

The Kolakoski transform of words

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Abstract

Starting from a self-referential and recursive definition of the Kolakoski sequence, we introduce the Kolakoski transforms of words on 2 numbers. We then make conjectures allowing us to propose an approach tackling the Keane's conjecture [Kea]¹ using a probability perturbation method or a deterministic perturbation method. This way we brought the problem from discrete to semi-continuous mathematics and chaotic dynamical systems.

Introduction

The Kolakoski word (or sequence) $(K(n))_{n \geq 1}$ is the sequence $A000002$ [Slo] and was studied in [Kol] by William Kolakoski in the 60's. In fact this sequence was considered earlier [Old] by Rufus Oldenburger in 1939. It is completely determined by the recursive and self referential scheme $K(1) = 1, K(2) = 2$ and for $n \geq 3$ by the following formula:

$$K(n) = w_0(k(n))$$

where

$$k(n) = \inf \{t \geq 1 \mid K(1) + \dots + K(t) \geq n\}$$

is the sequence $A156253$ [Slo] and where w_0 is the word $12121212\dots$ defined by $w_0(n) = \frac{3+(-1)^n}{2}$ for $n \geq 1$. The proof is omitted here and is easy. Next it is natural to generalise this self-referential construction using any primitive word on 2 numbers as follows. We may emphasise that we can work with integers but also with real values as we shall see. This study relies indeed more on analytic aspects of the transform than on combinatorial considerations related to words. To this end we define firstly the simple Kolakoski transform and thereafter the weighted Kolakoski transform of words.

¹Keane was apparently the first to suggest that the density of 1's is $\frac{1}{2}$ in the Kolakoski word. It is still an open question and probably the most important regarding the Kolakoski sequence.

The simple Kolakoski transform For a given word w on the alphabet $\{r, s\}$ where r and s are distinct positive reals we define the simple Kolakoski transform of the word w by $K_w(1) \in \{r, s\}, K_w(2) \in \{r, s\}$ and for $n \geq 3$ by

$$K_w(n) = w(k_w(n))$$

where $k_w(n) = \inf \{t \geq 1 \mid K_w(1) + \dots + K_w(t) \geq n\}$.

For our purpose it is worth to go further and we consider the following more general transform.

The weighted Kolakoski transform Let $u = (u_i)_{i \in \mathbb{N}}$ be a sequence of strictly positive reals². Then the weighted Kolakoski transform of the word w on the alphabet $\{r, s\}$ where r and s are distinct positive reals of weight u is defined by

$K_{w,u}(1) \in \{r, s\}, K_{w,u}(2) \in \{r, s\}$ and for $n \geq 3$ by

$$K_{w,u}(n) = w(k_{w,u}(n))$$

where $k_{w,u}(n) = \inf \{t \geq 1 \mid u_1 K_{w,u}(1) + \dots + u_t K_{w,u}(t) \geq n\}$.

Hence the simple Kolakoski transform of the word w is the weighted Kolakoski transform of constant weight $u = 1, 1, 1, 1, 1, \dots$

In the sequel we write **SKT** for the simple Kolakoski transform and **WKT** for the weighted Kolakoski transform.

Plan of paper

In the first section we make the conjecture (1) related to SKT on words on the alphabet $\{1, 2\}$ which is a generalisation of the Keane's conjecture. Another big amount of experiments led us to consider non constant weights as a possible tool to circumvent problems like the Keane's conjecture.

Hence in the second section we study a tractable WKT of the word w_0 and then we introduce a probability perturbation method on the weights. We make a general conjecture (2) related to words on $\{1, 2\}$ and allowing us to deal with the Keane's conjecture via the conjecture(4). The conjecture (3) is more specific and related to the word $w_1 := 13131313\dots$ on $\{1, 3\}$ for which the conjecture (2) clearly doesn't hold.

In the third section we consider a deterministic perturbation method related to the primitive word and not to the weight. The word w_1 seems then surprisingly connected to the Golden ratio $\Phi = \frac{1+\sqrt{5}}{2}$ allowing us to state the conjectures (5) and (6) related to the words w_1 and w_0 respectively. This yields the conjecture (7) related to w_0 and proving the Keane's conjecture. We provide a conjecture (8) generalising the conjecture (6).

²The weight u can be a word or any suitable sequence, i.e. the weight must allow us the computation of infinitely many terms of the transform. We could also consider a weight $u_{n,k}$ depending also on n but we don't consider this transform here.

In a fourth section we state a very ambitious conjecture (9) where we try to summarize our thoughts in a single and general conjecture related to a deterministic perturbation of the SKT of words w on a real alphabet $\{a, y\}$ with $a, y > 0$.

Finally in the fifth section we study the SKT of the perturbed words $w = y(y+1)y(y+1)y(y+1)\dots$ where $y > 0$ is a real value and provide the conjecture (10) yielding an explicit family of words satisfying a generalized Keane's conjecture.

More properties of the WKT will be described elsewhere [Clo] as well as the extension of the WKT to words over alphabet containing more than 2 numbers.

1 Generalisation of the Keane's conjecture

We generalise the Keane's conjecture to any SKT of words on the alphabet $\{1, 2\}$.

Conjecture (1)

Whatever the word w on the alphabet $\{1, 2\}$ you consider and verifying

$$\lim_{n \rightarrow \infty} \frac{w(1) + \dots + w(n)}{n} = \lambda$$

where $1 \leq \lambda \leq 2$ then the SKT of w always satisfies

$$\lim_{n \rightarrow \infty} \frac{K_w(1) + \dots + K_w(n)}{n} = \lambda \quad (1)$$

In particular, taking $w = w_0$ (so that $\lambda = \frac{3}{2}$) the conjecture (1) implies that

$$\lim_{n \rightarrow \infty} \frac{K(1) + \dots + K(n)}{n} = \frac{3}{2}$$

and the Keane's conjecture [Kea] would be true for the Kolakoski sequence.

Hundreds of experiments support the conjecture (1) and we provide thereafter 4 examples illustrating this fact.

Examples supporting the conjecture (1)

Taking a word w on the alphabet $\{1, 2\}$ satisfying

$$\lim_{n \rightarrow \infty} \frac{w(1) + \dots + w(n)}{n} = \lambda$$

we start from $K_w(1) = 1$ and $K_w(2) = 2$ and we compute the SKT of w . Then we plot for $1 \leq n \leq 30000$

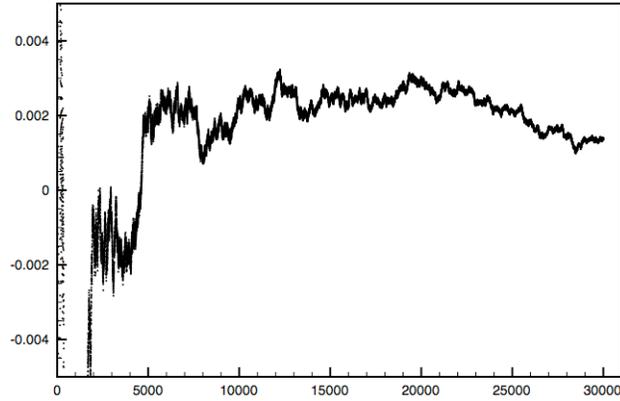
$$\frac{K_w(1) + \dots + K_w(n)}{n} - \lambda$$

Although we compute few terms in each case, the repeated observation that $\frac{K_w(1) + \dots + K_w(n)}{n} - \lambda$ stays close to zero supports somewhat the conjecture (1).

