Number of Stable Roommate's instances

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August 2022

Abstract

While studying *the stable roommates problem*, it appears that some instances only differ by a permutation, and it can affects some counting problems, for example the counting of the unstable instances. We show here a formula giving the number of non-isomorphic instances of *the stable roommates problem* of cardinality n. We also show that equivalence classes' cardinality under the permutations' action almost reach their maximum

1 Introduction

1.1 Definitions

Definition 1. *We will represent here an instance of the problem as a set of n n-uples, in which the first value is the agent's index, and the other values is the preference list itself.*

Example 1. *The following set :*
$\{(1,2,3,4),$
$(2,3,1,4),$
$(3,1,2,4),$
$(4,1,2,3)\}$
represents the instance of cardinality 4 *in which the agent* 1 *ranks* 2 *in first position, then* 3 *and finally* 4 *etc.*

Definition 2. *Let $I_n(SR)$ be the set of every instances of* the roommates problem *of cardinality n.*

Definition 3. *Given $\{(a_{i,k})_{k\in[\![1,n]\!]} | i \in [\![1,n]\!]\} = I \in I_n(SR)$ and $i \in [\![1,n]\!]$ a fixed integer, let $\sigma_i \in \mathfrak{S}_n$ be the only permutation such that $(\sigma_i(1), \sigma_i(2), ..., \sigma_i(n)) = (a_{i,k})$*

1.2 Motivations

At first sight, the number of instances of the problem of cardinality n appears to be $(n-1)!^n$ since an instance can be described as an $n \times n$ matrix with its first column fixed, and with each integer between 1 and n appearing once in each line. However, some distinct instances (with this counting method) only differ by a permutation.

Example 2. *The following instances*
$\{(1,2,3,4),$ $\{(2,1,3,4),$
$(2,3,1,4),$ $(1,3,2,4),$
$(3,1,2,4),$ $(3,1,2,4),$
$(4,1,2,3)\}$ $(4,1,2,3)\}$
only differ by the permutation $(1\ 2)$

We observe that these kind of instances have the same properties in a way that will be explained in 2.2.

Moreover, aside from the fact that exhaustive studies can be made more quickly, for orbits under a group action are of different cardinalities, it can be more relevant to count the number of orbits fulfilling a given condition, rather than counting the number of instances fulfilling that condition.

Example 3. • *The instances that do not admit a stable matching are, permutations excepted,* $\{(1, 2, 3, 4), (2, 3, 1, 4), (3, 1, 2, 4), (4, arbitrary)\}$

- *There are* 48 *instances that do not admit a stable matching, so* $\frac{1}{27}$ *of every instances.*
- *There are* 60 *instances non-isomorphic 2 by 2 and two of those do not admit a stable matching, so* $\frac{1}{30}$ *of every non-isomorphic instances do not admit a stable matching.*

Proof. • Given $I \in I_4(SR)$ an instance that does not admit a stable matching. The matching $\{\{1, 2\}, \{3, 4\}\}$ is not stable, without loss of generality we assume 1 and 3 form a blocking pair :
$(1, ..., 3, ..., 2, ...) \in I$
$(3, ..., 1, ..., 4, ...) \in I$
But $\{\{a, c\}, \{b, d\}\}$ is not stable either. We assume without loss of generality that 1 is part of a blocking pair, hence 2 is part of the blocking pair too.
$(1, 4, 3, 2) \in I$
$(4, ..., 1, ..., 2, ...) \in I$
Plus $\{\{1, 4\}, \{2, 3\}\}$ is not a stable matching. 1 can't be part of a blocking pair since 4 is ranked first by 1, hence 4 is part of a blocking pair.
$(4, 3, 1, 2) \in I$
$(3, 1, 4, 2) \in I$

Finally, $I =$
$\{(1, 4, 3, 2)$
$(2, \text{arbitrary})$
$(3, 1, 4, 2)$
$(4, 3, 1, 2)\}$

Which is differing from the expected instance only by the permutation (2 4).

- The agent that will be rejected by the other 3 : 4 possibilities. The cycle direction of the other 3 : $2! = 2$ possibilities. The preference list of the rejected one : $3! = 6$ possibilities. All in all $4 \cdot 2 \cdot 6 = 48$.
- We can show this result with tools developed in the following parts, the proof is available in appendix A.

