it follows that

$$V' = -\frac{V^2}{V_1 V_2 V_3 V_4} \left( \frac{abc}{6} \right)^2$$

since $V_i = -|x_i y_i z_i|$, etc.

If we designate by $A$, $B$, $C$, $D$ the areas of the faces $BCD$, $CDA$, $DAB$, $ABC$ and by $x$, $y$, $z$, $t$ the absolute normal tetrahedral coordinates of $O$ with respect to $ABCD$, we have

$$V' = -\frac{8V^3}{AxByczDlt} \left( \frac{abc}{6} \right)^2.$$

But we have $AxBy+Cz+Dt = 3V$. The maximum of $AxByCzDlt$ occurs for $Ax=By=Cz=Dt$, so that the point $O$ should coincide with the centroid of $ABCD$.

**Genetic Algebra**

4277 [1948, 34]. Proposed by C. D. Olds, San Jose State College, California

In a non-associative algebra, it is necessary to distinguish the possible interpretations of $x^n$. Thus, for example, in a non-commutative non-associative algebra $x^4$ can mean $x \cdot x^2$ or $x^2 \cdot x$. In a general non-commutative non-associative algebra the number of interpretations of $x^n$ is $2(2n-3)!/n!(n-2)!$. Is there a formula for the number of interpretations of $x^n$ in a general commutative non-associative algebra?

**Discussion by H. W. Becker, Santa Monica, California.** The expression

$$N_{n+1} = \frac{(2n)!}{n!(n+1)!} = \left( \begin{array}{c} 2n \\ n \end{array} \right) / (n+1)$$

has a wide variety of distinct combinatorial and graphical interpretations, of at least five different types. That the types are distinct is made evident upon attempting to enumerate the symmetrically different sequences or graphs of each type, for these in general yield unequal numbers.

Wedderburn [1] demonstrated the well known

$$N_1 = N_2 = 1, \quad N_n = \sum_{m=1}^{n} N_{n-m} \cdot N_m,$$

and showed that, upon omitting all terms of (2) having like form, we have

$$N_1' = N_2' = 1, \quad N_{2n+1}' = \sum_{m=1}^{n} N_{2n+1-m} \cdot N_m'$$

$$N_2' = 1, \quad N_{2n}' = N_n'(N_n' + 1)/2 + \sum_{m=1}^{n-1} N_{2n-m} \cdot N_m'.$$
Here $N'_n$ and $N_n$ are the numbers of commutative and non-commutative, non-associative products of $n$ factors.


In lieu of an exact formula for $N_n$, we seek an approximation. By Stirling's approximation to $n!$, (1) becomes

$$N_{n+1} = 4^n/(n + 1) \sqrt{n},$$

whence $N_{n+1}/N_n$ approaches 4 with increasing $n$. If we calculate $N_n$ and $N'_n$ for values of $n$ through $n=25$, and utilize an idea due to Motzkin [8], the following results are suggested empirically:

$$N_{n+1}/N'_n = R_n, \quad R_n/R_{n-1} = r, \quad r \rightarrow \infty = 1.612.$$

Thus

$$N'_{n+1}/N'_n = k_n, \quad k_n \rightarrow k = 4/\pi = 2.48.$$

If the number $k_n$ is defined so that the following equation is true, it appears as though there exists a number $k$, the limit of $k_n$, in terms of which the desired approximation of $N'_{n+1}$ may be expressed. We have then $k_n \approx 0.812$ and approximately

$$N'_{n+1} = k_n k^n/(n + 1) \sqrt{n} = N_{n+1}/(0.7)n^n.$$

This formula is suggested for what it may be worth, based as it is on inspection of a few values of $n$, and lacking any proof of the existence of the limits $t, k,$ or $k$. The detailed figures are shown in the table below. Note, however, that (5) and (8) have the same form, $U_n = ab^n/n^n$, as the approximations for the number of $n$-branch series-parallel passive circuits [9], and of $n$-branch trees [10].