Example n°1

$$w(n) = \lfloor (n+1)\sqrt{2} \rfloor - \lfloor n\sqrt{2} \rfloor \text{ so } \lim_{n \rightarrow \infty} \frac{w(1)+\dots+w(n)}{n} = \sqrt{2} .$$

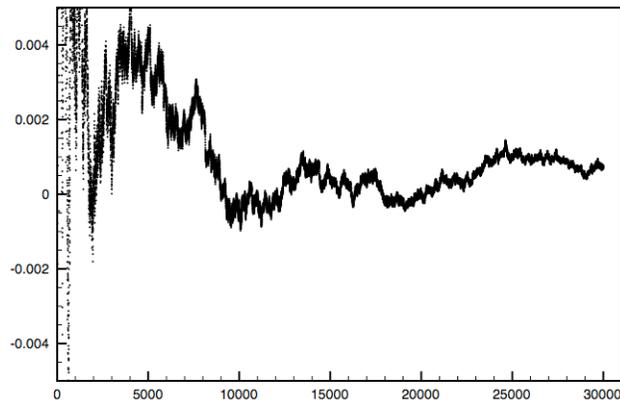
Plot of $\left(\frac{K_w(1)+\dots+K_w(n)}{n} - \sqrt{2} \right)$



Example n°2

$$w(n) = 2 \text{ if } n \text{ is squarefree } w(n) = 1 \text{ so } \lim_{n \rightarrow \infty} \frac{w(1)+\dots+w(n)}{n} = 1 + \frac{6}{\pi^2} .$$

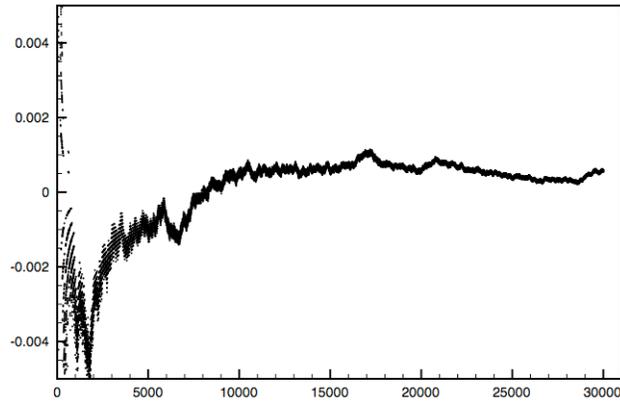
Plot of $\left(\frac{K_w(1)+\dots+K_w(n)}{n} - \left(1 + \frac{6}{\pi^2} \right) \right)$



Example n°3

$w(n) = 2$ if n is divisible by 3 $w(n) = 1$ so $\lim_{n \rightarrow \infty} \frac{w(1)+\dots+w(n)}{n} = \frac{4}{3}$.

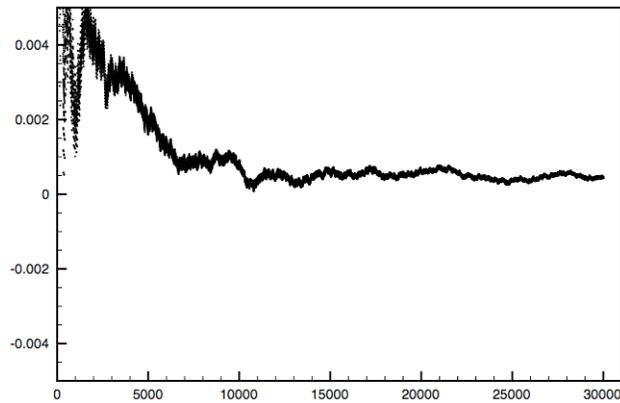
Plot of $\left(\frac{K_w(1)+\dots+K_w(n)}{n} - \frac{4}{3} \right)$



Example n°4

$w(n) = 2$ if n is divisible by 5 $w(n) = 1$ otherwise so $\lim_{n \rightarrow \infty} \frac{w(1)+\dots+w(n)}{n} = \frac{6}{5}$.

Plot of $\left(\frac{K_w(1)+\dots+K_w(n)}{n} - \frac{6}{5} \right)$



2 Perturbation of SKT

At first glance the SKT doesn't allow us to have any clue to understand better the Kolakoski word since it is quite apparent that working only with $\{1, 2\}$ hides what is going on. With the WKT however we were able to observe intrinsic asymptotic properties of the SKT using a perturbation method based on random weights. Then we checked out that these properties are not shared by the WKT of words on $\{1, 3\}$ giving credence to this approach to the Keane's conjecture. We state thereafter the conjecture (2) and suggest to tackle the Keane's conjecture via the conjecture (4). The conjecture (3) shows that the conjecture (2) doesn't work for words on the alphabet $\{1, 3\}$ like $w_1 = 13131313\dots$

2.1 A tractable weighted transform of w_0

Before stating the conjecture (2) let us consider a concrete example of weighted transform of w_0 which is well understood since it relies on something known.

Suppose the weight u is constant and $\forall n \geq 1, u_n = 2$. Then the WKT of w_0 with starting values $K_{w_0,u}(1) = K_{w_0,u}(2) = 1$ satisfies

- $K_{w_0,u}(n) = A157129(n)$ [Slo]

Next it is easy to see

- $K_{w_0,u}(n) = \frac{1}{2}A071928(n)$ [Slo]

where $A071928$ is the generalisation $\text{Kol}(2, 4)$ considered and solved in a paper of Bernd Sing [Sin]. Therefore we have (details omitted)

$$K_{w_0,u}(1) + \dots + K_{w_0,u}(n) = \frac{3}{2}n + O(1)$$

More precisely let $A(n) = 3n - 2(K_{w_0,u}(1) + \dots + K_{w_0,u}(n))$ then $A(n)$ satisfies a multiple recurrence relation modulo 12.

Namely for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$ we have $A(n) = 1, 2, 1, 0, 1, 2, 3, 4, 3, 2, 1$ and for $k \geq 1$ the following recurrence formulas hold:

- $A(12k) = A(4k)$
- $A(12k + 1) = A(4k + 1)$
- $A(12k + 2) = A(4k + 2)$
- $A(12k + 3) = A(4k + 2) - 1$
- $A(12k + 4) = A(4k + 2) - 2$
- $A(12k + 5) = A(4k + 2) - 1$
- $A(12k + 6) = A(4k + 2)$
- $A(12k + 7) = 4 - A(4k + 3)$

- $A(12k + 8) = 4 - A(4k + 4)$
- $A(12k + 9) = 4 - A(4k + 3)$
- $A(12k + 10) = 4 - A(4k + 2)$
- $A(12k + 11) = A(4k + 3)$

Hence one may notice this cousin sequence of the Kolakoski sequence satisfies the Keane's conjecture.

Thus the WKT of w_0 of constant weight $u = 2$ is much easier to handle than the WKT of w_0 of constant weight $u = 1$. This trivial observation led us to think about perturbing the constant weight $1, 1, 1, 1, 1, \dots$ by adding extra 2 's (or anything else) in some places according to a suitable rule. From a big amount of experiments it appears some random perturbations yield interesting behavior. Thus we make the following conjecture (2) using random weights u such that the probability that a term of u has value 1 can be very close to 1.