□

2 Group action

2.1 Definition of instances' permutations

We create a group action of $\mathfrak{S}_n$ on $I_n(SR)$, we define the following operation :

$$\mathfrak{S}_n \times I_n(SR) \longrightarrow I_n(SR)$$

$$\tau * \{(a_{i,1}, a_{i,2}, ..., a_{i,n})|i \in [\![1, n]\!]\} \longmapsto \{(\tau(a_{i,1}), \tau(a_{i,2}), ..., \tau(a_{i,n}))|i \in [\![1, n]\!]\}$$

We check the properties of a group action :

1. $id * \{(a_{i,1}, a_{i,2}, ..., a_{i,n})|i \in [\![1, n]\!]\} = \{(a_{i,1}, a_{i,2}, ..., a_{i,n})|i \in [\![1, n]\!]\}$
2. $(\tau \circ \tau') * \{\sigma_i|i \in [\![1, n]\!]\} = \{(\tau \circ \tau')(a_{i,1}), ..., (\tau \circ \tau')(a_{i,n}))|i \in [\![1, n]\!]\}$
$= \{\tau(\tau'(a_{i,1})), ..., \tau(\tau'(a_{i,1}))|i \in [\![1, n]\!]\} = \tau * \{\tau'(a_{i,1}), ..., \tau'(a_{i,1}))|i \in [\![1, n]\!]\}$

The defined operation is a group action. We want to count the number of distincts orbits under this group action, i.e. the cardinality of the quotient set $I_n(SR)/\mathfrak{S}_n$.

Definition 4. *Given* $\tau \in \mathfrak{S}_n, I \in I_n(SR)$ *and* M *a matching on* I.
We define $\tau * M = \{\{\tau(a), \tau(b)\}, \{a, b\} \in M\}$

2.2 Effect of a permutation on an instance

Theorem 1. *Given $I \in I_n(SR)$, $\tau \in \mathfrak{S}_n$ and M a matching on I.*
$\{a,b\}$ is a blocking pair on M if and only if $\{\tau(a),\tau(b)\}$ is a blocking pair on $\tau * M$.

Proof. We show only one implication, the other one can be obtained with the exact same argument with the permutation τ^{-1}.
Given I, τ, M, a and b as described above.
Let $\{a,b\}$ be a blocking pair on M, let $M(a), M(b)$ be the agents matched with a and b in M respectively.
In the instance I, the preference lists are :
$(a, ..., b, ..., M(a), ...) \in I$
$(b, ..., a, ..., M(b), ...) \in I$
Hence in the instance $\tau * I$:
$(\tau(a), ..., \tau(b), ..., \tau(M(a)), ...) \in \tau * I$
$(\tau(b), ..., \tau(a), ..., \tau(M(b)), ...) \in \tau * I$
However, by definition, $\tau(M(a)) = M(\tau(a))$ and $\tau(M(b)) = M(\tau(b))$, thus $\{\tau(a),\tau(b)\}$ is a blocking pair in $\tau * M$. □

Corollary 1. *Given $I \in I_n(SR)$, $\forall J \in \omega(I)$, I and J have the same properties of stability i.e. I admits a stable matching if and only if J admits a stable matching, more generally, J admits a minimum of k blockings pairs if and only if I admits a minimum of k blocking pairs.*

2.3 Properties of orbits and fixed points

Theorem 2. *Given $A = \{I \in I_n(SR), \sigma_1 = id\}, B = \{\omega(I), I \in A\}$ covers $I_n(SR)$*

Proof. It is known that the set of every orbits is a partition of $I_n(SR)$. We show that B contains every orbits. Given $I \in I_n(SR)$, $\tilde{I} = \sigma_1^{-1} * I \in \omega(I)$, $I \in \omega(\tilde{I})$ and by definition, the preference list of the agent 1 in $\tilde{I}$ is the uple $(1, 2, ..., n)$. □

Fixing the first line in addition to the first column, the number of such instances is $(n-1)!^{n-1}$ so $|A| = (n-1)!^{n-1} \geq |I_n(SR)/\mathfrak{S}_n|$

Lemma 1. *Given an instance $I \in I_n(SR)$: $\{id\} \subset Stab(I) \subset \{\sigma_i \circ \sigma_1^{-1}, i \in [\![1,n]\!]\}$.*

