Bulletin Calcutta M S 39 1947 1568 → 186

BU2

THE ASYMPTOTIC NUMBER OF THREE-DEEP LATIN RECTANGLES

By

S. M. Kerawala

(Received June 5, 1947)

A recent paper by Erdös and Kaplansky (1946) establishes the following asymptotic expansion for the number $f(n, k)$ of n by k Latin rectangles:

$$\frac{f(n,k)\exp(kC_2)}{(n!)^k} = 1 - \frac{k(k-1)(k-2)}{6n} + \frac{k(k-1)(k-2)(k^3-3k^2+8k-30)}{72n^2} + \ldots,$$

the existence of which was previously conjectured by Jacob (1930). For $k=3$, the expansion reduces to

$$\frac{f(n,3)\,e^3}{(n!)^3} = U_n = 1 - \frac{1}{n} - \frac{1}{2n^2} + \ldots$$

Comparing the value of

$$1 - \frac{1}{n} - \frac{1}{2n^2}$$

with the exact value of U_n given by me (1941) for $n \leq 25$, Erdös and Kaplansky show that the first three terms of their expansion provide numerical results accurate to four figures for $n \geq 20$, whereas MacMahon's (1915) operator or my difference-equation (Kerawala, 1941) or Riordan's combinatory formula (1945) would require heavier computations.

However, the method by which Erdös and Kaplansky derive their general asymptotic formula lacks the power for pushing their series any further. The method, if pursued, may with luck give the coefficient of n^{-3}, but there appears to be no hope of obtaining the coefficients of higher terms by their method. The purpose of the present paper is to derive the higher terms of the asymptotic series when $k=3$.

In my earlier paper (1941), I have shown that U_n satisfies the following difference equation:

$$U_n = \frac{(n-1)(n^2-2n+2)}{n^2(n-2)}U_{(n-1)} + \frac{(n^2-2n+2)}{n^2(n-1)}U_{(n-2)} + \frac{(n^2-2n-2)}{n^2(n-1)(n-2)^2}U_{(n-3)}$$

$$+ \frac{2(n^2-5n+3)}{n^2(n-1)(n-2)^2(n-3)}U_{(n-4)} - \frac{4}{n^2(n-2)^2(n-3)(n-4)}U_{(n-5)}$$

I substitute

$$U_n = 1 + \sum_{r=1}^{\infty}(a_r\, n^{-r}),$$

ip given

assuming the expansion convergent for $n > n_0$. On equating the coefficients of $n^{-2}, n^{-3}, n^{-4}, \ldots$ on both sides and simplifying, I obtain:

$$
\begin{aligned}
&1 + a_1 &&= 0,\\
&4 + 3a_1 + 2a_2 &&= 0,\\
&10 + 8a_1 + 6a_2 + 3a_3 &&= 0,\\
&20 + 23a_1 + 17a_2 + 10a_3 + 4a_4 &&= 0,\\
&26 + 68a_1 + 51a_2 + 32a_3 + 15a_4 + 5a_5 &&= 0,\\
&-32 + 197a_1 + 168a_2 + 104a_3 + 54a_4 + 21a_5 + 6a_6 &&= 0,\\
&-462 + 520a_1 + 596a_2 + 365a_3 + 193a_4 + 84a_5 + 28a_6 + 7a_7 &&= 0,\\
&-2664 + 1017a_1 + 2165a_2 + 1414a_3 + 723a_4 + 330a_5 + 123a_6 + 36a_7 + 8a_8 &&= 0,\\
&-12662 - 484a_1 + 7551a_2 + 5886a_3 + 2962a_4 + 1326a_5 + 528a_6 + 172a_7 + 45a_8 + 9a_9 &&= 0,\\
&-55648 - 22507a_1 + 22818a_2 + 24968a_3 + 13295a_4 + 5707a_5 + 2279a_6 + 801a_7 + 232a_8 &&\\
&\qquad + 55a_9 + 10a_{10} &&= 0,\\
&\text{etc.} \qquad \text{etc.} \qquad \text{etc.}
\end{aligned}
$$

Related to 3 line LR's Sequence type 2 #1568

On solving these, I find

$$
\begin{aligned}
1!\,a_1 &= -1, & 6!\,a_6 &= 6961,\\
2!\,a_2 &= -1, & 7!\,a_7 &= 38366,\\
3!\,a_3 &= 2, & 8!\,a_8 &= -1899687,\\
4!\,a_4 &= 49, & 9!\,a_9 &= -133065253,\\
5!\,a_5 &= 629, & 10!\,a_{10} &= -6482111309,\\
&\text{etc.} & &\text{etc.}
\end{aligned}
$$

Hence,

$$
\frac{f(n,3)e^3}{(n!)^3} = 1 - \frac{1}{n} - \frac{1}{2n^2} + \frac{2}{3!\,n^3} + \frac{49}{4!\,n^4} + \frac{629}{5!\,n^5} + \frac{6961}{6!\,n^6} + \frac{38366}{7!\,n^7} - \frac{1899687}{8!\,n^8} - \frac{133065253}{9!\,n^9} - \frac{6482111309}{10!\,n^{10}} - \text{etc.}
$$

The first three terms are identical with the three terms of the Erdös-Kaplansky expansion. The series may be pushed further to any desired extent.

Department of Mathematics,
Presidency College,
Calcutta

References

Dulmage, L., (1944), *American Mathematical Monthly*, **51**, 586-587.

Erdös P. and Kaplansky, I., (1946), *American Journal of Mathematics*, **68**, 230-236.

Jacob, S. M., (1930), *Proc. London Math. Soc.*, **31**, 329-354.

Kerawala, S. M., (1941), *Bull. Cal. Math. Soc.*, **33**, 119-127.

MacMahon, P. A., (1915), *Combinatory Analysis*, *Vol. I*, Cambridge, 246-263.

Riordan, J., (1944), *American Mathematical Monthly*, **51**, 450-452.

——, (1945), *American Mathematical Monthly*, **53**, 18-20.