In some way the SKT preserves randomness when we consider words on $\{1, 2\}$. There are many ways to define random weights able to support our conjectures but here we define the following simple ones which are easy to simulate using low discrepancy sequences.

Definition of our random weights Let $\alpha > 0$ be a real value. Let $f_\alpha > 0$ be a function of regular variation of index α^3 . Let δ be a function. Then $u_{f_\alpha, \delta} = (u_{f_\alpha, \delta}(n))_{n \geq 1}$ is the infinite random weight defined as follows:

- $u_{f_\alpha, \delta}(n) = \delta(n)$ with probability $\frac{1}{f_\alpha(n)}$ otherwise $u_{f_\alpha, \delta}(n) = 1$

2.2 Conjecture (2)

Let w be a word on $\{1, 2\}$ satisfying

$$\lim_{n \rightarrow \infty} \frac{w(1) + \dots + w(n)}{n} = \lambda$$

Suppose δ is any positive bounded function. Then $\forall \alpha \in]0, 1[$ and for any function of regular variation $f_\alpha > 0$ of index α the random WKT of w of weight $u_{f_\alpha, \delta}$ satisfies

$$\lim_{n \rightarrow \infty} \frac{K_{w, u_{f_\alpha, \delta}}(1) + \dots + K_{w, u_{f_\alpha, \delta}}(n)}{n} = \lambda \quad (2)$$

In the APPENDIX 1 we provide graphics supporting somewhat the conjecture (2) using the low discrepancy sequence $(\{n\sqrt{2}\})_{n \geq 1}$ to generate the weight $u_{f_\alpha, \delta}$ where $\{x\}$ is the fractional part of x .

³ f is a function of regular variation of index α if we have $f(x) = x^\alpha L(x)$ where L is slowly varying i.e. $\forall x > 0 \lim_{y \rightarrow \infty} \frac{L(xy)}{L(y)} = 1$

Remark From our conjecture (1) the SKT of any word w on $\{1, 2\}$ satisfies $\lim_{n \rightarrow \infty} \frac{K_w(1) + \dots + K_w(n)}{n} = \lambda$. From the previous conjecture (2) a suitable random WKT has the same property. However many WKT of words on $\{1, 2\}$ don't have the property. For instance experiments suggest that for words w on $\{1, 2\}$ satisfying $\lim_{n \rightarrow \infty} \frac{w(1) + \dots + w(n)}{n} = \lambda$ the self WKT of w (the WKT of w of weight w) starting with 1, 2 satisfies "almost surely"

$$\lim_{n \rightarrow \infty} \frac{K_{w,w}(1) + \dots + K_{w,w}(n)}{n} = \frac{3\lambda - 2}{\lambda}$$

In an other hand if you consider for a given integer $p \geq 2$ the WKT of w_0 of weight $u_p(n) = \frac{p}{p-1}$ if n is even and $u_p(n) = 1$ otherwise, starting with 1, 2, it seems we always have

$$\lim_{n \rightarrow \infty} \frac{K_{w_0, u_p}(1) + \dots + K_{w_0, u_p}(n)}{n} = \frac{3p - 2}{2p - 1}$$

So it is not $\frac{3}{2}$ except when $p \rightarrow \infty$. These observations and many others will be described in more details elsewhere. They underline that the conjecture (2) is far from being obvious and was stated after a lot of experiments.

2.3 Conjecture (3)

This conjecture is more specific in order to see the different behavior between WKT of $w_0 = 12121212\dots$ and WKT of $w_1 = 13131313\dots$

Note that the SKT of w_1 starting with 1, 3 is the Kolakoski (1, 3) sequence A064353 [Slo] for which the frequency of the number 3 is 0.6027847 not $\frac{1}{2}$ [Baa]. So we may expect that the conjecture (2) doesn't work for w_1 . It appears it is the case and moreover in some cases the limit (2) doesn't exist. So we conjecture the random WKT of w_1 considered above don't always satisfy (2). More precisely let u_α be the weight

- $u_\alpha(n) = 1 + \frac{1}{n}$ with probability $\frac{1}{n^\alpha}$ otherwise $u_\alpha(n) = 1$

Then $\forall \alpha \in]0, 1[$ the WKT of w_1 of weight u_α satisfies

$$\lim_{n \rightarrow \infty} \frac{K_{w_1, u_\alpha}(1) + \dots + K_{w_1, u_\alpha}(n)}{n} \neq 2 \quad (3)$$

In the APPENDIX 2 we provide graphics supporting clearly the conjecture (3) making again use of low discrepancy sequences to generate the weight u_α .

2.4 Conjecture (4)

Let w be a word on $\{1, 2\}$ satisfying

$$\lim_{n \rightarrow \infty} \frac{w(1) + \dots + w(n)}{n} = \lambda$$

Then from the conjecture (2) letting $\alpha \rightarrow 1$ and $\delta(n) = 2 - \alpha$ we have $u_{\alpha,\delta} \rightarrow 1, 1, 1, 1, 1, \dots$ and we claim the WKT of w of weight $u_{\alpha,\delta}$ converges toward the SKT of w as $\alpha \rightarrow 1$. Then we add that we can infer from the conjecture (2) that the SKT of w verifies

$$K_w(1) + \dots + K_w(n) = \lambda n + O\left(n^{1/2+\varepsilon}\right) \quad (4)$$

In particular this means that the Keane's conjecture is true taking $w = w_0$.

3 Perturbation of the initial words w_0 and w_1

Instead of perturbing the SKT using random weights, we can also perturbate the primitive word w in a simple deterministic way keeping the SKT. It turns out we unearth a surprising connection between the SKT of the perturbed word w_1 and the golden ratio $\Phi = \frac{1+\sqrt{5}}{2}$.

We recall we have $\lim_{n \rightarrow \infty} \frac{K_{w_1}(1) + \dots + K_{w_1}(n)}{n} = \lambda_1$ where $\lambda_1 = 2.205569$ is the unique real root of the cubic polynomial $x^3 - 2x^2 - 1$ which was proved by Bernd Sing in [Sin] and we guess that $\lim_{n \rightarrow \infty} \frac{K_{w_0}(1) + \dots + K_{w_0}(n)}{n} = 1.5$ from the Keane's conjecture. Hereafter the conjecture (5) relates a deterministic perturbation of w_1 to the outcome of this perturbation under the SKT. The conjecture (6) is the analogue of (5) for w_0 .

Conjecture (5)

Let $w_{1,x}$ be the sequence $w_{1,x}(n) = 1$ if n is odd and $w_{1,x}(n) = 3 + x$ if n is even. Then $\exists \eta_1 > 0$ such that for any $0 < \varepsilon < \eta_1$ there exists an integer value $N(\varepsilon)$ such that the averages of the SKT of $w_{1,\varepsilon}$ satisfies (with suitable starting values)

$$\lim_{n \rightarrow \infty} \frac{K_{w_{1,\varepsilon}}(1) + \dots + K_{w_{1,\varepsilon}}(n)}{n} = 2 + O(\varepsilon)$$

and for $n \leq N(\varepsilon)$

$$\left| \frac{K_{w_{1,\varepsilon}}(1) + \dots + K_{w_{1,\varepsilon}}(n)}{n} - \lambda_1 \right| < f_1(\varepsilon)$$

whereas the averages of the SKT of $w_{1,-\varepsilon}$ satisfies (with suitable starting values)

$$\lim_{n \rightarrow \infty} \frac{K_{w_{1,-\varepsilon}}(1) + \dots + K_{w_{1,-\varepsilon}}(n)}{n} = 2 + O(\varepsilon)$$

and for $n \leq N(\varepsilon)$ we have

$$\left| \frac{K_{w_{1,-\varepsilon}}(1) + \dots + K_{w_{1,-\varepsilon}}(n)}{n} - \Phi \right| < g_1(\varepsilon)$$

where f_1 and g_1 are 2 functions satisfying $\lim_{x \rightarrow 0} f_1(x) = \lim_{x \rightarrow 0} g_1(x) = 0$ and where we have $\lim_{\varepsilon \rightarrow 0} N(\varepsilon) = \infty$.