Proof. The proof of $id \in Stab(I)$ is straightforward. Given $\tau \in Stab(I)$, $\exists k \in [\![1,n]\!], \tau \circ \sigma_1 = \sigma_k \iff \tau = \sigma_k \circ \sigma_1^{-1}$. □

Lemma 2. $\frac{(n-1)!^{n-1}}{n} \leq |I_n(SR)/\mathfrak{S}_n| \leq (n-1)!^{n-1}$

Proof. According to lemma 2, $|\{id\}| \leqslant |Stab(I)| \leqslant |\{\sigma_i \circ \sigma_1^{-1}, i \in [\![1,n]\!]\}|$
$\Leftrightarrow 1 \leqslant |Stab(I)| \leqslant n \Leftrightarrow 1 \geq \frac{1}{|Stab(I)|} \geq \frac{1}{n} \Leftrightarrow \frac{|\mathfrak{S}_n|}{1} \geq \frac{|\mathfrak{S}_n|}{|Stab(I)|} \geq \frac{|\mathfrak{S}_n|}{n} \Leftrightarrow n! \geq |\omega(I)| \geq (n-1)!$
Thus :
$\frac{|I_n(SR)|}{n!} \leq |I_n(SR)/\mathfrak{S}_n| \leq \frac{|I_n(SR)|}{(n-1)!} \iff \frac{(n-1)!^{n-1}}{n} \leq |I_n(SR)/\mathfrak{S}_n| \leq (n-1)!^{n-1}$ □

2.4 Counting elements of $Fix(\tau)$

Given $n \in \mathbb{N}$, $\tau \in \mathfrak{S}_n$. The purpose of this part is counting the elements of the sets $Fix(\tau) = \{I \in I_n(SR), \tau * I = I\}$.

Theorem 3. *Given $\tau \in \mathfrak{S}_n$ let $(a_1, a_2, ..., a_k)$ be its cyclic type.*

- *If $a_1 = a_2 = ... = a_k$, $|Fix(\tau)| = (n-1)!^k = (n-1)!^{\frac{n}{a_1}}$*
- *Else, $|Fix(\tau)| = 0$*

Proof. • If $|Fix(\tau)| > 0$, $I \in I_n(SR), \tau * I = I$ exists. Plus, if there is $i, j \in [\![1, n]\!], a_i \nmid a_j$, let $A_1, A_2, ..., A_k$ be the associated set to the cycles $C_1, C_2, ..., C_k$ (respectively of length $a_1, a_2, ..., a_k$).
$\tau^{a_j} = ... \circ C_j^{a_j} \circ C_i^{a_j} = ... \circ id \circ C_i^{a_j}$ with $C_i^{a_j} \neq id$ however, let $A_i = \{x_{i,1}, x_{i,2}, ..., x_{i,a_i}\}$ and $A_j = \{x_{j,1}, x_{j,2}, ..., x_{j,a_j}\}$ such that $\tau^k(x_{j,1}) = x_{j,k+1}$. Hence :
$x_{j,1} = \tau(x_{j,a_j}) = \tau^{a_j}(x_{j,1}) = x_{j,1}$, so τ^{a_j} has to be equal to id, otherwise the image of the x_{j,a_j}'s preference list can't be $x_{j,1}$'s preference list. However, $\tau^{a_j} \neq id$ since $C_i^{a_j} \neq id$

- If each cycle of τ are indeed of same length, an instance I of $Fix(\tau)$ can be explicited :
Let a be the common length of the cycles.
Let us choose one index in each cycle whose agent's preference lists will be explicited arbitrarily. The other preference lists are the iterated image of those preference lists under the action of τ.
$I \in Fix(\tau)$ is straightforward to check :
$\forall x \in I, \exists i \in [\![0, a]\!] x = \tau^i * x_0$ where x_0 is a line initially fixed. Thus, $\tau * x = \tau^{i+1 moda} * x_0 \in I$.

