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A NEW METHOD OF INVERSION OF THE LAPLACE TRANSFORM*

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Introduction. In determining a function $r(t)$ from its Laplace transform $R(p)$

$$R(p) = \int_0^\infty e^{-pt} r(t)\, dt \tag{1}$$

one applies either a partial fraction expansion or an integration along some contour in the complex p-plane; one thus obtains $r(t)$ in terms of the poles and residues of $R(p)$, or from the values of $R(p)$ on a contour of the p-plane. Both methods have obvious disadvantages for a numerical analysis.

In the following we propose to develop a method for determining $r(t)$ in terms of the values of $R(p)$ on an infinite sequence of equidistant points

$$p_k = a + k\sigma \qquad k = 0, 1, \cdots, n, \cdots \tag{2}$$

on the real p-axis, where a is a real number in the region of existence of $R(p)$, and an arbitrary positive integer. That $R(p)$ is uniquely determined from its values at the above points, is known [1]. It should therefore be possible to express $r(t)$ directly in terms of $R(a + k\sigma)$. In this paper it will be shown that $r(t)$ can be written in the form

$$r(t) = \sum_{k=0}^{\infty} C_k \varphi_k(t), \tag{3}$$

where the φ_k's are known functions, and the constants C_k can readily be determined from the values of $R(p)$ at the points $a + k\sigma$.

The φ_k's can be chosen from several sets of complete orthogonal functions; in our discussion we shall use the familiar trigonometric set, the Legendre set and the Laguerre polynomials.

The trigonometric set. We introduce the variable θ defined by

$$e^{-\sigma t} = \cos\theta \qquad \sigma > 0. \tag{4}$$

The $(0, \infty)$ interval transforms into the interval $(0, \pi/2)$, and $r(t)$ becomes

$$r\left(-\frac{1}{\sigma} \ln \cos\theta\right).$$

For simplicity of notation we shall denote the above function by $r(\theta)$ using the same letter r.

The defining equation (1) takes the form

$$\sigma R(p) = \int_0^{\pi/2} (\cos\theta)^{(p/\sigma)-1} \sin\theta\, r(\theta)\, d\theta \tag{5}$$

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hence with

$$p = (2k + 1)\sigma \qquad k = 0, 1, 2, \cdots$$

we have

$$\sigma R[(2k + 1)\sigma] = \int_0^{\pi/2} (\cos\theta)^{2k} \sin\theta r(\theta)\, d\theta. \tag{6}$$

In the following we shall assume, without loss of generality, that $r(0) = 0$ subtracting, if necessary, a constant from $r(\theta)$. The function $r(\theta)$ can be expanded in the $(0, \pi/2)$ interval into an odd-sine series

$$r(\theta) = \sum_{k=0}^{\infty} C_k \sin(2k + 1)\theta. \tag{7}$$

This can of course be done by properly extending the definition of $r(\theta)$ in the $(-\pi, +\pi)$ interval.

We shall next determine the coefficients C_k . We have

$$(\cos\theta)^{2n} \sin\theta = \left(\frac{e^{j\theta} + e^{-j\theta}}{2}\right)^{2n} \frac{e^{j\theta} - e^{-j\theta}}{2j},$$

expanding in the right hand side and properly collecting terms we obtain

$$2^{2n}(\cos\theta)^{2n} \sin\theta = \sin(2n + 1)\theta + \cdots$$
$$+ \left[\binom{2n}{k} - \binom{2n}{k-1}\right] \sin[2(n-k) + 1]\theta + \cdots + \left[\binom{2n}{n} - \binom{2n}{n-1}\right] \sin\theta. \tag{8}$$

We next insert (7) and (8) into (6); because of the orthogonality of the odd sines in the $(0, \pi/2)$ interval and since

$$\int_0^{\pi/2} [\sin(2n + 1)\theta]^2\, d\theta = \frac{\pi}{4},$$

we have

$$\sigma R[(2n + 1)\sigma] = 2^{-2n} \frac{\pi}{4} \left\{\left[\binom{2n}{n} - \binom{2n}{n-1}\right] C_0 + \cdots \right.$$
$$\left. + \left[\binom{2n}{k} - \binom{2n}{k-1}\right] C_{n-k} + \cdots + C_n \right\}$$

hence with $n = 0, 1, 2, \cdots$ we obtain the system

$$\frac{4}{\pi} \sigma R(\sigma) = C_0,$$

$$2^2 \frac{4}{\pi} \sigma R(3\sigma) = C_0 + C_1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \tag{9}$$

$$2^{2n} \frac{4}{\pi} \sigma R[(2n + 1)\sigma] = \left[\binom{2n}{n} - \binom{2n}{n-1}\right] C_0 + \cdots$$
$$+ \left[\binom{2n}{k} - \binom{2n}{k-1}\right] C_{n-k} + \cdots + C_n.$$

Thus $R(\sigma)$ gives C_0, $R(3\sigma)$ give C_1 and each value of $R(p)$ at the points $(2k + 1)\sigma$ together with the coefficients $C_0, C_1 \cdots, C_{k-1}$, determines C_k. The system (9) can obviously be written in such a way as to give directly C_k in terms of $R(\sigma)$, $R(3\sigma)$, $\cdots$ alone, but not much is gained, since in a numerical evaluation of the C_k's equation (9) can be used as easily. Table 1 gives the numerical values of the coefficients of the C_k's in the right hand side of (9), for $k = 0, 1, \cdots, 10$.

