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it follows that

$$V' = \frac{-V^3}{V_1V_2V_3V_4}\left(\frac{abc}{6}\right)^2$$

since $V_1 = -|x_2y_3z_4|$, etc.

If we designate by A, B, C, D the areas of the faces BCD, CDA, DAB, ABC and by x, y, z, t the absolute normal tetrahedral coordinates of O with respect to $ABCD$, we have

$$V' = -\frac{81V^3}{AxByCzDt}\left(\frac{abc}{6}\right)^2.$$

But we have $Ax+By+Cz+Dt=3V$. The maximum of $AxByCzDt$ occurs for $Ax=By=Cz=Dt$, so that the point O should coincide with the centroid of $ABCD$.

Genetic Algebra

4277 [1948, 34]. *Proposed by C. D. Olds, San Jose State College, California*

In a non-associative algebra, it is necessary to distinguish the possible interpretations of x^n. Thus, for example, in a non-commutative non-associative algebra x^3 can mean $x\cdot x^2$ or $x^2\cdot x$. In a general non-commutative non-associative algebra the number of interpretations of x^n is $2(2n-3)!/n!(n-2)!$ Is there a formula for the number of interpretations of x^n in a general commutative non-associative algebra?

Discussion by H. W. Becker, Santa Monica, California. The expression

$$N_{n+1} = (2n)!/n!(n+1)! = \binom{2n}{n}/(n+1) \tag{1}$$

has a wide variety of distinct combinatorial and graphical interpretations, of at least five different types. That the types are distinct is made evident upon attempting to enumerate the symmetrically different sequences or graphs of each type, for these in general yield unequal numbers.

Wedderburn [1] demonstrated the well known

$$N_1 = N_2 = 1, \qquad N_n = \sum_{m=1}^{n} N_{n-m}\cdot N_m, \tag{2}$$

and showed that, upon omitting all terms of (2) having like form, we have

$$N'_1 = N'_3 = 1, \qquad N'_{2n+1} = \sum_{m=1}^{n} N'_{2n+1-m}\cdot N'_m, \tag{3}$$

$$N'_2 = 1, \qquad N'_{2n} = N'_n(N'_n+1)/2 + \sum_{m=1}^{n-1} N'_{2n-m}\cdot N'_m. \tag{4}$$

Here N_n' and N_n are the numbers of commutative and non-commutative, non-associative products of n factors.

Etherington [2] independently found generating functions for N_n and N_n' and gave an [illegible] as to why they yield a simple formula for the first but not the second. His further work [3, 4] founded a tradition in *genetic algebra* whose development at the hands of R. D. Schafer [5, 6, 7] reached a wide audience.

In lieu of an exact formula for N_n, we seek an approximation. By Stirling's approximation to $n!$, (1) becomes

$$N_{n+1} \approx 4^n/(n+1)\sqrt{\pi n}, \tag{5}$$

whence N_{n+1}/N_n approaches 4 with increasing n. If we calculate N_n and N_n' for values of n through $n=25$, and utilize an idea due to Motzkin [8], the following results are suggested empirically:

$$N_{n+1}/N_{n+1}' = R_n, \qquad R_n/R_{n-1} = r_n, \qquad r_n \to r = 1.612. \tag{6}$$

Thus

$$N_{n+1}'/N_n' = k_n, \qquad k_n \to k = 4/r = 2.48. \tag{7}$$

If the number h_n is defined so that the following equation is true, it appears as though there exists a number h, the limit of h_n, in terms of which the desired approximation of N_{n+1}' may be expressed. We have then $h_n \approx 0.812$ and approximately

$$N_{n+1}' = h_n k^n/(n+1)\sqrt{n} \approx N_{n+1}/(0.7)r^n. \tag{8}$$

This formula is suggested for what it may be worth, based as it is on inspection of a few values of n, and lacking any proof of the existence of the limits r, k, or h. The detailed figures are shown in the table below. Note, however, that (5) and (8) have the same form, $U_n \approx ab^n/n^{3/2}$, as the approximations for the number of n-branch series-parallel passive circuits [9], and of n-branch root-trees [10].

Other interpretations of the function (1) should be of interest. N_n is the number of planar rhyme schemes [11], such that there are no crossovers in the Puttenham diagram [12]. N_{n+1} is the number of ballot sequences in a two party election, such that the non-loser gets $n-1$ votes and is never behind his opponent, who may get anywhere from 0 to $n-1$ votes, Lucas [13], p. 164, *le scrutin du ballotage.* (See also p. 14, *marches du pion du jeu de dames*; p. 86, *les deux files de soldats*; and p. 87, *déplacements de la tour sur l'échiquier triangulaire.*)

N_{n+1} is the number of ways of decomposing an $(n+2)$-gon into triangles by $n-1$ non-intersecting diagonals, Lucas [13], pp. 90–96, 489. N_{n+1} is also the number of ways of joining $2n$ points around a circle by n non-intersecting chords, the c_n of Motzkin [8], which linearize to the configurations superieures of Touchard [14].

This problem 4277 may also be regarded as a sequel to problem 3954 of O. Ore [1941, 564], whose solutions provide a helpful background.

n	N_n	N'_n	[illegible]	k_n	h_n
1	1	1	1	1	.806
2	1	1	2	1	.69
3	2	1	1.25	2	.909
4	5	2	1.867	1.5	.791
5	14	3	1.5	2	.856
6	42	6	1.714	1.8333	.809
7	132	11	1.558	2.0909	.842
8	429	23	1.663	2	.83
9	1430	46	1.595	2.1304	.824
10	4862	98	1.635	2.1122	.814
11	16796	207	1.607	2.1787	.8175
12	58786	451	1.624	2.1796	.8119
13	2 08012	983	1.612	2.2167	.8139
14	7 42900	2179	1.617	2.2258	.811
15	26 74440	4850	1.612	2.2485	.8118
16	96 94845	10905	1.615	2.2587	.8108
17	353 57670	24631	1.613	2.274	.8109
18	1296 44790	56011	1.613	2.2837	.8105
19	4776 38700	1 27912	1.612	2.2949	.8108
20	17672 63190	2 93547	1.612	2.3034	.8108
21	65641 20420	6 76157	1.612	2.3121	.811
22	2 44662 67020	16 63372	1.6121	2.3194	.8112
23	9 14825 63640	36 26149	1.6118	2.3265	.8115
24	34 30596 13650	84 36379	1.6118	2.3328	.8119
25	128 99041 47324	196 80277	1.6117	2.3387	.8121

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