Let $L$ denote the positive octant of the regular $d$-dimensional cubic lattice. Each vertex $(j_1, j_2, \ldots, j_d)$ of $L$ is adjacent to all vertices of the form $(j_1, j_2, \ldots, j_k + 1, \ldots, j_d)$, $1 \leq k \leq d$. A $d$-partition of a positive integer $n$ is an assignment of nonnegative integers $n_{j_1, j_2, \ldots, j_d}$ to the vertices of $L$, subject to both an ordering condition

$$n_{j_1, j_2, \ldots, j_d} \geq \max_{1 \leq k \leq d} n_{j_1, j_2, \ldots, j_k+1, \ldots, j_d}$$

and a summation condition $\sum n_{j_1, j_2, \ldots, j_d} = n$. The summands in the $d$-partition are thus nonincreasing in each of the $d$ lattice directions. We agree to suppress all zero labels. A 1-partition is the same as an ordinary partition; a 2-partition is often called a plane partition and a 3-partition is often called a solid partition. Three sample plane partitions of $n = 26$ are

$$\begin{pmatrix} 8 \\ 9 \\ 9 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 2 & 2 & 1 \\ 4 & 2 & 1 & 1 \\ 5 & 3 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 7 & 6 & 4 & 4 & 3 & 1 & 1 \end{pmatrix}.$$

Let $p_d(n)$ denote the number of $d$-partitions of $n$. The generating functions [1]

$$1 + \sum_{n=1}^{\infty} p_1(n)x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + \cdots$$

$$= \prod_{m=1}^{\infty} (1 - x^m)^{-1},$$

$$1 + \sum_{n=1}^{\infty} p_2(n)x^n = 1 + x + 3x^2 + 6x^3 + 13x^4 + 24x^5 + 48x^6 + 86x^7 + 160x^8 + \cdots$$

$$= \prod_{m=1}^{\infty} (1 - x^m)^{-m}.$$
give rise to well-known asymptotics \([2, 3, 4, 5]\):

\[
p_1(n) \sim \frac{1}{4\sqrt{3n}} \exp \left( \pi \sqrt{\frac{2n}{3}} \right) \\
\sim (0.1443375672\ldots)n^{-1} \exp \left( (2.5650996603\ldots)n^{1/2} \right),
\]

\[
p_2(n) \sim \frac{\zeta(3)^{7/36}e^{\zeta'(-1)}}{2^{11/36}3\pi n^{25/36}} \exp \left( 3\zeta(3)^{1/3} \left( \frac{n}{2} \right)^{2/3} \right) \\
\sim (0.2315168134\ldots)n^{-25/36} \exp \left( (2.0094456608\ldots)n^{2/3} \right)
\]
as \(n \to \infty\), where \(\zeta(3) = 1.2020569031\ldots\) is Apéry’s constant \([6]\) and \(\zeta'(-1) = -0.1654211437\ldots = 2(-0.0827105718\ldots) = \ln(0.8475366941\ldots)\) is closely related to the Glaisher-Kinkelin constant \([7]\). Although an infinite product expression for the generating function \([1]\)

\[
1 + \sum_{n=1}^{\infty} p_3(n)x^n = 1 + x + 4x^2 + 10x^3 + 26x^4 + 59x^5 + 140x^6 + 307x^7 + 684x^8 + \cdots
\]

remains unknown, it is conjectured that \([8, 9]\)

\[
p_3(n) \sim \frac{C}{n^{61/96}} \exp \left( \frac{2^{7/4}\pi}{3^{3/4}45^{1/4}}n^{3/4} + \frac{\sqrt{15}\zeta(3)}{\sqrt{2}\pi^2}n^{1/2} - \frac{15^{5/4}\zeta(3)^2}{2^{7/4}\pi^5}n^{1/4} \right) \\
\sim C n^{-61/96} \exp \left( (1.7898156270\ldots)n^{3/4} + (0.335461354\ldots)n^{1/2} - (0.0414392867\ldots)n^{1/4} \right)
\]

for some constant \(C > 0\). The evidence for this asymptotic formula includes exact enumerations (for \(n \leq 68\)) and Monte Carlo simulation. See \([10, 11, 12, 13]\) for more about planar partitions and \([14, 15, 16, 17]\) for more about solid partitions.