Remark

Since $\lambda_1 \neq \Phi$ we can't infer from the conjecture (5) the result of Sing i.e. $\lim_{n \rightarrow \infty} \frac{K_{w_1}(1) + \dots + K_{w_1}(n)}{n} = \lambda_1$ but we now imagine what is going on in the Kolakoski sequence. The conjecture (6) is trying to reveal the existence of 2 adjacent sequences forcing the Keane's conjecture to be true.

Conjecture (6)

Let $w_{0,x}$ be the sequence $w_{0,x}(n) = 1$ if n is odd and $w_{0,x}(n) = 2 + x$ if n is even. Then $\exists \eta_0 > 0$ such that $\forall \varepsilon, 0 < \varepsilon < \eta_0$ the averages of the SKT of $w_{0,\varepsilon}$ satisfies with suitable starting values

$$\lim_{n \rightarrow \infty} \frac{K_{w_{0,\varepsilon}}(1) + \dots + K_{w_{0,\varepsilon}}(n)}{n} = \frac{3}{2} + O(\varepsilon)$$

whereas the averages of the SKT of $w_{0,-\varepsilon}$ satisfies with suitable starting values

$$\lim_{n \rightarrow \infty} \frac{K_{w_{0,-\varepsilon}}(1) + \dots + K_{w_{0,-\varepsilon}}(n)}{n} = \frac{3}{2} + O(\varepsilon)$$

Remark The conjecture (6) is appealing since there is only a unique outcome and allows us to make the following conjecture (7).

Conjecture (7)

From the conjecture (6) we claim that the average of the SKT of w_0 (wich is the Kolakoski sequence K) satisfies

$$\lim_{n \rightarrow \infty} \frac{K(1) + \dots + K(n)}{n} = \lim_{x \rightarrow 0} \left(\lim_{n \rightarrow \infty} \frac{K_{w_{0,x}}(1) + \dots + K_{w_{0,x}}(n)}{n} \right) = \frac{3}{2}$$

and the Keane's conjecture would be true. We provide experiments supporting the conjectures (5)(6) in the APPENDIX 3 and 4 respectively. Finally we state the conjecture (8) allowing us to prove the conjecture (1) generalising the conjecture (6).

Conjecture (8)

Let w be a word on the alphabet $\{1, 2\}$ satisfying

$$\lim_{n \rightarrow \infty} \frac{w(1) + \dots + w(n)}{n} = \lambda$$

and let w_x be the sequence $w_x(n) = 1$ if $w(n) = 1$ $w_x(n) = 2 + x$ if $w(n) = 2$.

Then $\exists \eta_w > 0$ such that $\forall \varepsilon, 0 < \varepsilon < \eta_w$ the SKT of w_ε satisfies

$$\lim_{n \rightarrow \infty} \frac{K_{w_\varepsilon}(1) + \dots + K_{w_\varepsilon}(n)}{n} = \lambda + O(\varepsilon)$$

and the SKT of $w_{-\varepsilon}$ satisfies

$$\lim_{n \rightarrow \infty} \frac{K_{w_{-\varepsilon}}(1) + \dots + K_{w_{-\varepsilon}}(n)}{n} = \lambda + O(\varepsilon)$$

From this we claim we can infer the conjecture (1) is true in the same way than the conjecture (7), i.e.

$$\lim_{n \rightarrow \infty} \frac{K_w(1) + \dots + K_w(n)}{n} = \lambda$$

Remark We believe that a key ingredient to tackle the Keane's conjecture is to understand better the SKT of $w_{1,x}$ and where the golden ratio comes from in the conjecture (5). In particular the limits observed in the conjecture (3) are certainly related to the Golden ratio and λ_1 too.

It seems quite possible to state a very general conjecture for the words on the alphabet $\{a, y\}$ where a, y are strictly positive distinct real values. In fact we should say hypothesis instead of conjecture since we speculate about the existence of 2 dual functions.

4 Conjecture (9) of dual functions

We state a very general conjecture regarding the simple Kolakoski transform (SKT) of any word on 2 strictly positive and distinct real numbers.

Let $a > 0$ be a fixed real value and $y > 0$ be a real value different from a . Let w denotes a word on $\{a, y\}$ such that the limit average

$$\lambda_a(y) := \lim_{n \rightarrow \infty} \frac{w(1) + \dots + w(n)}{n}$$

exists with $\min(a, y) \leq \lambda_a(y) \leq \max(a, y)$. Let us define the other limit average

$$\mu_a(y) := \lim_{n \rightarrow \infty} \frac{K_w(1) + \dots + K_w(n)}{n}$$

where K_w is a SKT of w with any suitable starting values $K_w(1) > 0$ and $K_w(2) > 0$ not necessarily in $\{a, y\}$. Next let us define

$$\mu_a^-(y) := \lim_{x \rightarrow y^-} \mu_a(x)$$

$$\mu_a^+(y) := \lim_{x \rightarrow y^+} \mu_a(x)$$

Then we conjecture that there are 2 dual real functions $f_{a,y}$ and $g_{a,y}$ associated to the SKT of $w_{a,y}$ yielding the following properties.

Case 1

The associated functions $f_{a,y}$ and $g_{a,y}$ have no positive real root. Then there is no attractive value for $\mu_a(x)$ as $x \rightarrow y$ and necessarily the average of the SKT converges toward the limit average of the primitive word, i.e.

$$\mu_a(y) = \lim_{x \rightarrow y} \mu_a(x) = \mu_a^+(y) = \mu_a^-(y) = \lambda_a(y)$$

Case 2

The associated functions $f_{a,y}$ and $g_{a,y}$ have each one a largest positive root $\alpha_{a,y}$ and $\beta_{a,y}$ respectively satisfying

$$\min(\alpha_{a,y}, \beta_{a,y}) \leq \lambda_a(y) \leq \max(\alpha_{a,y}, \beta_{a,y})$$

Then $\mu_a(y)$ equals $\alpha_{a,y}$ or $\beta_{a,y}$ which is an attractive value for $\mu_a(x)$ as $x \rightarrow y^-$ or $x \rightarrow y^+$ but not both. In other words a proposition is true among the 4 following ones:

1. $\mu_a(y) = \mu_a^-(y) = \alpha_{a,y} \Rightarrow \mu_a^+(y) = \beta_{a,y}$
2. $\mu_a(y) = \mu_a^-(y) = \beta_{a,y} \Rightarrow \mu_a^+(y) = \alpha_{a,y}$
3. $\mu_a(y) = \mu_a^+(y) = \alpha_{a,y} \Rightarrow \mu_a^-(y) = \beta_{a,y}$
4. $\mu_a(y) = \mu_a^+(y) = \beta_{a,y} \Rightarrow \mu_a^-(y) = \alpha_{a,y}$

Each of these 4 situations seems to occur quite randomly except for some specific words like the following ones.

