PROOF OF SLOANE'S OBSERVATION

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Before we state our result we fix the following notation. For any positive integer x with binary expansion $\sum_{i=0}^{L} \alpha_i 2^i$, let

$$\bar{x} := \sum_{i=0}^{L} \alpha_{L-i} 2^i.$$

Note that $\bar{x}$ is the integer obtained reversing the binary expansion of x. A crucial thing to note is that $\bar{x}$ is always an odd number (except when $x = 0$). Let A_n be the set of totally balanced integers with $2n$ digits in their binary expansion. It is clear that $|A_n| = C_n$ where C_n is n-th Catalan number. We observe that

$$\min A_n = (1010\ldots10)_2, \quad \max A_n = (1\ldots10\ldots0)_2.$$

Each set A_n is naturally ordered and let $\omega_{k,n}$ denote the k-th element in the set A_n, that is, $\omega_{k,n}$ is the k-th totally balanced integer with $2n$ binary digits.

Theorem 1. *Consider the sequence $\{\mu_{k,n} : 1 \leq k \leq C_n, n \in \mathbb{N}\}$ define by*

$$\mu_{k,n} = \frac{\overline{\omega_{k,n}}}{4^n}.$$

Then $\mu_{k,n}$ is the sequence in (2.8).

Proof. Let v_n denote the van der Corput sequence. From the definition of the sequence in (2.8), it is clear that the sequence can be relabeled by the index $\{(k,n) : 1 \leq k \leq C_n, n \in \mathbb{N}\}$ and

$$a_{k,n} = v_{\omega_{k,n}},$$

where $\omega_{k,n}$ is as defined above. It follows from the definition of van der Corput sequence that

$$v_{\omega_{k,n}} = \frac{\overline{\omega_{k,n}}}{4^n} = \mu_{k,n}.$$

□

Corollary 1. *The denominator in the sequence* (2.8) *is a power of* 4. *And each* 4^n *appears in the denominator exactly C_n times.*

Proof. First of all note that $\overline{\omega_{k,n}}$ is an odd number for each pair k and n. Therefore $\mu_{k,n} = \frac{\overline{\omega_{k,n}}}{4^n}$ is in the simplest form. The corollary now follows from Theorem 1. □

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