97650	
21	87318	93786	1 00485	
22	88803	95634	1 02718	
23	89700	96876	1 04328	
24	45150	97500	1 05300	
25	—	—	52975	
n	1176	1225	1275	

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