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As part of the MAA's celebration of its 100th anniversary, we have assembled a volume of essays from esteemed mathematicians. Every MAA member has received an electronic copy of A Century of Advancing Mathematics *(see p. 2). Physical copies of the book can be purchased on* maa.org. *Here is an excerpt of one of the articles. Read the complete essay in* A Century of Advancing Mathematics, *an MAA Press publication.*

Defying God: The Stanley-Wilf Conjecture, Stanley-Wilf Limits, and a Two-Generation Explosion of Combinatorics

By Eric S. Egge

In March of 2005, at the Third International Conference on Permutation Patterns in Gainesville, Florida, Doron Zeilberger declared that "Not even God knows $a_{1000}(1324)$." Zeilberger's claim raises thorny theological questions, which I am happy to ignore in this article, but it also raises mathematical questions. The quantity $a_{1000}(1324)$ is the one-thousandth term in a certain sequence $a_n(1324)$. God may or may not be able to compute the thousandth term in this sequence, but how far can mortals get? If we can't get beyond the 40th or 50th term, can we at least approximate the one-thousandth term? How fast does $a_n(1324)$ grow, anyway? And what does $a_n(1324)$ even mean?

The answers to these questions involve fast computers, fascinating mathematics, and remarkable human ingenuity. But their stories, which are ongoing, also reflect important undercurrents and developments that have influenced all of mathematics, but especially combinatorics, over the past two generations and more.

Knuth's Railroad Problems

The story of $a_{1000}(1324)$ begins with a gap in the railroad literature, which Donald Knuth began to fill in 1968 in the first edition of the first volume of his masterpiece, *The Art of Computer Programming.* In the second section of chapter 2, Knuth included several exercises exploring a problem involving sequences of railcars one can obtain using a turnaround. One of Knuth's exercises is equivalent to the following problem.

At dawn we have n railroad cars positioned on the right side of the track in Figure 1, numbered 1 through n from right to left. During the day we gradually move the cars to the left side of the track, by moving each car into and back out of the turnaround area. There can be any number of cars in the turnaround area at a time, and at the end of the day the cars on the left side of the track can be in many different orders. Each possible order determines a permutation of the numbers 1, 2, …, n. Show that a permutation $\pi_1; \ldots; \pi_n$ (this time reading from left to right along the tracks) is attainable in this way if and only if there are no indices $i < j < k$ such that $\pi_i < \pi_k < \pi_j$.

The solution to this problem is a fun exercise in careful bookkeeping. If such a subsequence exists, then

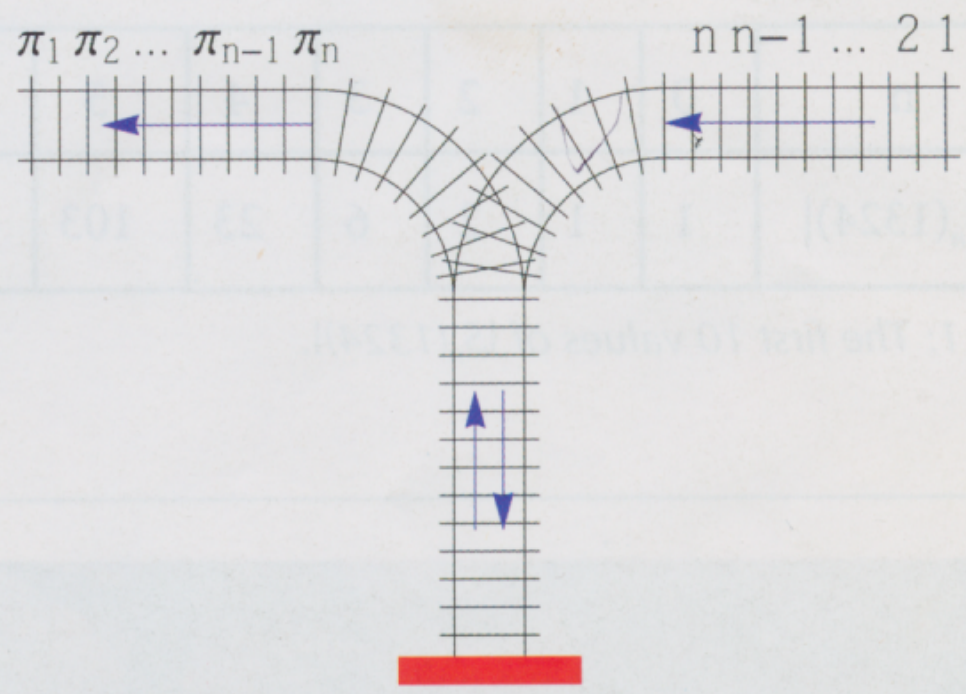

Figure 1. Knuth's railroad tracks.

consider the situation when car π enters the turnaround. Since π_i is the smallest of our three car numbers, cars π_j and π_k have already entered the turnaround, in that order. Furthermore, in order for car π_i to appear to the left of cars π_j and π_k, cars π_j and π_k must both still be in the turnaround when car π_i enters. But now cars π_j and π_k will leave the turnaround in the wrong order.

Conversely, suppose we have a target permutation $\pi_1, \ldots, \pi_n$ with no subsequence of the forbidden type. We can always move π_1 into position, and when car π_1 leaves the turnaround, the cars in the turnaround are, from bottom to top, n; $n-1, \ldots, \pi_1 + 1$. Now notice that π_2 cannot be larger than $\pi_1 + 1$, since this would mean π_1, π_2, and $\pi_1 + 1$ form a forbidden subsequence. So if π_2 is in the turnaround, then it is the top car there. Either way, we can move car π_2 into position. In general, if we have just moved car π_j into position, and b is the smallest entry greater than π_j which has not yet left the turnaround, then $\pi_{j+1} \le b$, since otherwise π would have a forbidden subsequence π_j, π_{j+1}, b. Therefore, if π_{j+1} has entered the turnaround then it is the top car there, and we can move it into place.

Knuth was interested in this railcars problem because it models the data structure commonly called a stack, which arises in numerous programming problems, so he introduced no particular notation for the permutations he obtained. Indeed, no general notation for these permutations appeared in print until 1985, when Simion and Schmidt published the first systematic study of permutations with forbidden subsequences of the type Knuth uses.

Today, if π and σ are permutations of lengths n and k respectively, then we say a subsequence of π of length k has *type* σ whenever its entries are in the same relative order as the entries of σ.

For example, the subsequence 829 of the permutation 718324695 has type 213, since its smallest entry is in the middle, its largest entry is last, and its middle entry is first. In this context, we say π avoids σ, or π is σ-avoiding, whenever π has no subsequence of type σ, and we write $S_n(\sigma)$ to denote the set of all permutations of length n which avoid σ. We might also say that σ is a *forbidden subsequence* or a *forbidden pattern.* With this terminology, the permutations Knuth obtains with his railcars are exactly the 132-avoiding permutations, and the term $a_{1000}(1324)$ that Zeilberger's God finds so perplexing is none other than the size of $S_{1000}(1324)$. In table 1 we have the first 10 values of $|S_n(1324)|$.

The rest of this essay is in A Century of Advancing Mathematics, published by MAA Press.

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n	0	1	2	3	4	5	6	7	8	9
$\lvert S_n(1324)\rvert$	1	1	2	6	23	103	513	2762	15793	94776

Table 1: The first 10 values of $|S_n(1324)|$.

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