Some conjectural supercongruences for the Apéry numbers

Peter Bala, Oct 27 2022

Let A(n) = A005259(n) and B(n) = A005258(n) denote the two types of Apéry numbers.

The sequences $\{\mathrm{A}(n)\}$ and $\{\mathrm{B}(n)\}$ together with the two sequences $\{\mathrm{A}(n-1)\}$ and $\{\mathrm{B}(n-1)\}$ of shifted Apéry numbers satisfy the supercongruences

$$u(p^r) \equiv u(p^{r-1}) \pmod{p^{3r}}$$

valid for all primes $p >= 5$ and all positive integers r [see, for example, Straub]. The purpose of this note is to list several sequences formed from the Apéry numbers and the shifted Apéry numbers that empirically appear to satisfy the following stronger supercongruences

$$\text{(S)} \quad \begin{array}{c} u(p) \equiv u(1) \pmod{p^5} \text{ for all primes } p >= 5 \\ u(p^r) \equiv u(p^{r-1}) \pmod{p^{3r+3}} \text{ for } r >= 2 \text{ and all primes } p >= 5. \end{array}$$

Note that to every linear combination $\{a\mathrm{U}(n) + b\mathrm{V}(n)\}$ in the table below that conjecturally satisfies (S) there is associated a companion product sequence $\left\{\mathrm{U}(n)^{a\mathrm{U}(1)}\mathrm{V}(n)^{b\mathrm{V}(1)}\right\}$ that also appears to satisfy (S).

Table of sequences $\{u(n)\}$ conjecturally satisfying (S)

	$u(n)$	OEIS reference
C1(a)	$\mathrm{A}(n) + 7\mathrm{A}(n-1)$	A212334
C1(b)	$\mathrm{A}(n)^5\mathrm{A}(n-1)^7$	A357507
C2(a)	$\mathrm{B}(n) + \mathrm{B}(n-1)$	A352655
C2(b)	$\mathrm{B}(n)^3\mathrm{B}(n-1)$	A357506
C3(a)	$5\mathrm{A}(n) - 14\mathrm{B}(n)$	A357567
C3(b)	$\dfrac{\mathrm{A}(n)^{25}}{\mathrm{B}(n)^{42}}$	not listed

	$u(n)$	OEIS reference
C4(a)	$5\mathrm{A}(n-1)-2\mathrm{B}(n-1)$	A357956
C4(b)	$\mathrm{A}(n-1)^5-\mathrm{B}(n-1)^2$	A357957
C5(a)	$5\mathrm{A}(n)+14\mathrm{B}(n-1)$	A357958
C5(b)	$\mathrm{A}(n)^{25}\mathrm{B}(n-1)^{14}$	not listed
C6(a)	$5\mathrm{A}(n-1)+2\mathrm{B}(n)$	A357959
C6(b)	$\mathrm{A}(n-1)^5B(n)^6$	A357960

References

[1] Armin Straub, Multivariate Apéry numbers and supercongruences of rational functions Algebra & Number Theory, Vol. 8, No. 8 (2014), pp. 1985-2008; arXiv:1401.0854 [math.NT], 2014.