### 0.1. Hardy-Ramanujan-Rademacher.

The Hardy-Ramanujan-Rademacher formula for \(p_1(n)\) is a spectacular exact result \([18, 19, 20, 21, 22, 23, 24, 25, 26]\):

\[
p_1(n) = \frac{\pi}{2^{5/4}3^{3/4}} \left( n - \frac{1}{24} \right)^{-3/4} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_{3/2} \left( \sqrt{\frac{2\pi}{3k}} \sqrt{n - \frac{1}{24}} \right)
\]

where

\[
I_{3/2}(x) = \sqrt{\frac{2x}{\pi}} \left( \frac{\cosh(x)}{x} - \frac{\sinh(x)}{x^2} \right)
\]
is the modified Bessel function of order $3/2$,

$$A_k(n) = \sum_{\substack{\gcd(h,k)=1, \\
1 \leq h < k}} \omega_{h,k} \exp\left(-\frac{2\pi i n h}{k}\right),$$

and $\omega_{h,k} = \exp(\pi i s(h, k))$ is the unique $24k^{th}$ root of unity with Dedekind sum

$$s(h, k) = \sum_{m=1}^{k-1} \left(\frac{m}{k} - \left\lfloor \frac{m}{k} \right\rfloor - \frac{1}{2}\right) \left(\frac{hm}{k} - \left\lfloor \frac{hm}{k} \right\rfloor - \frac{1}{2}\right).$$

For example,

$$A_1(n) = 1, \quad A_2(n) = (-1)^n, \quad A_3(n) = 2 \cos\left(\frac{\pi(12n - 1)}{18}\right),$$

$$A_4(n) = 2 \cos\left(\frac{\pi(4n - 1)}{8}\right), \quad A_5(n) = 2 \cos\left(\frac{\pi(2n - 1)}{5}\right) + 2 \cos\left(\frac{4\pi n}{5}\right).$$

Defining

$$c = \sqrt{\frac{2}{3}} \pi, \quad \lambda(n) = \sqrt{n - \frac{1}{24}},$$

$$\mu(n) = c\lambda(n), \quad A_k^*(n) = A_k(n)/\sqrt{k},$$

we have the following variations:

$$p_1(n) = \frac{1}{2^{1/2}\pi} \sum_{k=1}^{\infty} A_k(n) k^{1/2} \frac{d}{dn} \left[ \frac{\sinh\left(c\lambda(n)/k\right)}{\lambda(n)} \right]$$

$$= 2 \frac{3^{1/2}}{24n - 1} \sum_{k=1}^{\infty} A_k^*(n) \left[ \left(1 - \frac{k}{\mu(n)}\right) \exp\left(\frac{\mu(n)}{k}\right) + \left(1 + \frac{k}{\mu(n)}\right) \exp\left(-\frac{\mu(n)}{k}\right) \right].$$

In contrast, the original Hardy-Ramanujan formula is only an asymptotic expansion:

$$p_1(n) \sim \frac{1}{2^{3/2}\pi} \sum_{k=1}^{\infty} A_k(n) k^{1/2} \frac{d}{dn} \left[ \frac{\exp\left(c\lambda(n)/k\right)}{\lambda(n)} \right]$$

$$\sim 2 \frac{3^{1/2}}{24n - 1} \sum_{k=1}^{\infty} A_k^*(n) \left(1 - \frac{k}{\mu(n)}\right) \exp\left(\frac{\mu(n)}{k}\right),$$

which was later proved to be divergent by Lehmer [27, 28, 29]. Therefore Rademacher’s contribution was the identification of a small additional term that forces the original Hardy-Ramanujan series to converge.
A third formula for \( p_1(n) \):
\[
p_1(n) \sim \frac{\pi}{2^{5/4}3^{3/4}} \lambda(n)^{-3/2} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_{-3/2} \left( \frac{c\lambda(n)}{k} \right)
\]

appears in Almkvist [30, 31] and is a consequence of a more general theory (to be discussed shortly). The only difference between this formula and the Hardy-Ramanujan-Rademacher formula is that \( I_{-3/2} \) appears rather than \( I_{3/2} \). It is believed to be divergent, but this has not yet been proved. For practical purposes, using the modified Bessel function of order \(-3/2\):
\[
I_{-3/2}(x) = \sqrt{\frac{2x}{\pi}} \left( \frac{\sinh(x)}{x} - \frac{\cosh(x)}{x^2} \right)
\]
gives only slightly different numerical results (for large \( \sqrt{n}/k \)).

