# Counting Transitive Subgroups of $S_{n}$ John Erickson, PhD 

## Rapid Introduction

## A definition, a Theorem, and a Problem/Conjecture

I am usually most interested in how mathematical ideas come about and how to best explain them. I therefore tend to spend a lot of time on background material, and I will do so later, but I will uncharacteristically get to the point immediately in the event that the reader is not interested or doesn't need the background or just wants to flesh out the ideas for themselves.

Definition: We define the sequence $t(n)$ to be the number of distinct transitive subgroups of $S_{n}$ where we regard conjugate subgroups as distinct.

I have now computed $t(n)$ for $n \leq 31$. The sequence $t(n)$ has just been added as A337015 at the OEIS but we will refer to it as $t$ in what follows. On the other hand, if we only count subgroups "up to conjugacy" then the number of transitive subgroups of $S_{n}$ is the sequence A002106 at the OEIS (Online Encyclopedia of Integer Sequences). The first 13 terms of A002106 starting with first term $n=1$ are $\{1,1,2,5,5,16,7,50,34,45,8,301,9\}$.
 to be the number of all distinct subgroups of $S_{n}$ where we regard conjugate subgroups as distinct.

The proof of this theorem is included below in a later section.
Problem/Conjecture: If A005432 $(n)$ is the OEIS sequence defined to be the number of all distinct subgroups of $S_{n}$ then for $n \geq 1$

$$
\ln \left(\frac{A 005432(n)}{t(n)}\right)>\frac{n-1}{2} \text { if } n \text { is prime and } \ln \left(\frac{A 005432(n)}{t(n)}\right)<\frac{n-1}{2} \text { if } n \text { is composite }
$$

This conjecture is true for $n \leq 18$ which isn't exactly an impressive amount of data. Therefore the problem is to compute more terms of A005432 ( $t$ is more tractable) and find a counterexample, or provide a proof, or revise the conjecture to something you can prove in light of the additional data. I see no intuitive reason why this statement should be true but I am not an expert in this area.

Note: There will be a small amount of repetition of the above material in the following sections.

## Outline \& A Few Notes

We will give a brief outline of the development of the ideas.

- We discuss partitions of sets. Most of this is fairly standard.
- We discuss the orbits of subgroups of $S_{n}$.
- We define a sequence $t(n)$ and compute the values four different ways.
- We prove a theorem connecting A005432(n) and $t(n)$.
- We pose a problem/conjecture connecting A005432(n) and $t(n)$.

Note: In theory, the mathematical ideas are independent of the coding, but they wouldn't have come about but for an experimental and computational viewpoint. I wrote some not particularly optimized Mathematica and GAP code but in the end I needed code from an expert. The GAP codes are simple C style procedural style program. Unless, I am citing someone else's code, the Mathematica programs were developed by just "rolling up" the preceding sequences of calculations into a function and the main idea to be understood is that \% simply represents the output of the preceding calculation. It's a very nice way to do "rapid prototyping" and conduct experiments.

Note: The Mathematica code below relied on arrays of subgroups like, for example, allSubgroupsOfS4 which contained all the subgroups of $S_{4}$ and a massive array of arrays called allSubgroupsArray containing all the subgroups of $S_{1}, S_{2}, S_{3}, S_{4}, S_{5}, S_{6}$, and $S_{7}$. It's defined in a separate file which needs to be initialized in the same kernel session as this file. I could not find a subgroup function in Mathematica so I made the subgroups of $S_{n}$ manually for $n \leq 4$, but I used GAP for $5 \leq n \leq 7$ and then massaged the output to the form that Mathematica uses. This was the first time I had used GAP.

Note: This little monograph, like the book it is the last chapter of, is deliberately written in a casual, non-academic style, as the objective is more to show "how the sausage is made" rather than hiding all the struggle involved in doing mathematics and then presenting a pristine final product.

## Partitions of Sets

What could be simpler than taking a set of distinct objects and breaking it into parts? A lot of things are simpler actually. The first issue to settle is if the set of objects are distinguishable or not. If they are indistinguishable then order doesn't matter, but if they are distinguishable, perhaps with labels, then we need to decide if order matters to us or not. If order matters, at what level does it matter? The order of the parts or the order of the objects within the parts. Multiple clear choices must be made. For example, if we use numbers as our labels, but don't care about order, then we consider $\{\{1,2\},\{3,4\}\},\{\{2,1\},\{3,4\}\}$, and $\{\{3,4\},\{2,1\}\}$ as the same partitions of $\{1,2,3,4\}$ but different from $\{\{1,3\},\{2,4\}\}$. Our primary focus here will ultimately be on unordered partitions of sets of distinct distinguishable objects which we will model with $\{1,2,3, \ldots, n\}$.

We will investigate and review this topic by playing with Wolfram code we found online. Some of what follows is included as motivational and is a bit repetitive. The underlying theme is that when you count
the elements of a set, you can do so by adding up the counts of the disjoint subsets whose union forms the set, and that, moreover, you can do it again to achieve a finer granularity of decomposition.

