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Some problems of non-associative combinations (I)

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The problems considered here are essentially algebraic; but it is convenient to begin with a picturesque formulation.

Let a convex polygon cut out of paper be cut along a diagonal; it is thus divided into two convex polygons. Either of these may then again be cut along a diagonal making three convex polygons; and the process may be continued until only triangles are left, or terminated earlier, as desired. When r cuts have been made, the original polygon has been dissected into $r+1$ sub-polygons.

Such a dissection will be called a *partition* of the polygon. Geometrically, a partition may be described as a set of r diagonals which do not intersect in the interior. The polygon itself $(r = 0)$ is included among its partitions. We may enquire in how many ways a partition can be made for a given polygon, with perhaps some restriction on the kinds of sub-polygon (triangles, quadrilaterals, etc.) which may be left.

This is essentially a problem of *non-associative combinations*[1], or combinations with brackets inserted; for it will be shown that if the given polygon has $n+1$ sides, the partitions may be described algebraically as the different ways of inserting brackets in a product $\alpha\beta\gamma \ldots.$ of n factors in a given order; which will be called *associations* of the n factors.

[1] Of a more general kind than those considered in my paper under this title, *Proc. Roy. Soc., Edin.*, 59, 1939, 153-162.

To see this, select a particular side of the given polygon as *base*, and let the other sides be labelled α, β, γ, ... Since the order in which the cuts are performed is not taken into account, we may proceed as follows. Regard the given polygon as an elastic band stretched tightly round $n+1$ pins at its vertices, the two base pins being kept fixed and the others removed in r stages, corresponding to the r cuts in a suitable order. At each stage two or more sides λ, μ, ... collapse on to a new side, previously a diagonal: if this new side is labelled as a product $\lambda\mu$ ...., then ultimately the base itself will be labelled as a non-associative product containing as factors the n sides α, β, γ, ... of the original polygon in order, the manner in which they are associated being determined by the partition. (See the example in figure 1.)

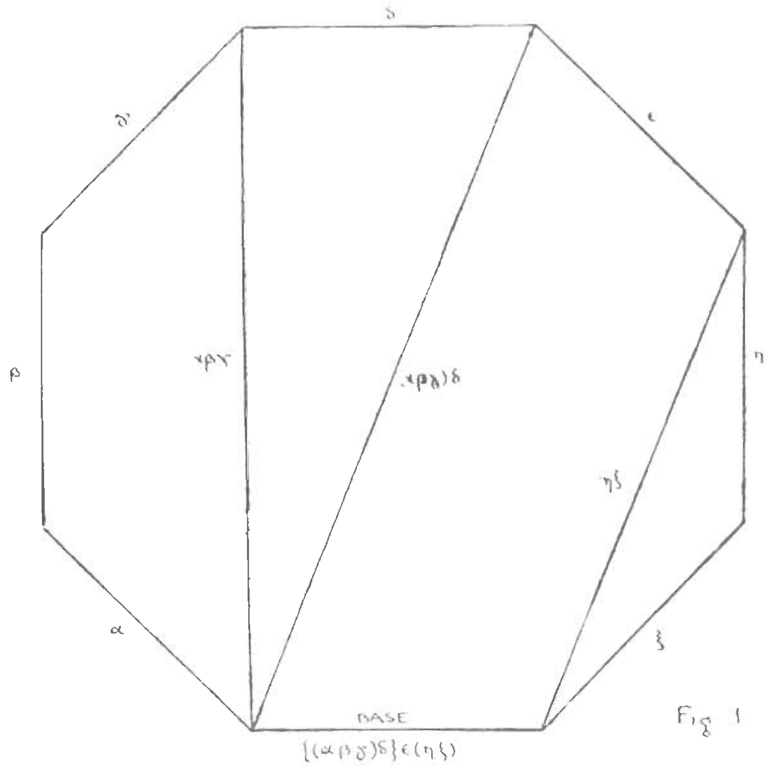

Fig 1

Conversely, any manner of inserting r pairs of brackets in the product $\alpha\beta\gamma$ .... corresponds to one definite partition of the given polygon with chosen base by r cuts, if the following points are observed:

(i) The brackets must be effective; *i.e.*, unnecessary brackets as in $(\alpha\beta\gamma\ldots.)$ enclosing the complete product, $\alpha(\beta)\gamma\ldots.$, or $\alpha((\beta\gamma))\delta\ldots.$, are not to be counted. They would be like cutting the polygon along the base, along a side, or twice along the same diagonal.