□

Example 4. *For $n = 6$, $\tau = (1\ 2\ 3) \circ (4\ 5\ 6)$. By fixing the preference lists of the agents 1 and 4 arbitrarily :*
$(1, 2, 3, 4, 5, 6), (4, 2, 6, 1, 5, 3) \in I \tau \circ (1, 2, 3, 4, 5, 6) = (2, 3, 1, 5, 6, 4) \in I$
$\tau^2 \circ (1, 2, 3, 4, 5, 6) = (3, 1, 2, 6, 4, 5) \in I$
$\tau \circ (4, 2, 6, 1, 5, 3) = (5, 3, 4, 2, 6, 1) \in I$
$\tau^2 \circ (4, 2, 6, 1, 5, 3) = (6, 1, 5, 3, 4, 2) \in I$
And the instance
$I = \{(1, 2, 3, 4, 5, 6), (2, 3, 1, 5, 6, 4), (3, 1, 2, 6, 4, 5), (4, 2, 6, 1, 5, 3), (5, 3, 4, 2, 6, 1), (6, 1, 5, 3, 4, 2)\}$
is indeed in $Fix(\tau)$.

3 Counting instances

3.1 Cardinality of the quotient set

Theorem 4 (1)**.** *Given $k, n \in \mathbb{N}^2$, let $P_{k,n}$ be the number of partitions of $[\![1, n]\!]$ where each part is of length k.*

$$P_{n,n} = 1$$

$$P_{1,n} = 1$$

$$P_{k,n} = \binom{n-1}{k-1} P_{k,n-k}$$

Or :

$$P_{k,n} = \prod_{i=1}^{\frac{n}{k}} \binom{ki-1}{k-1} = \frac{n!}{k!^{\frac{n}{k}} (\frac{n}{k})!}$$

Proof. <u>Base case</u> $P_{n,1} = P_{n,n} = 1$: There is only one way to share $[\![1, n]\!]$ into n parts of size 1, and one way to share the same set in one part of size n.
<u>Induction step</u> Assuming $P_{n,k} = \prod_{i=1}^{\frac{n}{k}} \binom{ki-1}{k-1}$, thus :
$P_{n+k,k} = \binom{n+k-1}{k-1} P_{n,k} = \binom{n+k-1}{k-1} \prod_{i=1}^{\frac{n}{k}} \binom{ki-1}{k-1} = \prod_{i=1}^{\frac{n+1}{k}} \binom{ki-1}{k-1}$ Or :
$\prod_{i=1}^{\frac{n}{k}} \binom{ki-1}{k-1} = \prod_{i=1}^{\frac{n}{k}} \frac{k}{ki} \binom{ki}{k} = \prod_{i=1}^{\frac{n}{k}} \frac{1}{i} \prod_{i=1}^{\frac{n}{k}} \frac{(ki)!}{k!(k(i-1))!} = \frac{n!}{(\frac{n}{k})!} \prod_{i=1}^{\frac{n}{k}} \frac{1}{k!} = \frac{n!}{(\frac{n}{k})! k!^{\frac{n}{k}}}$ □

Theorem 5. *Given $n \in \mathbb{N}$, there is exactly*

$$|I_n(SR)/\mathfrak{S}_n| = \sum_{k|n} \frac{k^k (n-1)!^k}{n^k k!}$$

instances of cardinality n non-isomorphic 2 by 2.

Proof. According to theorem 4, the problem is reduced to count $|Fix(\tau)|$ for every τ whose cyclic type contains only a fixed number k i.e. where τ can be written as $\frac{n}{k}$ disjoints k-cycles. By definition, there is $P_{n,k}$ partition of $[\![1,n]\!]$ respecting the condition forementioned. Plus, for such a partition, there are $(k-1)!^{\frac{n}{k}}$ permutations whose cycle notation gives a cycle in each parts of the given partition. Finally, there are $P_{n,k}(k-1)!^{\frac{n}{k}}$ permutations τ such that $|Fix(\tau)| > 0$, and in this case, $|Fix(\tau)| = (n-1)!^{\frac{n}{k}}$. By applying the Burnside formula :

$$|I_n(SR)/\mathfrak{S}_n| = \frac{1}{|\mathfrak{S}_n|}\sum_{\tau\in\mathfrak{S}_n}|Fix(\tau)| = \frac{1}{|\mathfrak{S}_n|}\sum_{k|n}\sum_{\tau\in\mathfrak{S}_n\text{product of }k\text{-cycles}}|Fix(\tau)|$$

