OEIS A350189

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ABSTRACT. This is an illustration of the graphs generated in A350189 if the binary matrices are interpreted as incidence matrices of vertex-labeled graphs with undirected edges forbiding 3 types of subgraphs.

1. Binary Matrices with Forbidden 2×2 All-1 Submatrix

Sequence [3, A350189] counts symmetric binary $n \times n$ matrices $A_{i,j} = A_{j,i}$ with forbidden 2×2 submatrices where all 4 entries equal 1. So $A_{i,j} \in \{0, 1\}$, $1 \le i, j \le n$ and the forbidden binary submatrix could be defined as

$$\mathbb{I}_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{1}$$

Let $w = \sum_{i,j} A_{i,j}$ be the number of 1's (the binary weight) of the matrices. Then the triangle of the number of these matrices is Table 1.

2. Labeled graphs with simple edges

In the standard association with incidence matrices, each matrix represents an incidence matrix of a vertex-labeled graph with n vertices and undirected edges. Loops are allowed (from 1's on the matrix diagonal) and the graphs may have multiple components. The number of edges E in these graphs is the trace of the matrix (loops) plus half the number of 1's outside the diagonal.

The forbidden submatrices where all four 1 are outside the diagonal forbid the usual 4-cycles, if one of the four 1 is on the diagonal forbid 3-cycles with a loop, and if two 1 are on the diagonal forbid edges with two loops at both ends: Figure 1.

This leads to a resummation of the entries in each row and to the counts of Table 2.

Date: April 1, 2022.
2020 *Mathematics Subject Classification.* Primary 05C38.

$n\backslash w$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1	1	1											
2	1	2	2	2									
3	1	3	6	10	9	9	4						
4	1	4	12	28	46	72	80	80	60	16			
5	1	5	20	60	140	296	500	780	1005	1085	992	560	170

TABLE 1. The number of symmetric binary $n \times n$ matrices with w 1's and no $\mathbb{I}_2$ submatrix [3, A350189].

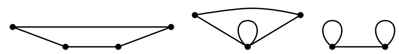


FIGURE 1. Forbidden subgraphs by the criterion of forbidden $\mathbb{I}_2$ sub-incidence-matrices

$V\backslash E$	0	1	2	3	4	5	6	7	8	9	10	11
0	1											
1	1	1										
2	1	3	3									
3	1	6	15	17	3							
4	1	10	45	114	153	72	4					
5	1	15	105	445	1200	1938	1535	370	5			
6	1	21	210	1315	5610	16449	31639	35610	18510	2850	6	
7	1	28	378	3255	19740	87255	280280	632265	932715	791910	314097	39249
8	1	36	630	7112	57603	350616	1628606	5738780	14962897	27554968	33237848	23408560

TABLE 2. The number $L(V, E)$ of labeled graphs with V vertices and E edges without subgraphs of the types of Figure 1.

Example 1. *The column $L(V, 0) = 1$ counts the one labeled graph with all vertices disconnected.*

Example 2. *The column $L(V, 1)$ counts the labeled graph with an edge connecting two distinct out of $\binom{V}{2}$ vertices and the V labeled graphs with a loop at one vertex, $L(V, 1) = \binom{V}{2} + V = \binom{1+V}{2}$.*

Example 3. *$L(V, 2) = 3\binom{V+2}{4}$, see [3, A050534], first adds two more virtual vertices to the set of V labeled vertices, selects 4 vertices of this super-set, sorts them by label like $v_1 < v_2 < v_3 < v_4$ and draws two edges in the 3 possible ways: $(v_1 - v_2, v_3 - v_4)$ or $(v_1 - v_3, v_2 - v_4)$ or $(v_1 - v_4, v_2 - v_3)$. Then edges to any of the 2 virtual vertices are interpreted to loops.*

The row sums are [3, A352258]

$$L(V) \equiv \sum_{E\ge 0} L(V, E) = 1, 2, 7, 42, 399, 5614, 112221 \ldots \quad (2)$$

3. CONNECTED GRAPHS WITH SIMPLE EDGES

A further reduction of the data is to count only the *connected* graphs with V vertices and E edges out of these, Table 3, with row sums

$$C(V) \equiv \sum_{E\ge 0} C(V, E) = 1, 2, 3, 16, 156, 2392, 53448, 1657190, 68876776 \ldots \quad (3)$$

The sequences $L(V)$ and $C(V)$ are a pair of EXP/LOG transforms [1], and the bivariate generating functions are related as described by Gilbert [4].