(ii) Pairs of brackets must not overlap as in $\alpha(\beta[\gamma)\delta]$; for this would determine a set of diagonals intersecting in the interior of the polygon.

We have thus set up a one-one correspondence between all possible partitions of a convex polygon with $n+1$ sides, and all possible associations of n similar objects; or, what comes to the same thing, of n dissimilar objects in a prescribed order. Combinations of this kind were represented by Cayley[1] as *trees*. Figure 2 will make this

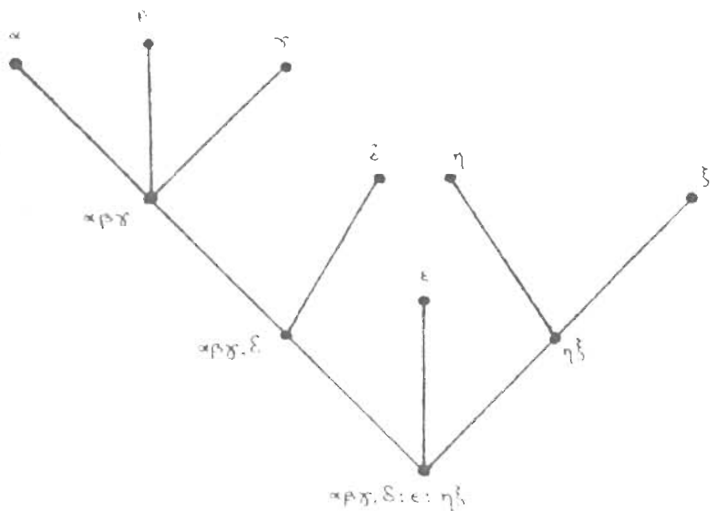

Fig. 2

representation clear without further explanation. For the present purpose, corresponding to the restriction (i) above, we must consider only trees which at every knot bifurcate at least. (In some of Cayley's investigations, knots such as ┿ were permitted.) There is

[1] *E.g.*, *Phil. Mag.*, 13 (1857), 172-176.

also exhibited in figure 2 a convenient notation for dispensing with brackets when writing non-associative products. Dots are placed between the grouped factors, more dots implying more delay in combination.

Consider now various special cases of the problem. In each case the required number is denoted by A_n. Taking

$$A_1 = 1,$$

the enumeration is made with the help of the generating function

$$f(x) = \Sigma A_n x^n \qquad (n = 1, 2, 3, \ldots . \infty).$$

Case 1 is equivalent to the first, and Case 5 to the second, of Schröder's *Vier combinatorische Probleme*[1]. Case 1 was discussed in a series of papers by Lamé, Catalan, Rodrigues, Binet[2], and has been touched on by Cayley[3], Wedderburn[4], Etherington[5]. The other cases as far as I know are new. The solution of the general problem, Case 4, is completed in the next Note. The connection between the geometrical and algebraical problems was noticed by Catalan for Case 1 only.

Case 1. *Partition into triangles.*

In my paper (1939, *loc. cit.*) I confined attention to non-associative products in which factors are combined only two at a time. The manner of association of the factors was called the *shape* of the product. In the case when multiplication is non-commutative as well as non-associative, shapes form a special class of the associations considered above. They correspond to partitions of a convex polygon into triangles. They also correspond to trees bifurcating at every knot, which I called *pedigrees*.

The enumeration in this case is given by the following formulae. Considering how the product of n factors may be built up,

$$\begin{aligned} A_n &= A_1 A_{n-1} + A_2 A_{n-2} + \ldots . + A_{n-1} A_1 \qquad (n > 1) \\ &= \text{the coefficient of } x^n \text{ in } f(x)^2; \end{aligned}$$

whence $$f(x) = x + f(x)^2,$$
i.e., $$f(x)^2 - f(x) + x = 0.$$

[1] *Zeits. Math.*, 15 (1870), 361-376.

[2] *Journ. de Math.*, 3, 4 (1838-39), 505, etc.

[3] *Phil. Mag.*, 13 (1859), 374-378.

[4] *Ann. Math.* (2), 24 (1922), 121-140.

[5] *Loc. cit.*, and *Math. Gaz.*, 21 (1937), 36-39.