Illustration of the conjecture

We consider $a = 1$ and the words $w = 1y1y1y1y1y1y\dots$ where $y > 1$ is a real value. Then we claim that we are in the case 2 only if y is an odd integer value. So that we have two possibilities.

Possibility 1 $y \notin \{2k+1 \mid k \in \mathbb{N}\}$ Then for any $y > 0$ real value but not an odd integer the associated functions $f_{a,y}$ and $g_{a,y}$ have no positive real root and we have from the conjecture (9)

$$\mu_1(y) = \lim_{x \rightarrow y} \mu_1(x) = \mu_1^+(y) = \mu_1^-(y) = \lambda_1(y) = \frac{1+y}{2}$$

Possibility 2 $y \in \{2k+1 \mid k \in \mathbb{N}\}$ Then the associated functions $f_{1,y}$ and $g_{1,y}$ are polynomials with integer coefficients of degree y with a largest real positive root $\alpha_{1,y}$ and $\beta_{1,y}$ respectively satisfying

$$\beta_{1,y} < \frac{1+y}{2} < \alpha_{1,y}$$

and we have

$$\mu_1(y) = \mu_1^+(y) = \alpha_{1,y} \Rightarrow \mu_1^-(y) = \beta_{1,y}$$

For instance if $y = 3$ we claim that $f_1(y) = x^3 - 2x^2 - 1$ and $g_1(y) = x^3 - 2x^2 + 1$ so that $\alpha_{1,3} = 2.205\dots$ and $\beta_{1,3} = \Phi$.

We fill thereafter a table with 20000 computed terms for the limits.

y	$\mu_1(y)$	$\mu_1^+(y)$	$\mu_1^-(y)$	$\frac{1+y}{2}$
2.7	1.849	1.849	1.850	1.850
2.8	1.899	1.898	1.899	1.900
2.9	1.948	1.947	1.948	1.950
3.0	2.205	2.205	1.618	2.000
3.1	2.049	2.048	2.049	2.050
3.2	2.099	2.102	2.099	2.100
3.3	2.155	2.153	2.155	2.150

Apart $y = 3$ we have roughly the same limit in each line. The case $y = 2$ yields the truth of the Keane's conjecture. This case provide examples of words falling mostly in case 1 of the conjecture (9). However there are families of words which seem to fit mostly the case 2 as shown thereafter.

5 Conjecture (10)

We can bring this study in the realm of dynamical systems. Indeed there is experimental evidence showing the limits defined in the conjecture (9) are somewhat sensitive to the value of a and y . However we think it is not a chaotic dynamical system since the Kolakaski map we will introduce below doesn't seem to diverge. This way we were able to exhibit a family of words which satisfy always the case 1 of the conjecture (9). Since this family encapsulates w_0 it could be interesting to explore this idea further. To see this let us consider words w on the alphabet $\{y, y+1\}$ defined by $w = y(y+1)y(y+1)y(y+1)\dots$. SKT of K_w with suitable starting values (the Kolakoski sequence is obtained for $y = 1$). Then we define similarly as above:

$$\mu(y) := \lim_{n \rightarrow \infty} \frac{K_w(1) + \dots + K_w(n)}{n}$$

and

$$\mu^-(y) := \lim_{x \rightarrow y^-} \mu(x)$$

$$\mu^+(y) := \lim_{x \rightarrow y^+} \mu(x)$$

We are looking here for properties of the so called Kolakoski map

$$y \rightarrow (\mu(y), \mu^+(y), \mu^-(y))$$

where $y > \frac{1}{2}$ since the map seems always well defined .

Conjecture (10)

We claim that for any $y > \frac{1}{2}$ the image of y under this map exists and y is in one of the 3 sets F_1, F_2, F_3 defined as follows:

$$F_1 = \left\{ y > \frac{1}{2} \mid \mu(y) = \mu^+(y) = \mu^-(y) = y + \frac{1}{2} \right\}$$

$$F_2 = \left\{ y > \frac{1}{2} \mid \mu(y) = \mu^-(y) < \mu^+(y) \right\}$$

$$F_3 = \left\{ y > \frac{1}{2} \mid \mu(y) = \mu^+(y) > \mu^-(y) \right\}$$

Here a table showing this fact using colors: red for $y \in F_1$, blue for F_2 and green for F_3 .

y	$\mu(y)$	$\mu^+(y)$	$\mu^-(y)$	$y + 0.5$
0.50	1.000	1.000	undefined	1.000
0.55	1.000	1.072	1.000	1.050
0.60	1.099	1.099	1.099	1.100
0.65	1.126	1.170	1.126	1.150
0.70	1.155	1.239	1.155	1.200
0.75	1.342	1.342	1.140	1.250
0.80	1.299	1.299	1.299	1.300
0.85	1.330	1.369	1.330	1.350
0.90	1.363	1.436	1.363	1.400
0.95	1.433	1.468	1.433	1.450
1.00	1.500	1.499	1.499	1.500
1.05	1.534	1.566	1.5345	1.550
1.10	1.568	1.631	1.568	1.600
1.15	1.634	1.664	1.634	1.650
1.20	1.700	1.700	1.700	1.700

So we see the set F_1 contains isolated values and the set F_3 seems almost empty. What is striking is that the red lines appear with a regular frequency. So we conjecture that for $k \geq 1$ integer value we have

$$y = \frac{3+k}{5} \in F_1$$

In particular for $k = 2$ the word $y(y+1)y(y+1)\dots = 12121212\dots = w_0$. Next the above claim means that we have $\mu(1) = \frac{3}{2}$. Thus the Keane's conjecture would be true again.

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APPENDIX 1

Experimental support of the conjecture (2)

We provide experiments supporting the conjecture (2) for words w on $\{1, 2\}$

$$\lim_{n \rightarrow \infty} \frac{w(1) + \dots + w(n)}{n} = \lambda$$

For computational purposes we compute our random weight $u_{\alpha, \delta} = (u_{\alpha, \delta})_{n \geq 1}$ as follows:

- $u_{\alpha, \delta}(n) = \delta(n)$ if $\{n\sqrt{2}\} < n^{-\alpha}$ and $u_{\alpha, \delta}(n) = 1$ otherwise

where $\{\}$ is the fractional part function. Indeed the sequence $\{nx\}$ (where x is irrational) is a low discrepancy sequence and produces a quasi random sequence. For instance suppose u is a sequence such that the n -th term is different from 1 with probability $P(n)$ and equals $\delta(n)$ otherwise. Then the sequence v defined by

- $v(n) = \delta(n)$ if $\{nx\} < P(n)$ and $v(n) = 1$ otherwise

is a quasi random sequence which mimics well u for $0 < \alpha \leq 1$.

Conventions

We choose $\delta(n) = 1 + \frac{1}{n}$ and start with $K_{w, u_{\alpha, \delta}}(1) = w(1)$ and $K_{w, u_{\alpha, \delta}}(2) = w(2)$. Then we compute $K_{w, u_{\alpha, \delta}}$ the WKT of w of weight $u_{\alpha, \delta}$ for various word w and $\alpha = \frac{1}{4}$, $\alpha = \frac{1}{2}$ and $\alpha = \frac{3}{4}$.