From a representation point of view one can count only one of the unlabeled graphs as a representative of the automorphism group of each of the labeled graphs which leads to Table 4. Apparently the row sums are given by the number of unlabeled 2-trees with $V+2$ nodes [3, A054581], but that might be mere coincidence.

$V\backslash E$	0	1	2	3	4	5	6	7	8	9	10	11	12	
0	1													
1	1	1												
2	0	1	2											
3	0	0	3	10	3									
4	0	0	0	16	76	60	4							
5	0	0	0	0	125	787	1125	350	5					
6	0	0	0	0	0	1296	10356	22860	16110	2820	6			
7	0	0	0	0	0	0	16807	165364	513660	627725	293580	39207	847	
8	0	0	0	0	0	0	0	262144	3103640	12762624	23523920	20425160	7745304	102

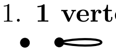
TABLE 3. The number $C(V, E)$ of connected labeled graphs with V vertices and E edges without subgraphs of the types of Figure 1.

$V\backslash E$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1	1	1											
2	0	1	1										
3	0	0	1	3	1								
4	0	0	0	2	5	4	1						
5	0	0	0	0	3	13	15	7	1				
6	0	0	0	0	0	6	29	51	37	12	1		

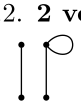
TABLE 4. The number $C_u(V, E)$ of connected unlabeled graphs with V vertices and E edges without subgraphs of the types of Figure 1.

The following graphs illustrate the unlabeled graphs enumerated in Table 4. If one strikes all graphs with loops, the connected squarefree graphs enumerated in Table 6 remain.

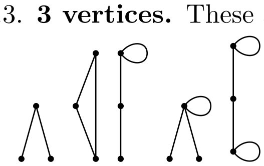
3.1. **1 vertex.** These are the 2 graphs with 1 vertex:



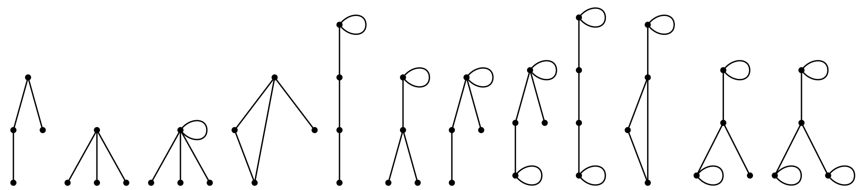
3.2. **2 vertices.** These are the 2 graphs with 2 vertices:



3.3. **3 vertices.** These are the 5 graphs with 3 vertices:

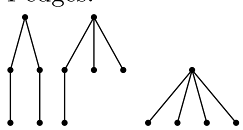


3.4. **4 vertices.** These are the 12 graphs with 4 vertices:

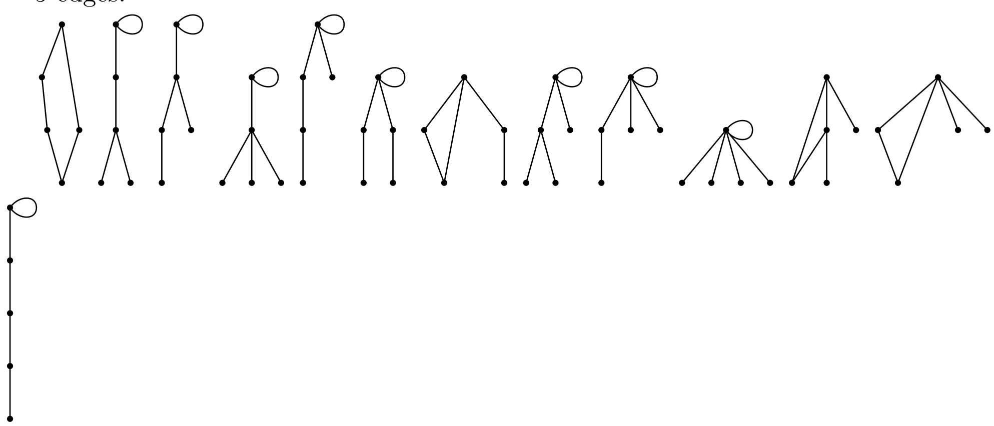


3.5. **5 vertices.** These are the 39 graphs with 5 vertices:

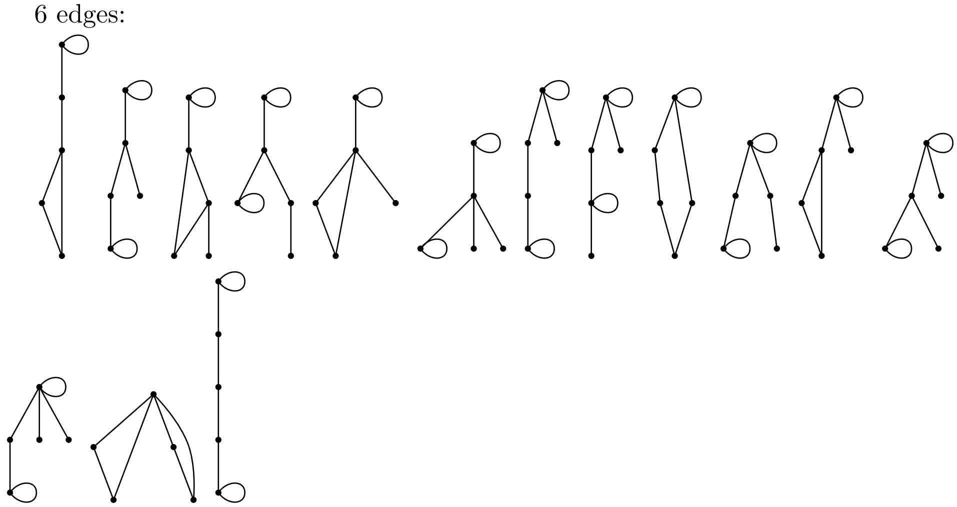
4 edges:



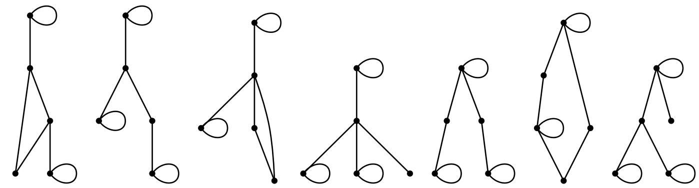
5 edges:



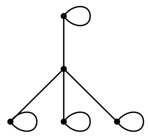
6 edges:



7 edges:

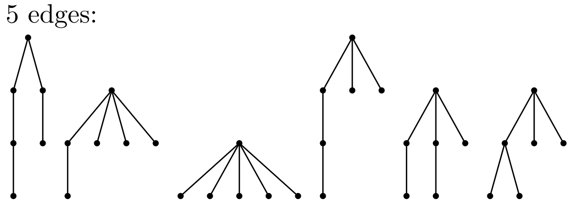


8 edges:

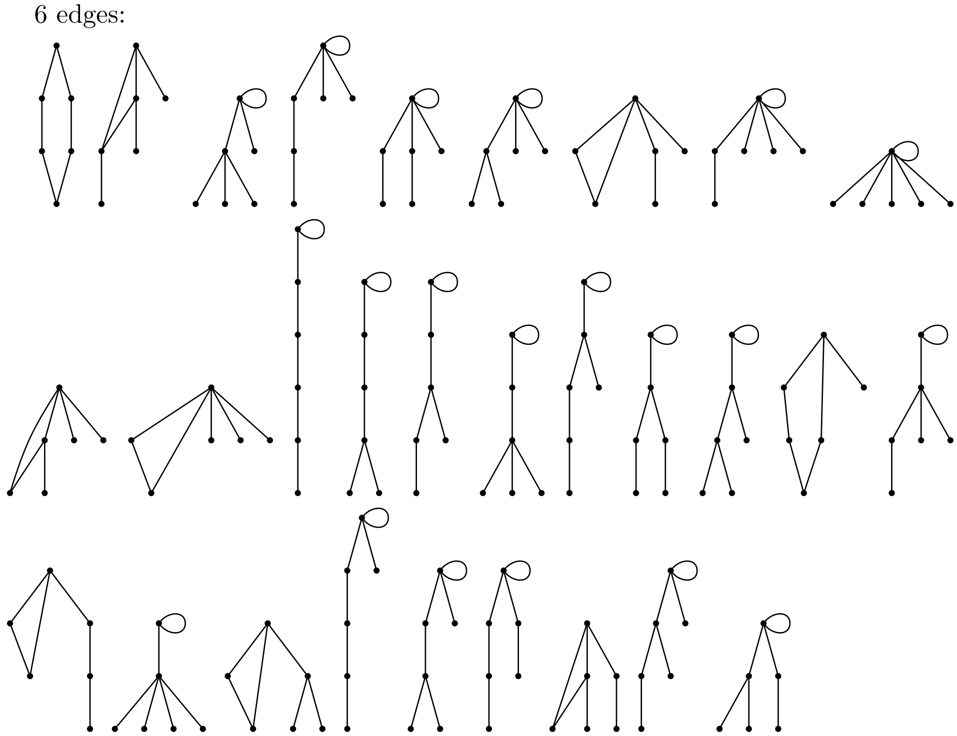


3.6. **6 vertices.** These are the 136 graphs with 6 vertices:

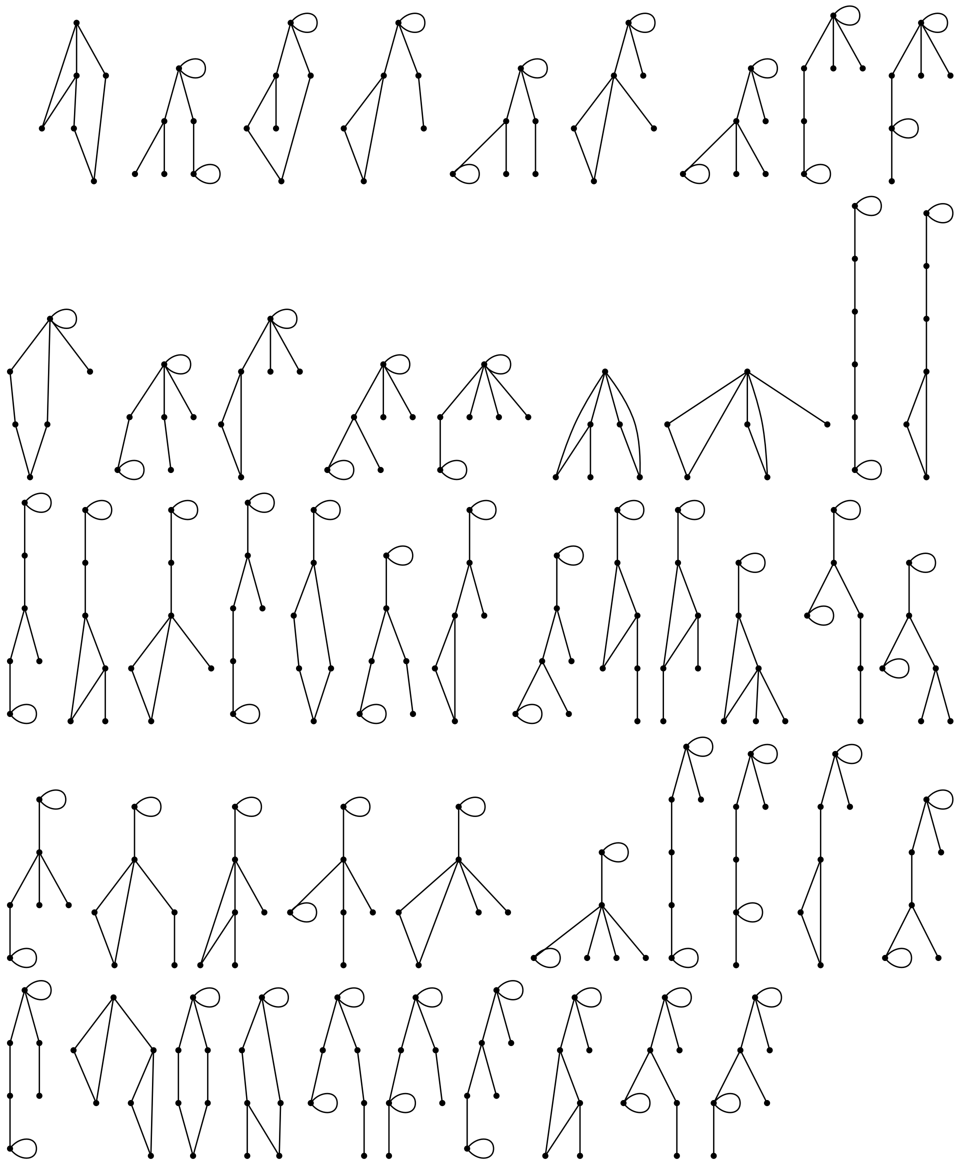
5 edges:



