The number of fixed density necklaces for \( n = 2, 3, 4, 8, 14, 22, 40, 88, 188, 392, 812, 1622, 3112 \). This is sequence \( \text{A000013} \) in Neil J. Sloane's \textit{OEIS} of integer sequences.

The number of unlabelled necklaces for \( n = 2, 3, 1, 1, 1, 2, 4, 9, 18, 36, 90, 186, 335, 603, 1111, 2112 \). This is sequence \( \text{A000029} \) in Neil J. Sloane's \textit{OEIS} of integer sequences.

There are explicit formulas for both of these sequences, below, for any alphabet.

\[
\mu_{n,d} = \frac{1}{n!} \sum_{j=1}^{n} \mu_j \mu_{n-j}
\]

\[
\lambda_{n,d} = \frac{1}{n!} \sum_{j=1}^{n} \lambda_j \mu_{n-j}
\]

\textbf{Unlabelled necklaces}

Unlabelled necklaces are an equivalence class of necklaces under rotation.

Given a string of length \( n \), one can create a necklace by rotating the string that many places. The resulting necklace is equivalent to the original necklace.

Every necklace corresponds to a De Bruijn sequence, which is a shortest possible string containing every \( k \)-ary string of length \( n \) exactly once.

\textbf{Necklaces with Fixed Density}

In many applications not all necklaces are required, but rather only those with a fixed density. In the more general case, one may want a list of necklaces where the number of occurrences for every character is fixed.

To count fixed density necklaces we let \( N_{n,d_1,\ldots,d_k} \) denote the number of necklaces composed of \( n \) occurrences of the symbol \( a \) for \( i=1,\ldots,k \). Let the density of the necklace \( \frac{n_1}{n} + \frac{n_2}{n} + \ldots + \frac{n_k}{n} = d \). It is known that

\[
N_{n,d_1,\ldots,d_k} = \frac{1}{n!} \sum_{j_1+\ldots+j_k=n} \mu_{j_1} \mu_{j_2} \cdots \mu_{j_k} \binom{n}{j_1,\ldots,j_k}
\]

To get the number of fixed density necklaces with length \( n \) and density \( d \), we sum over all possible values of \( n_1, n_2, \ldots, n_k \).

The number of fixed density necklaces is counted similarly.

\[
N_{j_1+\ldots+j_k=n} \mu_{j_1} \mu_{j_2} \cdots \mu_{j_k} \binom{n}{j_1,\ldots,j_k}
\]

For the binary case, these formulas simplify as follows:

\[
N_{n,d} = \frac{1}{n!} \sum_{j_1+\ldots+j_k=n} \mu_{j_1} \mu_{j_2} \cdots \mu_{j_k}
\]
The number of necklaces with no 00 subsequence for n = 1, 2, 3, 4, 5, 8, 10, 15, 19, 31, 41, 64, 90. This is sequence A000358 in Neil J. Sloane’s database of integer sequences.

Bracelets

A bracelet is a necklace that can be turned over.

The formula for a bracelet is:

\[ B(n) = \binom{N}{k} + \binom{N}{k + 1} / 2 \]

where \( B(n) \) is the number of bracelets.

For n = 1, 2, 3, 4, 6, 8, 13, 18, 30, 46, 78, 126, 224, 380, 687, 1224, 2250, 4120, 7630, 14308, 26931. This is sequence A001371 in Neil J. Sloane’s database of integer sequences.

The number of necklaces with no forbidden sequence to be of the form 00 is precisely the set: {0101, 0111, 111}. Notice that this is precisely the set of necklaces that start with 01. The bracelet number is therefore \( B(2k) = \binom{2k}{k} + \binom{2k}{k + 1} / 2 \).

Programs available:

- Necklaces: Lyndon words, De Bruijn sequences
- Fixed density necklaces
- Fixed-length necklaces and Lyndon words
- Lyndon words and necklace factorizations
- Lyndon words
- Lyndon words and necklace factorizations

For more information about the number of Lyndon words with given trace, visit Neil J. Sloane’s database of integer sequences. This is sequence A005979 in Neil J. Sloane’s database of integer sequences.