Presentation of results

For each word w we plot on the same graphic the function of n

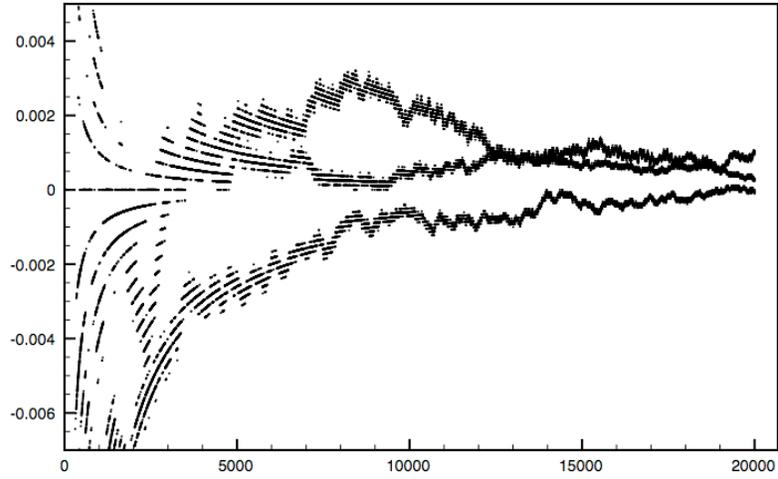
$$\frac{K_{w, u_{\alpha, \delta}}(1) + \dots + K_{w, u_{\alpha, \delta}}(n)}{n} - \frac{w(1) + \dots + w(n)}{n}$$

for $\alpha = 0.25$ $\alpha = 0.5$ $\alpha = 0.75$ and for $1 \leq n \leq 20000$.

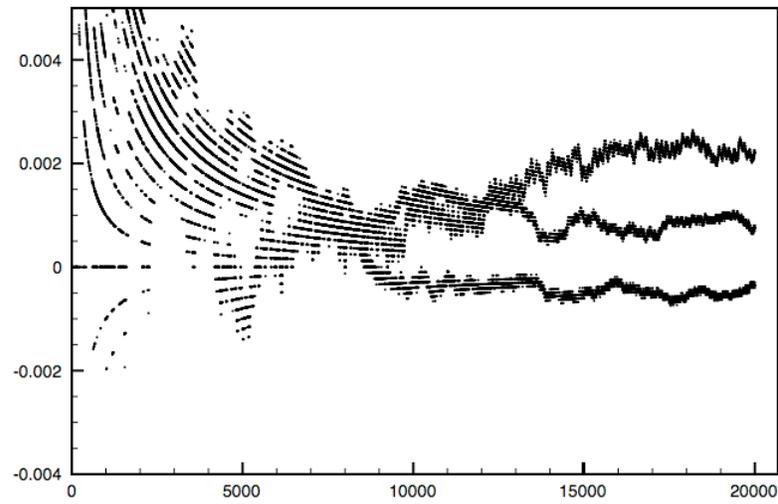
Comment

Although the convergence to zero is chaotic it seems likely that each case satisfies the conjecture (2).

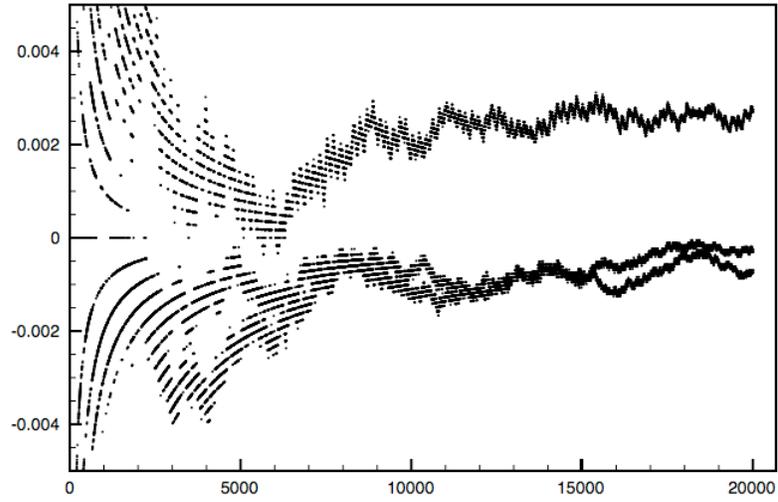
The word w_0



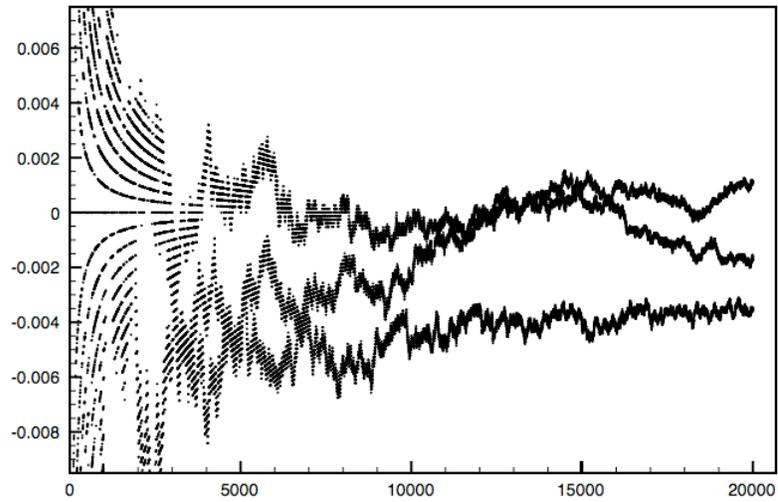
The word $w(n) = 2$ if $n \equiv 0 \pmod{3}$ [3]



The word $w(n) = 2$ if $n \equiv 0 \pmod{4}$



The word $w(n) = 2$ if n is squarefree



APPENDIX 2

Experimental support of the conjecture (3)

We provide experiments supporting the conjecture (3) for the word $w_1 = 13131313\dots$ for which

$$\lim_{n \rightarrow \infty} \frac{w_1(1) + \dots + w_1(n)}{n} = 2$$

For computational purposes we compute our random weight $u_{\alpha, \delta} = (u_{\alpha, \delta})_{n \geq 1}$ like in the APPENDIX 1.

- $u_{\alpha, \delta}(n) = \delta(n)$ if $\{n\sqrt{2}\} < n^{-\alpha}$ and $u_{\alpha, \delta}(n) = 1$ otherwise

Conventions

We choose again $\delta(n) = 1 + \frac{1}{n}$ and start with $K_{w_1, u_{\alpha, \delta}}(1) = 1$ and $K_{w_1, u_{\alpha, \delta}}(2) = 3$. Then we compute $K_{w_1, u_{\alpha, \delta}}$ the WKT of w_1 of weight $u_{\alpha, \delta}$ for $\alpha = \frac{1}{4}$, $\alpha = \frac{1}{2}$ and $\alpha = \frac{3}{4}$.

