Mess Math 32 (1902)

A PROBLEM ON ARRANGEMENTS.

By *H. M. Taylor*.

THE following paper is the result of an attempt to solve the problem:—

"To find the number of different ways in which a party of n married couples can be seated round a table, the ladies and gentlemen being seated alternately, and no man being next his wife."

We shall consider here only arrangements in which the ladies and gentlemen are seated alternately, and we shall call those in which no man is next his wife *perfect* arrangements; those in which one man and one only is next his wife *singly-imperfect* arrangements; and those in which two men and two only are next their wives *doubly-imperfect* arrangements.

Let us denote the n couples by the letters Aa, Bb, Cc, Dd, ..., Zz, the ladies being distinguished by small letters.

Let $f(n)$ be the number of ways of seating the n ladies so as to form perfect arrangements, the n gentlemen being assumed already seated in some definite, say alphabetical, order. Then the total number of ways of seating the n couples, taking into consideration all possible arrangements of the n gentlemen among themselves, will be

$$(n-1)!\,f(n).$$

Let $\phi(n)$ be the number of ways of seating the ladies so as to form singly-imperfect arrangements, the n gentlemen being assumed as before seated in alphabetical order, one named lady only being next her husband, and she on a definite side of him, right or left; and let $\psi(n)$ be the number of ways of seating the ladies so as to form doubly-imperfect arrangements, one definite lady being next her husband on a definite side of him, and some one other lady and only one being next her husband, the side being indefinite.

If in any perfect arrangement of the n couples Aa, Bb, Cc, ... we interchange b with the lady between B and C, we shall get an arrangement of the n couples, which may be either singly-imperfect or doubly-imperfect, since b is necessarily next B, and the lady whom b has been changed with may be or may not be next her husband. If from such an arrangement we

withdraw the couple Bb we obtain an arrangement of the $n-1$ couples Aa, Cc, Dd, ... which may be either perfect, singly-imperfect, or doubly-imperfect. Conversely it follows that every perfect arrangement of the n couples can be obtained from some arrangement of the $n-1$ couples Aa, Cc, Dd, ... perfect, singly-imperfect, or doubly-imperfect by introducing the couple Bb between A and C and then interchanging b with some other lady.

From any perfect arrangement of the $n-1$ couples such as $AxC\ldots LcM\ldots$ we can obtain a perfect arrangement of the n couples by introducing the couple Bb between x and C and then interchanging b with any other lady except x or c, viz. in $n-3$ ways.

Hence from perfect arrangements of $n-1$ couples we obtain $(n-3)f(n-1)$ different perfect arrangements of the n couples.

From a singly-imperfect arrangement of the $n-1$ couples such as $AcC\ldots$ we can by introducing the couple Bb between c and C and then interchanging b with any other lady except c obtain $n-2$ different perfect arrangements of the n couples.

In this way we can obtain $(n-2)\,\phi(n-1)$ different perfect arrangements of the n couples.

From a singly-imperfect arrangement such as either $AaC\ldots$ or $AxCc\ldots$ we cannot by introducing the couple Bb and then interchanging b with a or c respectively obtain a perfect arrangement of the n couples. If, however, we take any one of the remaining $2n-5$ forms of singly-imperfect arrangements $AxCdD\ldots$; $AxCyDd\ldots$; ...; $AxC\ldots Zz$ or $AxC\ldots Za$ we can by introducing the couple Bb between A and C and then interchanging b with the lady who is the cause of the imperfection of the arrangement obtain one perfect arrangement of the n couples.

Hence we can obtain from singly-imperfect arrangements of the $n-1$ couples

$$(n-2)\,\phi(n-1)+(2n-5)\,\phi(n-1) \text{ or } (3n-7)\,\phi(n-1)$$

different perfect arrangements.

Again, from any doubly-imperfect arrangement of the $n-1$ couples of the form $AcC\ldots LlM\ldots$ we can by introducing the couple Bb between c and C and then interchanging b and l obtain a perfect arrangement of the n couples. We can therefore obtain a perfect arrangement of the n couples from a doubly-imperfect arrangement of the $n-1$ couples in $\psi(n-1)$ ways.