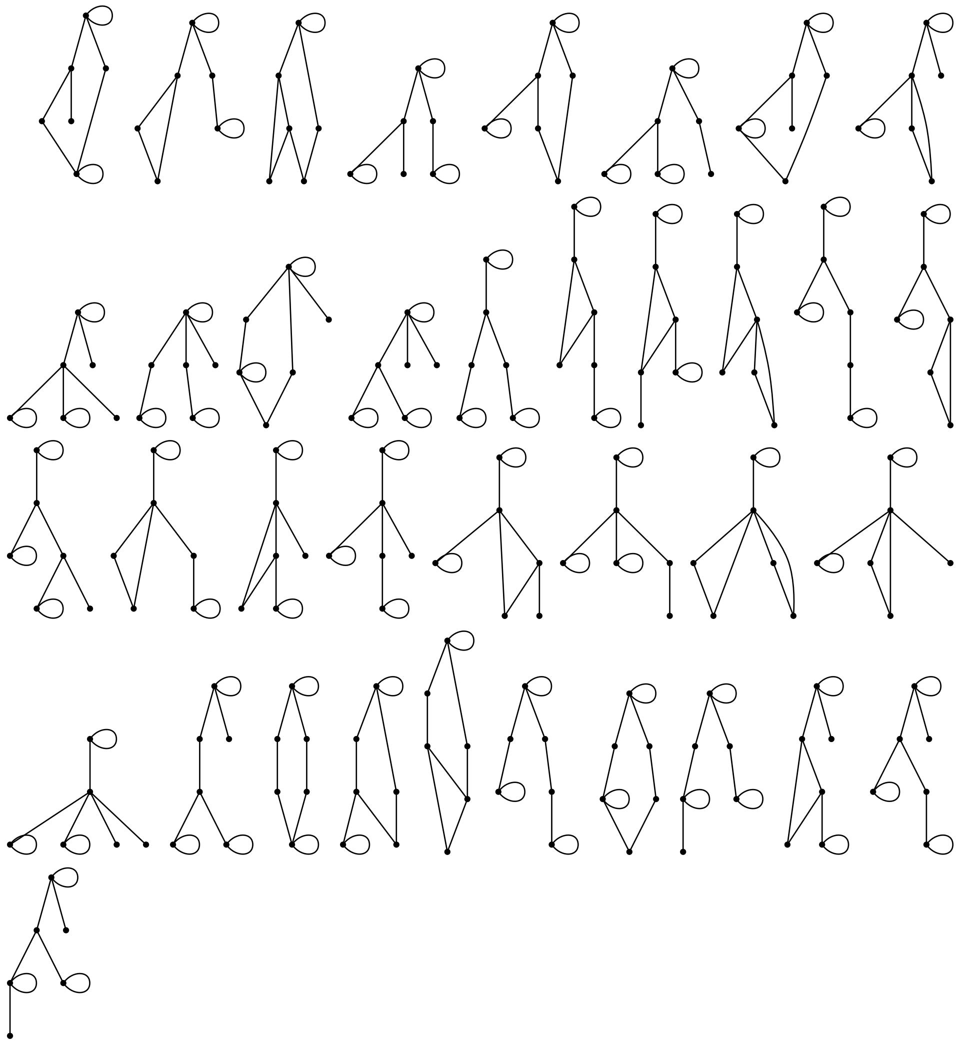
6 edges:



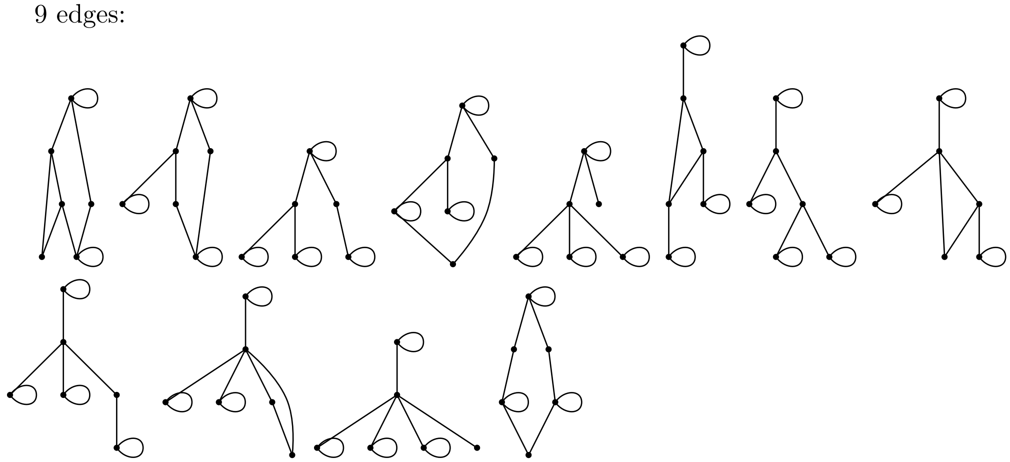
7 edges:



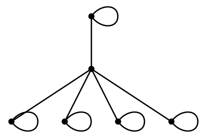
8 edges:



9 edges:



10 edges:



$V \backslash E$	0	1	2	3	4	5	6	7	8	9	10	11	12	
0	1													
1	1													
2	0	1												
3	0	0	3	1										
4	0	0	0	16	12									
5	0	0	0	0	125	162	15							
6	0	0	0	0	0	1296	2580	900						
7	0	0	0	0	0	0	16807	47715	35595	6615				
8	0	0	0	0	0	0	0	262144	1006488	1270080	619920	90720		
9	0	0	0	0	0	0	0	0	4782969	23859108	44893170	39867660	15892065	199

TABLE 5. The number $C_{\not L}(V, E)$ of connected labeled loopless squarefree graphs with V vertices and E edges. Row sums [3, A345248].

$V \backslash E$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1	1												
2	0	1											
3	0	0	1	1									
4	0	0	0	2	1								
5	0	0	0	0	3	4	1						
6	0	0	0	0	0	6	9	4					
7	0	0	0	0	0	0	11	24	17	5			

TABLE 6. The number $C_{\not L u}(V, E)$ of connected unlabeled loopless squarefree graphs with V vertices and E edges. Row sums [3, A077269]

4. Loopless Connected Squarefree Graphs with Simple Edges

The data of Table 3 can be further restricted to traceless incidence matrices, i.e., graphs without loops, in Table 5. See [3, A191966] for the last entries in each row. The values on the diagonal of Table 3 and 5 are the labeled trees [3, A000272].

References

1. Mira Bernstein and Neil J. A. Sloane, *Some canonical sequences of integers*, Lin. Alg. Applic. **226–228** (1995), 57–72, (E:) [2]. MR 1344554
2. Richard A. Brualdi, *From the editor-in-chief*, Lin. Alg. Applic. **320** (2000), no. 1–3, 209–216. MR 1796542
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4. E. N. Gilbert, *Enumeration of labelled graphs*, Can. J. Math. **8** (1956), 405–411. MR 0081470

URL: `http://www.mpia.de/~mathar`

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