Presentation of results

We plot on the same graphic the functions of n

$$\frac{K_{w_1, u_{\alpha, \delta}}(1) + \dots + K_{w_1, u_{\alpha, \delta}}(n)}{n} - 2$$

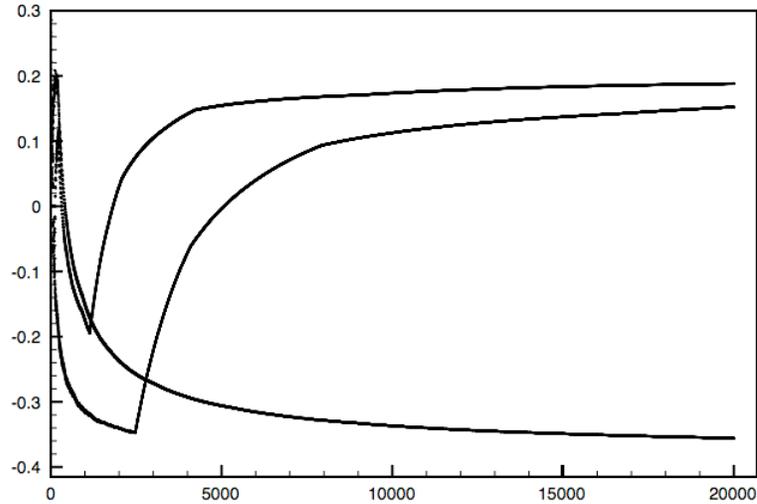
for $\alpha = 0.25$ $\alpha = 0.5$ $\alpha = 0.75$ and $1 \leq n \leq 20000$.

Comment

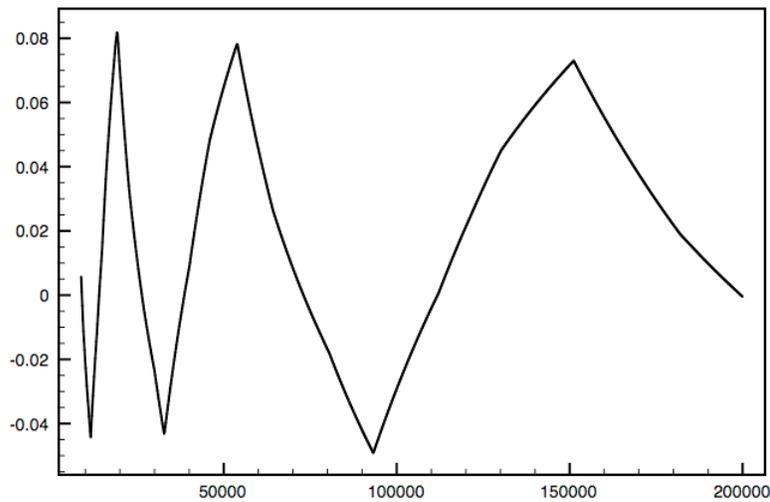
None of them seems to converge to zero supporting the conjecture (3). For $\alpha = 0.25$ and $\alpha = 0.5$ the graphs seem to converge smoothly toward a value ~ 0.2 . For $\alpha = 0.75$ the graph seems to converge smoothly toward another value ~ -0.4 . This is a completely different situation than with the previous random WKT of words w on the alphabet $\{1, 2\}$.

We add at the end an extra example showing the randomWKT transform of w_1 could even diverge.

Plot of $\frac{K_{w_1, u_{\alpha, \delta}}(1) + \dots + K_{w_1, u_{\alpha, \delta}}(n)}{n} - 2$



It can be even worse if you take $\delta(n) = 1 + \frac{1}{n^\alpha}$ instead of $1 + \frac{1}{n}$. For instance we plot thereafter $\frac{K_{w_1, u_{\alpha, \delta}}(1) + \dots + K_{w_1, u_{\alpha, \delta}}(n)}{n} - 2$ for $\alpha = \frac{1}{2}$ and for $n = 1$ up to 200000.



It seems the graph oscillates around zero and doesn't converge to zero. This phenomenon should be interesting to understand in order to prove the conjecture (3) and to handle the conjecture (2).

APPENDIX 3

Experimental support of the conjecture (5)

Let $w_{1,x}$ be the sequence $w_{1,x}(n) = 1$ if n is odd and $w_{1,x}(n) = 3 + x$ if n is even.

Presentation of results

Then we plot for each $\varepsilon \in \{0.1, 0.01, 0.001\}$ on the same figure

$$\frac{K_{w_{1,\varepsilon}}(1) + \dots + K_{w_{1,\varepsilon}}(n)}{n}$$

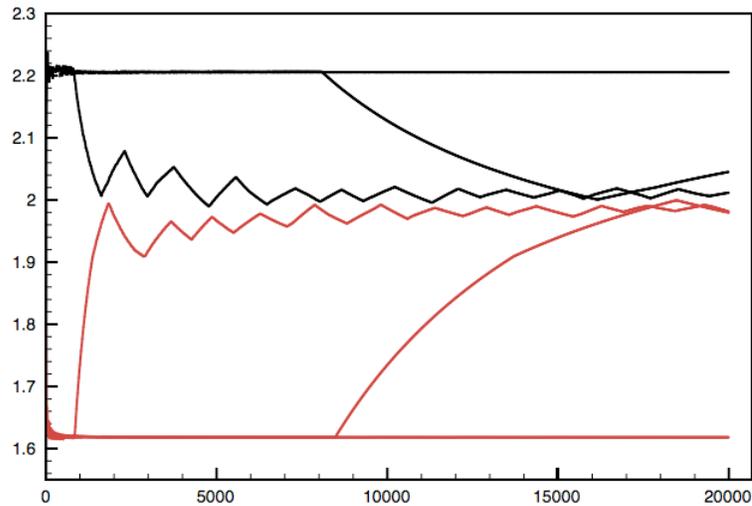
with color black and

$$\frac{K_{w_{1,-\varepsilon}}(1) + \dots + K_{w_{1,-\varepsilon}}(n)}{n}$$

with color red.

Comment

One can see the black graphics are closer longer to λ_1 as $\varepsilon \rightarrow 0$ and the red ones are closer longer to Φ until a value of $n = N(\varepsilon)$ which is growing to ∞ . Then black and red graphics are adjacent around 2. This supports the conjecture (5).



APPENDIX 4

Experimental support of the conjecture (6)

Let $w_{0,x}$ be the sequence $w_{0,x}(n) = 1$ if n is odd and $w_{0,x}(n) = 3 + x$ if n is even.

Presentation of results

For $\varepsilon = 0.001$ and $\varepsilon = 0.0001$ we plot for each ε on the same graphic

$$\frac{K_{w_{0,\varepsilon}}(1) + \dots + K_{w_{0,\varepsilon}}(n)}{n}$$

and

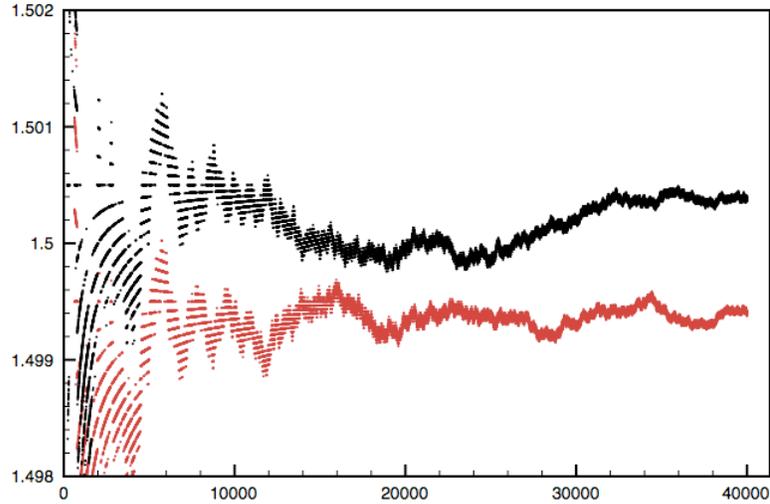
$$\frac{K_{w_{0,-\varepsilon}}(1) + \dots + K_{w_{0,-\varepsilon}}(n)}{n}$$

with colors black and red respectively.