$$= \frac{1}{n!}\sum_{k|n}(n-1)!^{\frac{n}{k}}(k-1)!^{\frac{n}{k}}P_{n,k} = \frac{1}{n!}\sum_{k|n}(n-1)!^{\frac{n}{k}}(k-1)!^{\frac{n}{k}}\prod_{i=1}^{\frac{n}{k}}\binom{k-1}{ki-1}$$

$$= \frac{1}{n!}\sum_{k|n}(n-1)!^{\frac{n}{k}}(k-1)!^{\frac{n}{k}}\frac{n!}{k!^{\frac{n}{k}}(\frac{n}{k})!} = \sum_{k|n}\frac{(n-1)!^{\frac{n}{k}}}{k^{\frac{n}{k}}(\frac{n}{k})!} = \sum_{k|n}\frac{(n-1)!^k}{\left(\frac{n}{k}\right)^k k!}$$

□

The 12 firsts terms :

1. 1
2. 1
3. 2
4. 60
5. 66360
6. 4147236820
7. 19902009929142960
8. 10325801406739620796634430
9. 7761071385712793470699048910192684 80
10. 1091106884155713164803449157423087261531243719417 6
11. 35994624192947744840352324616606223433003529309090909090909420800
12. 3416108471291036068187007799193040991104571529753514021493002352099210640712524 6400

3.2 Asymptotic analysis

Theorem 6. *The cardinality of $|I_n(SR)/\mathfrak{S}_n|$ is asymptotically similar to the infimum :*

$$|I_n(SR)/\mathfrak{S}_n| \sim \frac{(n-1)!^{n-1}}{n}$$

Proof. Given $k|n, k\neq n$:
$\frac{\left(\frac{(n-1)!^k}{\left(\frac{n}{k}\right)^k k!}\right)}{\left(\frac{(n-1)!^{n-1}}{n}\right)} = \frac{k^k}{n^{k-1}(n-1)!^{n-1-k}} < \frac{n^k}{n^{k-1}(n-1)!^{n-1-k}} = \frac{n}{(n-1)!^{n-1-k}}$
Since $k\geq n, k\neq n$ and $n-1\nmid n$, $n-1-k\geq 1$, so for $n>2$, $(n-1)!^{n-1-k}\geq(n-1)!$, thus
$\frac{n}{(n-1)!^{n-1-k}} \leq \frac{n}{(n-1)!} \longrightarrow 0$
Hence, $\forall k|n, k\neq n$, $\frac{(n-1)!^k}{\left(\frac{n}{k}\right)^k k!} = o\left(\frac{(n-1)!^{n-1}}{n}\right)$. Finally :

$$\sum_{k|n}\frac{(n-1)!^k}{\left(\frac{n}{k}\right)^k k!} = \frac{(n-1)!^{n-1}}{n} + \sum_{k|n,k\neq n}\frac{(n-1)!^k}{\left(\frac{n}{k}\right)^k k!} = \frac{(n-1)!^{n-1}}{n} + \sum_{k|n,k\neq n}o(\frac{(n-1)!^{n-1}}{n})$$

$$= \frac{(n-1)!^{n-1}}{n} + o(\frac{(n-1)!^{n-1}}{n}) \sim \frac{(n-1)!^{n-1}}{n}$$

□

A Appendix

Proof. 3^{rd} point of example 3 :
Given the instances
$I =$
$\{(1, 2, 3, 4),$
$(2, 3, 1, 4),$
$(3, 1, 2, 4),$
$(4, 1, 2, 3)\}$

$J =$
$\{(1, 2, 3, 4),$
$(2, 3, 1, 4),$
$(3, 1, 2, 4),$
$(4, 1, 2, 3)\}$

It is straightforward to observe that every $\sigma_i, i \in [\![2, 4]\!]$ are not in the stabilizer, so $Stab(I) = Stab(J) = \{id\}$, thus $|\omega(I)| = \frac{|\mathfrak{S}_n|}{|Stab(I)|} = \frac{4!}{1} = 24$. Hence let E be the set of every instances of cardinality 4 that does not admit a stable matching : $\omega(I) \sqcup \omega(J) \subset E$ and $|\omega(I) \sqcup \omega(J)| = |E|$ so $\omega(I) \sqcup \omega(J) = E$. □

B References

1. cf. A200472 by Denis P. Walsh