Other interpretations of the function (1) should be of interest. $N_n$ is the number of planar rhyme schemes [11], such that there are no crossovers in the Puttenham diagram [12]. $N_{n+1}$ is the number of ballot sequences in two party election, such that the non-loser gets $n-1$ votes and is never behind his opponent, who may get anywhere from 0 to $n-1$ votes, Lucas [13], p. 164, le scrutin du ballottage. (See also p. 14, marches du pion du jeu de dames; p. 86, les deux fîles de soldats; and p. 87, déplacements de la tour sur l'échiquier triangulaire.)

$N_{n+1}$ is the number of ways of decomposing an $(n+2)$-gon into triangles by $n-1$ non-intersecting diagonals, Lucas [13], pp. 90–96, 489. $N_{n+1}$ is also the number of ways of joining $2n$ points around a circle by $n$ non-intersecting chords, the $c_n$ of Motzkin [8], which linearize to the configurations superieurs of Touchard [14].

---

1. J. H. M. Wedderburn
2. I. H. M. Etherington
5. R. D. Schafer, Amer. Math. Monthly
6. ———, Amer. Math. Monthly
7. ———, Amer. Math. Monthly
12. G. Puttenham, The Art of Pariology
Solutions

[December,
inative and non-commutative,
ing functions for \( N_n \) and \( N'_n \),
simple formula for the first
tradition in genetic algebra
[5, 6, 7] reached a wide audi-
approximation. By Stirling's

If we calculate \( N_n \) and \( N'_n \),
to Motzkin [8], the follow-

\[
r_n \rightarrow r = 1.612.
\]

\[
\lim_{n \to \infty} \frac{N_n}{N'_n} = 2.48.
\]

This problem 4277 may also be regarded as a sequel to problem 3954 of O. Ore [1941, 564], whose solutions provide a helpful background.

\[
\begin{array}{cccccc}
\hline
n & N_n & N'_n & r_n & k_n & h_n \\
\hline
1 & 1 & 1 & 1 & 0.806 \\
2 & 1 & 2 & 1 & 0.69 \\
3 & 2 & 1.25 & 2 & 0.909 \\
4 & 5 & 1.867 & 1.5 & 0.791 \\
5 & 14 & 3 & 1.5 & 2 & 0.856 \\
6 & 42 & 1.714 & 1.8333 & 0.809 \\
7 & 132 & 1.558 & 2.0909 & 0.842 \\
8 & 429 & 2.003 & 0.83 \\
9 & 1430 & 4.595 & 2.1304 & 0.824 \\
10 & 4862 & 1.635 & 2.1122 & 0.814 \\
11 & 16796 & 1.607 & 2.1787 & 0.8175 \\
12 & 58786 & 1.624 & 2.1796 & 0.8119 \\
13 & 208012 & 1.612 & 2.2167 & 0.8139 \\
14 & 742900 & 1.617 & 2.2258 & 0.811 \\
15 & 2674440 & 1.612 & 2.2485 & 0.8118 \\
16 & 964845 & 1.615 & 2.2587 & 0.8108 \\
17 & 35357670 & 1.613 & 2.274 & 0.8109 \\
18 & 129644790 & 1.613 & 2.2837 & 0.8105 \\
19 & 477638700 & 1.612 & 2.2949 & 0.8108 \\
20 & 1767263190 & 1.612 & 2.3034 & 0.8108 \\
21 & 6564120420 & 1.612 & 2.3121 & 0.811 \\
22 & 24466267020 & 1.612 & 2.3194 & 0.8112 \\
23 & 91482563640 & 1.612 & 2.3265 & 0.8115 \\
24 & 343059613650 & 1.612 & 2.3328 & 0.8119 \\
25 & 1289904147324 & 1.617 & 2.3387 & 0.8121 \\
\hline
\end{array}
\]

4. ————, ibid., 61 (1941) 24-42.