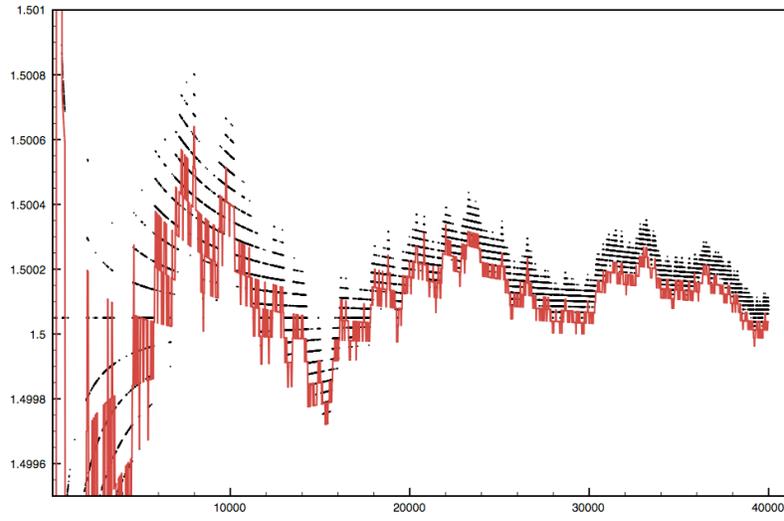
Comment

One can see the black and red graphics are closer to $\frac{3}{2}$ as $\varepsilon \rightarrow 0$. The distance to $3/2$ seems also to behave like $O(\varepsilon)$ as $n \rightarrow \infty$ supporting somewhat the whole conjecture (6).

$\varepsilon = 0.001$



$$\varepsilon = 0.0001$$



APPENDIX 5

Experimental support of the conjecture (9)

Let $w_{0,x}$ be the sequence $w_{0,x}(n) = 1 + x$ if n is odd and $w_{0,x}(n) = 2(1 + x)$ if n is even.

Presentation of results

For $\varepsilon = 0.1, 0.01, 0.001, 0.0001$ we plot for each ε on the same graphic

$$\frac{K_{w_{0,\varepsilon}}(1) + \dots + K_{w_{0,\varepsilon}}(n)}{n}$$

and

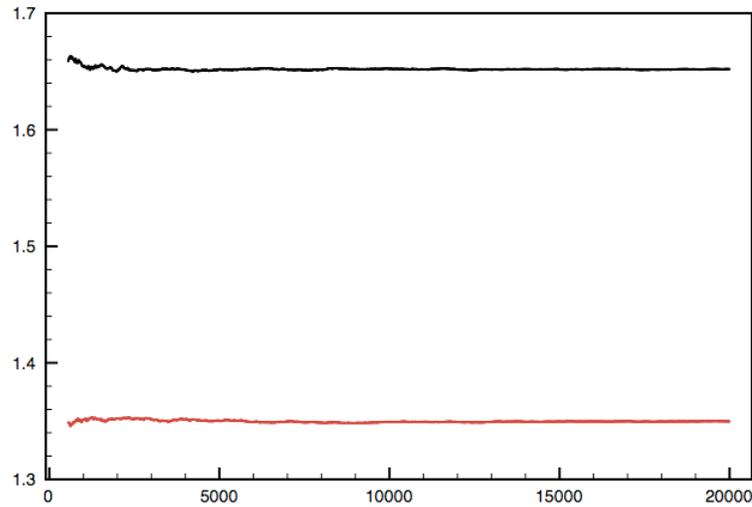
$$\frac{K_{w_{0,-\varepsilon}}(1) + \dots + K_{w_{0,-\varepsilon}}(n)}{n}$$

with colors black and red respectively.

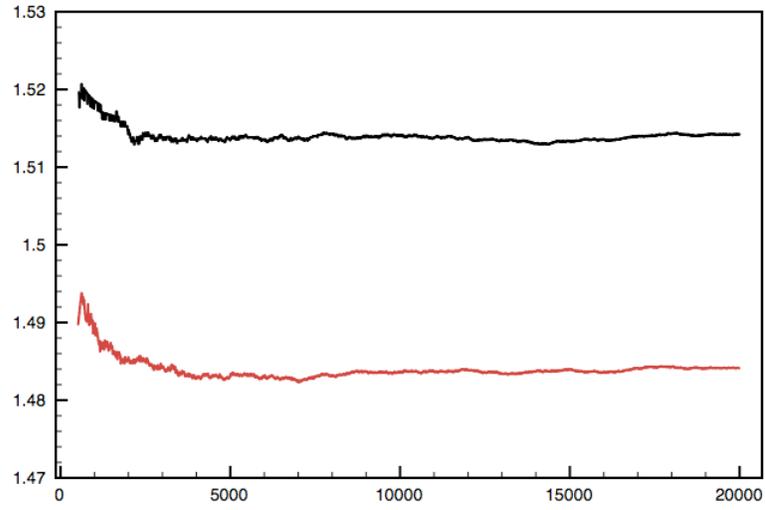
Comment

The fact these are 2 adjacent averages seeming to converge toward $\frac{3}{2}(1 + \varepsilon)$ and $\frac{3}{2}(1 - \varepsilon)$ respectively supports the conjecture (9). However the chaotic aspect of the graphs when $\varepsilon \rightarrow 0$ can't tell much.

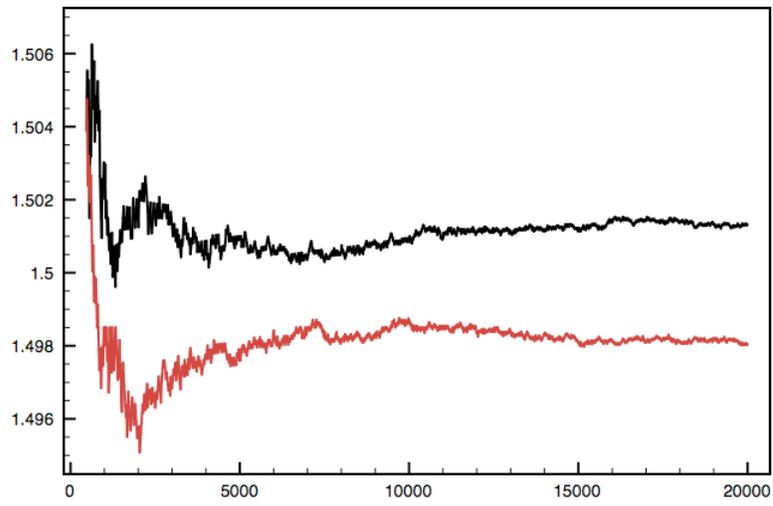
$$\varepsilon = 0.1$$



$\varepsilon = 0.01$



$\varepsilon = 0.001$



$\varepsilon = 0.0001$

