Notes on A368737

Peter Bala, Jan 2024

Pillai's gcd sum function $\mathrm{P}(n)$ (A018804) is defined as

$$\mathrm{P}(n) = \sum_{k=1}^{n} \gcd(k, n). \tag{1}$$

The identity

$$\mathrm{P}(n) = \sum_{d|n} \mathrm{phi}(d) \frac{n}{d}, \tag{2}$$

where $\mathrm{phi}(n)$ denotes Euler's totient function (A000010), expresses $\mathrm{P}(n)$ as the Dirichlet convolution of two multiplicative functions. Hence $\mathrm{P}(n)$ is a multiplicative function.

We define $\mathrm{P}_3(n)$ by

$$\mathrm{P}_3(n) = \sum_{k=1}^{n} \gcd(3k, n). \tag{3}$$

This is A368737 in the OEIS. We shall show that $\mathrm{P}_3(n)$ is also a multiplicative function. We require the following elementary properties of the gcd function:

1) if m is a positive integer, then $\gcd(ma, mb) = m \gcd(a, b)$.

2) if $\gcd(a, c) = 1$ then $\gcd(ab, c) = \gcd(b, c)$.

3) the identity $\gcd(k + n, n) = \gcd(k, n)$ shows that the gcd function, qua function of k, is periodic with period n.

4) the gcd function is a multiplicative function in the following sense: if b and c are coprime, then $\gcd(a, bc) = \gcd(a, b)\gcd(a, c)$.

Proposition 1.

(i)

$$\mathrm{P}_3(3n + 1) = \mathrm{P}(3n + 1) \tag{4}$$

(ii)

$$\mathrm{P}_3(3n + 2) = \mathrm{P}(3n + 2) \tag{5}$$

(iii)

$$\mathrm{P}_3(3n) = 9\mathrm{P}(n) \tag{6}$$

Proof.

(i) From (3)

$$\begin{aligned}
\mathrm{P}_3(3n+1) &= \sum_{k=1}^{3n+1} \gcd(3k, 3n+1) \\
&= \sum_{k=1}^{3n+1} \gcd(k, 3n+1) \\
&= \mathrm{P}(3n+1).
\end{aligned}$$

(ii) Similarly, one can show that $\mathrm{P}_3(3n+2) = \mathrm{P}(3n+2)$.

(iii) From (3)

$$\begin{aligned}
\mathrm{P}_3(3n) &= \sum_{k=1}^{3n} \gcd(3k, 3n) \\
&= 3\sum_{k=1}^{3n} \gcd(k, n) \\
&= 9\sum_{k=1}^{n} \gcd(k, n) \\
&= 9\mathrm{P}(n),
\end{aligned}$$

where, in the penultimate step, we made use of the periodicity of the gcd function as a function of k. ■

Proposition 2.

$$\mathrm{P}_3(n) = \sum_{d|n} \gcd(3, d)\mathrm{phi}(d)\frac{n}{d}\,. \tag{7}$$

Proof.

If $n \equiv 1 (\mathrm{mod}\ 3)$ then (7) follows from (2) and (4). If $n \equiv 2 (\mathrm{mod}\ 3)$ then (7) follows from (2) and (5). Suppose now n is a multiple of 3, say $n = 3N$. We need to prove that

$$\mathrm{P}_3(3N) = \sum_{d|3N} \gcd(3, d)\mathrm{phi}(d)\frac{3N}{d}\,.$$

By (6),

$$\begin{aligned} \mathrm{P}_3(3N) &= 9\mathrm{P}(N) \\ &= 9\sum_{d|N} \mathrm{phi}(d)\frac{N}{d}, \end{aligned}$$

by (2). Therefore the proposition will be established if we can show that following identity holds for all positive integer N:

$$\sum_{d|3N} \gcd(3,d)\frac{\mathrm{phi}(d)}{d} = 3\sum_{d|N} \frac{\mathrm{phi}(d)}{d}. \tag{8}$$

We separately evaluate the left and right sides of (8) and show that they are equal. Firstly, we evaluate the right side of (8).

Let $N = 3^k N'$, where k is the highest power of 3 dividing N, so that $\gcd(3, N') = 1$. The divisors of N are of the form $e, 3e, 3^2e, ..., 3^k e$ where e runs through the divisors of N'. Note that all the divisors e of N' are coprime to 3.

Hence

$$\begin{aligned} 3\sum_{d|N} \frac{\mathrm{phi}(d)}{d} &= 3\left(\sum_{e|N'} \frac{\mathrm{phi}(e)}{e} + \sum_{e|N'} \frac{\mathrm{phi}(3e)}{3e} + \cdots + \sum_{e|N'} \frac{\mathrm{phi}\left(3^k e\right)}{3^k e}\right) \\ &= \sum_{e|N'} \frac{\mathrm{phi}(e)}{e}\left(3 + 3\frac{\mathrm{phi}(3)}{3} + \cdots + 3\frac{\mathrm{phi}\left(3^k\right)}{3^k}\right) \\ &= \sum_{e|N'} \frac{\mathrm{phi}(e)}{e}\left(3 + 3\frac{(3-1)}{3} + \cdots + 3\frac{\left(3^k - 3^{k-1}\right)}{3^k}\right) \\ &= (2k+3)\sum_{e|N'} \frac{\mathrm{phi}(e)}{e}, \end{aligned}$$

where we made use of the multiplicativity of Euler's phi function and the values $\mathrm{phi}\left(3^j\right) = 3^j - 3^{j-1}$.

We now evaluate the left side of (8). The divisors of $3N$ are of the form $e, 3e, 3^2e, ..., 3^{k+1}e$ where e runs through the divisors of N'.

Hence

$$\begin{aligned}
\sum_{d|3N} \gcd(3,d)\frac{\mathrm{phi}(d)}{d} &= \left(\sum_{e|N'} \frac{\mathrm{phi}(e)}{e} + 3\sum_{e|N'} \frac{\mathrm{phi}(3e)}{3e} + \cdots + 3\sum_{e|N'} \frac{\mathrm{phi}\left(3^{k+1}e\right)}{3^{k+1}e}\right) \\
&= \sum_{e|N'} \frac{\mathrm{phi}(e)}{e}\left(1 + 3\frac{\mathrm{phi}(3)}{3} + \cdots + 3\frac{\mathrm{phi}\left(3^{k+1}\right)}{3^{k+1}}\right) \\
&= \sum_{e|N'} \frac{\mathrm{phi}(e)}{e}\left(1 + 3\frac{(3-1)}{3} + \cdots + 3\frac{\left(3^{k+1}-3^{k}\right)}{3^{k+1}}\right) \\
&= (2k+3)\sum_{e|N'} \frac{\mathrm{phi}(e)}{e},
\end{aligned}$$

as before for the rhs of (8). Thus (8) is established. This completes the proof of the Proposition. ■

Corollary. The arithmetical function $\mathrm{P}_3(n)$ is multiplicative since it is the Dirichlet convolution of a pair of multiplicative functions.