Analogous series exist for plane partitions. The terms involve neither exponentials nor Bessel functions, but rather a new function
\[
g(x, \gamma) = \sum_{\nu=0}^{\infty} \frac{x^{2\nu+\gamma-1}}{\nu!\Gamma(2\nu+\gamma)}
\]
that satisfies the third-order differential equation
\[
x g'''(x, \gamma) - (\gamma - 3)g''(x, \gamma) - 2g(x, \gamma) = 0
\]
(the derivatives are taken with respect to \( x \)) as well as
\[
g'(x, \gamma) = g(x, \gamma - 1), \quad 2g(x, \gamma + 2) + (\gamma - 1)g(x, \gamma) = xg(x, \gamma - 1).
\]
A heuristic argument in [30, 31] gives that
\[
p_2(n) \sim \varphi_1(n) + \varphi_2(n) + \varphi_3(n) + \cdots
\]
as \( n \to \infty \), where
\[
\varphi_1(n) = \zeta(3)^{13/24} e^{\zeta'(-1)} \sum_{k=0}^{\infty} a_{2k} \zeta(3)^k \Gamma \left( n\sqrt{\zeta(3)}, -\frac{1}{12} - 2k \right)
\]
and \( a_{2k} \) is the coefficient of \( x^{2k} \) in the Maclaurin series of
\[
h(x) = \exp \left( -\sum_{j=1}^{\infty} \frac{2(2j+1)!\zeta(2j)\zeta(2j + 2)}{j(2\pi)^{4j+2}} x^{2j} \right),
\]
\[ \varphi_2(n) = (-1)^n 2^{-5/3} \zeta(3)^{7/12} e^{2\phi(-1)} \sum_{k=0}^{\infty} b_{2k} \left( \frac{\zeta(3)}{8} \right)^k g \left( n \sqrt{\frac{\zeta(3)}{8}} \cdot \frac{1}{6} - 2k \right) \]

and \( b_{2k} \) is the coefficient of \( y^{2k} \) in the Maclaurin series of

\[ \frac{h(2y)^5}{h(y)h(4y)^2} \]

and so forth. The additional terms \( \varphi_3(n), \varphi_4(n) \) appear in [30] and \( \varphi_5(n), \varphi_6(n) \) appear in [31]. Taken together, these terms provide remarkably accurate estimates of \( p_2(n) \). Govindarajan & Prabhakar [32] revisited Almkvist’s results, using a modified function

\[ \hat{g}(x, \gamma) = \frac{1}{2} \sum_{\nu=0}^{\infty} \frac{x^{\nu}}{\nu! \Gamma((3 - \gamma + \nu)/2)} \]

that seems better behaved than \( g(x, \gamma) \) and evidently does for \( p_2(n) \) akin to what Rademacher’s modification of Hardy-Ramanujan did for \( p_1(n) \).

0.2. Addendum. Recent Monte Carlo work indicates that [33]

\[ \lim_{n \to \infty} n^{-3/4} \ln (p_3(n)) \approx 1.822 > 1.789... = \frac{2^{7/4} \pi}{3^{9/4} 5^{1/4}}. \]

contradicting [8, 9]. The asymptotics of solid partitions appear to differ sharply from those of line and plane partitions; in addition to sub-leading terms of order \( n^{1/2}, n^{1/4} \) and \( \ln(n) \), there seems to be an oscillatory function at the \( n^{-1/4} \) level. Theory lags far behind numerical experimentation here. Let

\[ 1 + \sum_{n=1}^{\infty} q(n)x^n = 1 + x + 4x^2 + 10x^3 + 26x^4 + 59x^5 + 141x^6 + 310x^7 + 692x^8 + \cdots \]

\[ = \prod_{m=1}^{\infty} (1 - x^m)^{-m(m+1)/2} . \]

Although the MacMahon conjecture is incorrect \( (p_3(n) \neq q(n) \text{ for } n > 5) \), there is still a possibility that \( p_3(n) \sim q(n) \) as \( n \to \infty \). The conjectured asymptotics for \( p_3(n) \) given earlier are validated asymptotics for \( q(n) \). In a recent breakthrough, Kotesovec [34] deduced that the multiplicative constant \( C \) for \( q(n) \) is

\[ 2^{-157/96} 15^{-13/96} \exp \left( -\frac{\zeta(3)}{8\pi^2} + \frac{75\zeta(3)^3}{2\pi^8} + \frac{\zeta'(-1)}{2} \right) \pi^{1/24} = 0.2135951604... \]

and we look forward to seeing underlying details.
Let us consider one of many possible variations on 1-partitions. Define $\hat{p}_1(n)$ to be the number of partitions of $n$ into integers, each of which may occur only an odd number of times. It can be shown that

$$\hat{p}_1(n) \sim \frac{B}{2\pi n} \exp\left(2B\sqrt{n}\right)$$

where

$$B^2 = \frac{\pi^2}{12} + \int_0^1 \frac{\ln(1 + x - x^2)}{x} \, dx = \frac{\pi^2}{12} + 2\ln(\varphi)^2$$

$$= \frac{\pi^2}{12} + 0.4631296411... = (1.1338415562...)^2$$

and $\varphi = (1 + \sqrt{5})/2$ is the Golden mean.

References