179
271
(26)

We have now obtained all the possible perfect arrangements, and we have not counted any one of them twice. This appears from the fact that every perfect arrangement of the n couples leads to some arrangement of the $n-1$ couples of one of the three kinds described, and that a unique form; and therefore in the converse process no two different arrangements of the $n-1$ couples of the three kinds described could lead to the same perfect arrangement of the n couples.

We may therefore now write down the equation

$$f(n) = (n-3) f(n-1) + (3n-7) \phi(n-1) + \psi(n-1) \ldots (1).$$

In a similar manner we see that singly-imperfect arrangements of the n couples of the form $AaBbC\ldots$ can be obtained from any perfect arrangement of the $n-1$ couples of the form $AaC\ldots$ or from any singly-imperfect arrangement of the form $AaC\ldots$ by the simple introduction of the couple Bb between a and C or between a and C respectively, and in no other way, and that all of them will be different.

Hence $\qquad \phi(n) = f(n-1) + \phi(n-1) \ldots\ldots\ldots\ldots\ldots (2).$

Similarly we see that doubly-imperfect arrangements of the n couples can be obtained by the simple introduction of the couple Bb in any singly-imperfect arrangement of the $2n-3$ forms $AaC\ldots$; $AaCc\ldots$; $AaCcD\ldots$; $\ldots$; $AaC\ldots Zz$; $AaC\ldots Z$; or in any doubly-imperfect arrangement of the form $AaC\ldots LlM\ldots$.

Therefore

$$\psi(n) = (2n-3) \phi(n-1) + \psi(n-1) \ldots\ldots\ldots (3).$$

By elimination of pairs of the functions f, ϕ, ψ from the equations of the form (1), (2), (3) we obtain the equations

$$nf(n+2) - (n^2+n+1) f(n+1)$$
$$- (n^2+n+1) f(n) - (n+1) f(n-1) = 0 \ldots (4),$$
$$\phi(n+2) - n\phi(n+1) - n\phi(n) - \phi(n-1) = 0 \ldots (5),$$
$$(2n-1)(2n-3)\psi(n+2) - (n+1)(2n-3)^2 \psi(n+1)$$
$$-(n-2)(2n+1)^2 \psi(n) - (2n-1)(2n+1)\psi(n-1) = 0 \ldots (6).$$

We might solve the equations (4), (5), (6) independently.

On account of the simpler form of equation (5) we choose that one for solution, although the method used is equally applicable to either of the others. By eliminating the functions $\phi(n+1)$, $\phi(n)\ldots$, $\phi(5)$, $\phi(4)$ from the equation (5) and the

next $n-2$ equations of the same form we obtain as the general solution of equation (5)

$$\phi(n+2)$$
$$= \left| \begin{array}{llllll} n, & n, & 1, & \ldots & & \\ -1, & n-1, & n-1, & 1, & \ldots & \\ & -1, & n-2, & n-2, & 1 \ldots & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ & & \ldots & -1, & 4, & 4, \ \phi(3), \\ & & \ldots & & -1, & 3, \ 3\phi(3) + \phi(2) \\ & & \ldots & & & -1, \ 2\phi(3) + 2\phi(2) + \phi(1) \end{array} \right|.$$

If we adopt as the values of $\phi(1)$, $\phi(2)$, $\phi(3)$ their actual values 1, 0, 0 respectively, we obtain the general solution for ϕ as applicable to the problem in hand.

When the function ϕ has been found the other functions will be given by the equations

$$f(n) = \phi(n+1) - \phi(n)$$

and $\qquad \psi(n) = \phi(n+1) - (n-2)\phi(n) + \phi(n-1).$

The values of the functions for the first ten values of n are given in the following table

n	$f(n)$	$\phi(n)$	$\psi(n)$	$(n-1)!\,f(n)$
1	-1	1	0	-1
2	0	0	1	0
3	1	0	1	2
4	2	1	1	12
5	13	3	8	312
6	80	16	35	9600
7	579	96	211	410800
8	4738	675	1459	23,879520
9	43387	5413	11584	1749,363840
10	439792	48800	103605	159591,720960.

(179) ~~(271) (426)~~

basic basic