$C_n = \frac{1}{n+1}\binom{2n}{n}$

Catalan numbers

This, with $f(0) = 0$, gives

$$f(x) = \tfrac{1}{2}\{1 - (1 - 4x)^{\frac{1}{2}}\},$$

yielding on expansion

$$A_n = \frac{(2n-2)!}{(n-1)!\,n!} = \frac{1}{n}\binom{2n-2}{n-1}$$

$= 1,\ 1.\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430,\ 4862,\ \ldots.$ for ~ 108 ✓

$n = 1, \ldots., 10, \ldots.$

Case 2. *Partition into quadrilaterals.*

This corresponds to trees which trifurcate at each knot; and to the associations formed by combining factors always three at a time, *e.g.*, $\alpha\beta\gamma : \delta\,.\,\epsilon\zeta\eta\,.\,\theta : \iota$. For this to be possible, the total number of factors must be *odd*, since at each combining operation the number of factors left is reduced by two. Similarly a polygon to be partitioned into quadrilaterals must have an *even* number of sides. Thus

$$A_2 = A_4 = A_6 = \ldots. = 0.$$

Proceeding as before,

$$A_n = \Sigma A_i A_j A_k \qquad (i + j + k = n,\ n > 1)$$
$$= \text{the coefficient of } x^n \text{ in } f(x)^3; \qquad (n > 1)$$
$$f(x) = x + f(x)^3:$$

and so the generating function is that root of the cubic equation

$$f(x)^3 - f(x) + x = 0$$

for which $f(0) = 0$. To calculate the values of A_1, A_3, A_5, ... in succession, we may use the method of successive approximations:

$$\begin{aligned} f(x) &= x + f(x)^3 & &= x + (x + \ldots.)^3 \\ &= x + x^3 + \ldots. & &= x + (x + x^3 + \ldots.)^3 \\ &= x + x^3 + 3x^5 + \ldots. & &= x + (x + x^3 + 3x^5 + \ldots.)^3, \text{ etc.} \end{aligned}$$

Continuing, it will be found that for $n = 1, 3, 5, \ldots., 19, \ldots.$

$A_n = 1, 1, 3, 12, 55, 273, 1428, 7752, 43263, 246675, \ldots.$ —— 1764

Case 3. *Partition into triangles or quadrilaterals.*

This corresponds to trees which divide into either two or three branches at each knot; and to associations formed by combining factors either two or three at a time. (*E.g.*, see the figures.) We have, for $n > 1$,

$$A_n = \Sigma A_i A_j + \Sigma A_k A_l A_m \qquad (i + j = k + l + m = n)$$
$$= \text{the coefficient of } x^n \text{ in } f(x)^2 + f(x)^3.$$

Therefore $\qquad f(x) = x + f(x)^2 + f(x)^3,$

and the generating function is that root of this equation for which $f(0) = 0$. By successive approximations,

$$f(x) = x + (x + \ldots.)^2 + (x + \ldots.)^3 \qquad = x + x^2 + \ldots.$$
$$= x + (x + x^2 + \ldots.)^2 + (x + x^2 + \ldots.)^3 = x + x^2 + 3x^3 + \ldots., \text{etc.};$$

and we find that for $n = 1, 2, \ldots., 10, \ldots.$

$$A_n = 1, 1, 3, 10, 38, 154, 645, 2853, 12844, 58985, \ldots.$$

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Case 4. *Generalisation.*

Suppose we wish to enumerate the partitions of a convex $(n + 1)$-gon with the restriction imposed that the final sub-polygons shall be all either $(a + 1)$-gons or $(b + 1)$-gons or $(c + 1)$-gons, etc., where $a, b, c, \ldots.$ are given positive integers. Following the method of previous cases, we arrive at the result that the generating function $f(x)$ is determined by

$$f^a + f^b + f^c + \ldots. - f + x = 0, \; f(0) = 0.$$

An explicit expression for A_n is given in formula (2) of the Note which follows.

Case 5. *The unrestricted problem.*

If no restrictions are imposed on the partitions,

$$f = x + f^2 + f^3 + f^4 + \ldots. \text{ to } \infty$$
$$= x + f^2/(1 - f).$$

Hence $\qquad 2f^2 - (1 + x)f + x = 0.$

This agrees with Schröder's result, found in a more complicated way. He deduced

$$f(x) = \tfrac{1}{4}\{1 + x - (1 - 6x + x^2)^{\frac{1}{2}}\},$$

and hence

$$A_n = \tfrac{1}{4}(-1)^{n-1} \Sigma \binom{\frac{1}{2}}{n - a} \binom{n - a}{a} 6^{n-2a}$$

where $\qquad a = 0, 1, 2, \ldots.; \; 2a \leqq n; \; n > 1.$

For $n = 1, 2, 3, \ldots., 10, \ldots.$ this gives the sequence

$$A_n = 1, 1, 3, 11, 45, 197, 903, 4279, 20793, 41128, \ldots.$$

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