An Explicit Formula for the Euler zigzag numbers (Up/down numbers) from power series

In this paper, I will derive an explicit formula for the Euler zigzag numbers (Up/down numbers). Euler zigzag number is the number of alternating permutation in a set. Therefore the explicit formula of Euler numbers (Secant numbers) and Bernoulli numbers are found as well. The formula involves two finite sum.

Introduction

Euler zigzag numbers, $A_n$, is the number of alternating permutation of the set \{1,2,...,n\}. And it is well known that:

$$\sec x + \tan x = \sum_{n=0}^{\infty} \frac{A_n}{n!} x^n$$

In the following section, I am going to derive an explicit formula of $A_n$ by using power series expansion.

Integrand of sec(x) + tan(x)

Let's consider the integrand of sec(x) + tan(x):

$$\int (\sec x + \tan x) dx$$

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Let:

\[ f(x) = -\ln(1 - \sin x) \]

I will do a power expansion of the function \( f(x) \), and a finite sum explicit formula for \( A_n \) can be found by some simplification.

**Power Series Expansion of \( f(x) \)**

\[
\begin{align*}
\sum_{n=1}^{\infty} \frac{A_{n-1}}{n!} x^n &= \int_0^x \cos y \, dy \\
\sum_{n=1}^{\infty} \frac{A_{n-1}}{n!} x^n &= \int_0^x \frac{1 + \sin y}{\cos y} \, dy \\
&= \int_0^x \frac{(1 + \sin y)(1 - \sin y)}{\cos y(1 - \sin y)} \, dy \\
&= \int_0^x \frac{\cos y \, dy}{1 - \sin y} \\
&= -\ln(1 - \sin x)
\end{align*}
\]

Now, I am going to simplify it, and show that it is actually \( e^{-\frac{\lambda}{2}} e^A e^B \) if \([A,B]=\lambda\).

Solving linear non-homogeneous ordinary differential equation with variable coefficients with operator method

Solving a partial difference equation in 2 variables with operator method

Solving recurrence equation with indexes from negative infinity to positive infinity

Reducing a partial difference equation into a partial differential equation and solving for the generating function using method of characteristics

Binomial Expansion for non-commutative elements \((A+B)^n\) where \([A, B] = \lambda\)

Finding nth derivative of the function \( \sec x + \tan x \) and partial difference equation

A procedure to list all derangements of a multiset

A free ebook about generating function, "generatingfunctionology"
equals to:

\[ A_j = \sum_{n=1}^{j+1} \sum_{k=0}^{n} \frac{C_n^j (n - 2k)^{j+1} (-1)^k}{2^n j^n n} \]

**Simplification**

In order to reduce the infinite sum to a finite sum, I first let:

\[ B_n^j = \sum_{k=0}^{n} C_n^j (n - 2k)^{j} (-1)^k \]

So that:

\[ A_j = \sum_{n=1}^{\infty} \frac{B_n^j}{2^n j^n n} \]

I observed that:

\[ B_n^j = 0 \text{ if } n > j \]

To show that, let's define the translational operator \( D \) such that:

\[
\begin{align*}
D a_n &= a_{n+1} \\
D^{-1} a_n &= a_{n-1}
\end{align*}
\]

Then:

\[ B_n^j = \left( \sum_{k=0}^{n} C_n^j (-1)^k D^{-2k} \right) n^j \]

\[ = (1 - D^{-2})^n n^j \]

\[ = (1 + D^{-1})^n (1 - D^{-1})^n n^j \]

Here, \( \nabla = 1 - D^{-1} \) is the backward difference operator.

Now, consider:

\[ \nabla n^j = n^j - (n - 1)^j \]

\[ = n^j - \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} n^k \]

\[ = - \sum_{k=0}^{j-1} \binom{j}{k} (-1)^{j-k} n^k \]

The result is a polynomial of degree \( j - 1 \). We can see that, if we apply backward difference operator to a polynomial, its degree decreases by 1. Therefore, if we apply the backward difference operator for a number of time which is larger than the degree of a polynomial, the result is zero.

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As a result, we have \( B_n^j = 0 \) if \( n > j \). Therefore:

\[
A_j = i^{j+1} \sum_{n=1}^{j+1} \sum_{k=0}^{n} \binom{n}{k} (n - 2k)^{j+1} (-1)^k \frac{1}{2n^k k}
\]

**Explicit Formula for Euler number**

Euler number \( E_n \) is given by the generating function:

\[
\frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n
\]

And it is given by:

\[
\begin{align*}
E_{2n} &= i \sum_{k=1}^{n+1} \sum_{j=0}^{k} \binom{k}{j} (-1)^j (k-2j)^{2n+1} 2^{j+k} k \\
E_{2n+1} &= 0
\end{align*}
\]

**Explicit Formula for Bernoulli Numbers**

From Wikipedia, we know:

\[
B_{2n} = \frac{(-1)^n - 1}{4^{2n} - 2} A_{2n-1}
\]

Therefore:

\[
\begin{align*}
B_0 &= 1 \\
B_1 &= -\frac{1}{2} \\
B_{2n} &= \frac{2}{2^{2n}-2^n} \sum_{k=1}^{2n} \sum_{j=0}^{k} \binom{k}{j} (-1)^j (k-2j)^{2n} 2^{j+k} k & \text{for } n > 0 \\
B_{2n+1} &= 0 & \text{for } n > 0
\end{align*}
\]

**Conclusion**

I have found out an simple formula for the Euler zigzag number:

\[
A_n = i^{n+1} \sum_{k=1}^{n+1} \sum_{j=0}^{k} \binom{k}{j} (-1)^j (k-2j)^{n+1} \frac{1}{2^{k} j k}
\]
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