TABLE 1

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differences of bin coeffs

n	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
0	1										
1	1	1									
2	2	3	1								
3	5	9	5	1							
4	~~10~~ 14	28	20	7	1						
5	42	90	75	35	9	1					
6	132	297	275	154	54	11	1				
7	429	1001	1001	637	273	77	13	1			
8	1430	3432	3640	2548	1260	440	104	15	1		
9	4862	11934	13260	9996	5508	2244	663	135	17	1	
10	16796	41990	48450	38760	23256	10659	3705	950	170	19	1

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Thus a method of analysis has resulted which compares sometimes favorably with the known methods of numerical evaluation of $r(t)$. Indeed the computation of $R((2k + 1)\sigma)$ presents no difficulty, and the C_k's can be readily determined from (9); the trigonometric functions are available, hence $r(\theta)$ can be computed with any desired accuracy from the series (7). In a numerical evaluation of $r(\theta)$ one computes the finite sum

$$r_N(\theta) = \sum_{k=0}^{N} C_k \sin (2k + 1)\theta \tag{10}$$

of the first $N + 1$ terms of (7); as N tends to infinity $r_N(\theta)$ tends to $r(\theta)$. The nature of the approximation is well known from the theory of Fourier series [2]; $r_N(\theta)$ and $r(\theta)$ are related by the equation

$$r_N(\theta) = \frac{4}{\pi} \int_0^{\pi/2} r(y) \frac{\sin [\frac{1}{2}(4N + 3)(\theta - y)]}{\sin \frac{1}{2}(\theta - y)} dy, \tag{11}$$

thus the approximating function $r_N(\theta)$ is the average of $r(\theta)$ with the Fourier kernel

$$\frac{\sin [\frac{1}{2}(4N + 3)(\theta - y)]}{\sin \frac{1}{2}(\theta - y)}$$

as the weighting factor. From $r(\theta)$ one can readily obtain $r(t)$ with the change of variable established by (4); however, Eq. (7) can be written directly in the time domain. Indeed since

$$\frac{\sin n\theta}{\sin \theta} = U_n(x) \qquad \cos \theta = x,$$

where $Un(x)$ are the Tchebycheff sine-polynomials of order n and

$$\sin \theta = (1 - e^{-2\sigma t})^{1/2}$$

we have from (7)

$$r(t) = (1 - e^{-2\sigma t})^{1/2} \sum_{k=0}^{\infty} C_k U_{2k}(e^{-\sigma t}). \tag{12}$$

The choice of σ depends on the interval $(0, T)$ in which $r(t)$ is best to be described; if it is chosen so that

$$e^{-\sigma T} = \tfrac{1}{2}$$

then the $(0, T)$ interval transforms into the $(0, \pi/3)$ interval. If a detailed description of $r(t)$ is desired both near the origin and for large values of t, then the function can be evaluated twice with two different values of σ.

The above provides a simple proof of the announced theorem that the Laplace transform $R(p)$ is uniquely determined from its values at the sequence

$$p_k = a + k\sigma \qquad k = 0, 1, \cdots, n, \cdots \tag{2}$$

of equidistant points on the real p-axis. This proof uses the well-known orthogonality and completeness of the trigonometric set. Indeed $r(\theta)$, and hence $r(t)$, is completely determined from the coefficients C_k of (7); these coefficients can be determined from $R(a + k\sigma)$; knowing $r(t)$ one clearly has $R(p)$ therefore $R(p)$ is uniquely determined from its values at the points (2).

The Legendre set. We shall next expand $r(t)$ into a series of Legendre polynomials. We introduce the logarithmic time-scale x defined by

$$e^{-\sigma t} = x \qquad \sigma > 0. \tag{13}$$

The $(0, \infty)$ interval transforms into the interval $(1, 0)$: again we shall denote the function

$$r\left(-\frac{1}{\sigma} \ln x\right)$$

by $r(x)$. Equation (1) takes the form

$$\sigma R(p) = \int_0^1 x^{(p/\sigma)-1} r(x)\, dx \tag{14}$$

from which we obtain with $p = (2k + 1)\sigma$,

$$\sigma R[(2k + 1)\sigma] = \int_0^1 x^{2k} r(x)\, dx. \tag{15}$$

Thus the value of the function $R(p)$ at the point $[(2k + 1)\sigma]$ gives the $2k$th moment of the function $r(x)$ in the $(0, 1)$ interval

It is known that the Legendre polynomials $P_k(x)$ form a complete orthogonal set in the $(-1, 1)$ interval; We extend the definition of $r(x)$ in the $(-1, 1)$ interval by making

$$r(-x) = r(x).$$