Derivation of the recurrence in A348533

$T(m+1,k) = \frac{k\cdot T(m,k)+e}{k-1}$, $S(m+1,k) = 1 + (S(m,k) + e - 1) \bmod T(m+1,k)$

with e=−p if S(m,k)>p and e=k−1−p otherwise, p = T(m,k) mod (k−1)

Start: T(1,k)=S(1,k)=1.

I) Basic recurrence for any number of people

N people, every k−th person is removed, survivor: s(N)

The first line in the following table shows the order of N people after the first victim, say number x, has been removed. The counting is continued with number x+1. The second line shows the initial order of N−1 people.

N :	**x+1**	**x+2**	**..**	**N**	**1**	**2**	**..**	**x-2**	**x-1**
N-1:	**1**	**2**	**..**	**N-x+1**	**N-x+2**	**N-x+3**	**..**	**N-2**	**N-1**

If k = 0 mod N then x = N else x = k mod N. Merged:

(1) x = 1 + (k−1) mod N

For cyclic reasons, s(N−1) and s(N) must be located in the same column.

This yields s(N) = 1 + (s(N−1) + x – 1) mod N.

Inserting formula (1) leads to the basic recurrence

(2) s(N) = 1 + (s(N−1) + k – 1) mod N, starting with s(1) = 1.

Example: k=4

s(2)= 1+ 4 mod 2 = 1

s(3)= 1+ 4 mod 3 = 2

s(4)= 1+ 5 mod 4 = 2 etc.

More terms:

```
  N = 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22
s(N)= 1,1,2,2,1,5,2,6,1, 5, 9, 1, 5, 9,13, 1, 5, 9,13,17,21, 3
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II) Arithmetic progression of survivers' numbers

For k=4, the numbers of people T(m,k) with survivor's number S(m,k)< k are (see table above):

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T(m,4)= 1,2,3,4,5,7,9,12,16,22,..
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with
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S(m,4)= 1,1,2,2,1,2,1, 1, 1, 3,..
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Generally, for T(m,k)≤N<T(m+1,k), s(N) is an arithmetic progression with difference k, starting with S(m,k): s(N)=S(m,k)+k·(N−T(m,k)).

Example k=4, N=2021

We find T(25,4)=1635≤2021<2180=T(26,4) with S(25,4)=2.

Survivor's number: 2+4·(2021−1635)=1546.

III) Recurrence for T(m,k) and S(m,k)

The progression collapses when N=T(m+1,k) because of "mod N" in formula (2). A new progression starts with

$$s(N)=S(m+1,k)=S(m,k)+k\cdot(T(m+1,k)-T(m,k))-T(m+1,k)\ (<k)$$

For this, we have to presume $m\geq k$, otherwise the progression is empty.
With $e(m)=S(m+1,k)-S(m,k)$, we yield

$$e(m)=k\cdot(T(m+1,k)-T(m,k))-T(m+1,k)\quad\Rightarrow T(m+1,k)=\frac{k\cdot T(m,k)+e(m)}{k-1}$$

Let $p(m)=T(m,k) \bmod (k-1)$. Then, as $T(m+1,k)$ is an integer:

(3a) $e(m)=-p(m)$ if $S(m,k)>p(m)$ or $e(m)=k-1-p(m)$ otherwise.

Note that $0<S(m,k)<k$ and $0<S(m+1,k)<k$ implies $0\leq \mathrm{abs}(e(m))<k-1$.

Hence, with (3a), we yield the recurrence:

(3b) $T(m+1,k)=\frac{k\cdot T(m,k)+e(m)}{k-1}$

(3c) $S(m+1,k)=S(m,k)+e(m)$

or (3d) $S(m+1,k)=1+(S(m,k)+e(m)-1) \bmod T(m+1,k)$

For $m\geq k$, (3c) and (3d) are equivalent, but for $m<k$, only (3d) takes into account the condition

(4) $S(m+1,k)\leq T(m+1,k)$

which is necessary because the survivor's place number cannot exceed the number of people.
We start the recurrence with m=1 and T(1,k)=S(1,k)=1 and then use (3abd).
End of derivation.

IV) Comparison with similar sequences

Example k=4:

(5a) $(T(m,k),S(m,k))=(1,1),(2,1),(3,2),(4,2),(5,1),(7,2)\ \ldots$

Formally, we can disregard (4), using (3c) instead of (3d), and start with $\tilde{S}(1,k)=k$.
This yields:

(5b) $(\tilde{T}(m,k),\tilde{S}(m,k))=(1,4),(1,3),(1,2),(\mathbf{1},\mathbf{1}),(2,3),(\mathbf{2},\mathbf{1}),(\mathbf{3},\mathbf{2}),(\mathbf{4},\mathbf{2}),(\mathbf{5},\mathbf{1}),(\mathbf{7},\mathbf{2})\ \ldots$

Note that (5a) is a subsequence of (5b).
$\tilde{T}(m,k)=1,1,1,1,2,2,3,4,5,7\ \ldots$ can be compared with A072493, defined by

(6a) $a(m)=\left\lceil\sum_{i=1}^{m-1}\frac{a(i)}{k-1}\right\rceil$ with (6b) $a(1)=1$.

(6a) can be written as

$$Z(m)=\sum_{i=1}^{m-1}a(i),\ a(m)=\left\lceil\frac{Z(m)}{k-1}\right\rceil=\frac{Z(m)+b(m)}{k-1}\text{ with }0\leq b(m)<k-1.$$

For k=4, we find

(7) $(a(m),b(m))=(1,3),(1,2),(1,1),(1,0),(2,2),(2,0),(3,1),(4,1),(5,0),(7,1)\ \ldots$

Comparison: (8) $a(m)=\tilde{T}(m,k)$, $b(m)=\tilde{S}(m,k)-1$

This is true for any $k>1$ (proof by full induction).
Therefore, the sequences a(m) and T(m, k) can be made equal by removing repeated terms in a(m).