## First Approach

If you Google "generate all partitions of a set Wolfram" the first link you get leads to a page titled "Generate All Partitions of a List" with the following wickedly slick code from the Wolfram help files where the only modification we have made is to use $\{1,2,3,4,5\}$ instead of $\{a, b, c, d, e\}$. First we show an example of using FoldPairList with TakeDrop as follows along with IntegerPartitions
$\ln [\rho]=$ FoldPairList[TakeDrop, Range[5], \{2, 2, 1\}]
Out $[\rho=\{\{1,2\},\{3,4\},\{5\}\}$
Flatten[Permutations /@ IntegerPartitions [5], 1]
Out[ $\cdot]=\{\{5\},\{4,1\},\{1,4\},\{3,2\},\{2,3\},\{3,1,1\},\{1,3,1\},\{1,1,3\},\{2,2,1\},\{2,1,2\}$, $\{1,2,2\},\{2,1,1,1\},\{1,2,1,1\},\{1,1,2,1\},\{1,1,1,2\},\{1,1,1,1,1\}\}$

Now we use these constructions to generate all partitions of $\{1,2,3,4,5\}$.
In[ $]$ ]: Block [\{n=5\}, FoldPairList[TakeDrop, Range[n], \#] \& /@
Flatten [Permutations /@ IntegerPartitions [n] , 1]] // Column
$\{\{1,2,3,4,5\}\}$
$\{\{1,2,3,4\},\{5\}\}$
$\{\{1\},\{2,3,4,5\}\}$
$\{\{1,2,3\},\{4,5\}\}$
$\{\{1,2\},\{3,4,5\}\}$
$\{\{1,2,3\},\{4\},\{5\}\}$
$\{\{1\},\{2,3,4\},\{5\}\}$
$\{\{1\},\{2\},\{3,4,5\}\}$
$\{\{1,2\},\{3,4\},\{5\}\}$
$\{\{1,2\},\{3\},\{4,5\}\}$
$\{\{1\},\{2,3\},\{4,5\}\}$
$\{\{1,2\},\{3\},\{4\},\{5\}\}$
$\{\{1\},\{2,3\},\{4\},\{5\}\}$
$\{\{1\},\{2\},\{3,4\},\{5\}\}$
$\{\{1\},\{2\},\{3\},\{4,5\}\}$
$\{\{1\},\{2\},\{3\},\{4\},\{5\}\}$
It should be made clear however that this isn't necessarily all partitions. For example where is the partition $\{\{3,4\},\{1,2,5\}\}$ ? Obviously, the explanation is that this is really only all partitions of 5 indistinguishable objects--despite the fact that we have apparently labeled them with numbers. The give away is that all of the numbers are in increasing order, so if we were to erase all of the numbers we don't lose any information about the partition. Therefore this is not really what we want, but it's still interesting! Let's count the number of partitions as a function of the input length.

In[ $[\mathrm{f}=$ : Table[Length[FoldPairList[TakeDrop, Range[n], \#] \& /@
Flatten[Permutations /@ IntegerPartitions[n], 1]], \{n, 1, 5\}]
Out [ $-=\{1,2,4,8,16\}$
|n[ $[$ ] $=$ FindSequenceFunction [\%, n$]$ // TraditionalForm
Out[o]//TraditionalForm=
$2^{n-1}$
Wow, that's awfully nice and simple! Of course, the reason is simple: each of the above partitions really only depends on choosing a subset from the $n-1$ spaces, or commas, to split $\{1,2,3, \ldots, n\}$ and there are $2^{n-1}$ such subsets. Since the partitions are obtained by mapping
FoldPairList[TakeDrop, Range[n], \#] \& the number of such partitions is simply the cardinality of Flatten[Permutations/@ IntegerPartitions[n], 1] obviously
m[ 0 ]: $=$ Table[Length[Flatten[Permutations /@ IntegerPartitions[n], 1]], \{n, 1, 5\}]
Out[ -$]=\{1,2,4,8,16\}$
Let's focus on $n=4$ as an example
$\ln [-]:=$ Flatten [Permutations /@ IntegerPartitions [4], 1]
Out $-=\{\{4\},\{3,1\},\{1,3\},\{2,2\},\{2,1,1\},\{1,2,1\},\{1,1,2\},\{1,1,1,1\}\}$
One partition of length 1,3 partitions of length 2,3 partitions of length 3 , and 1 partition of length 4 : the binomial coefficients!
$\operatorname{mn}[\mathrm{f}]:=$ Table[Length[Flatten [Permutations /@ IntegerPartitions[n, \{k\}], 1]], \{n, 1, 5\}, \{k, 1, n\}] // Column
\{1\}
$\{1,1\}$
Out $[0=\{1,2,1\}$
$\{1,3,3,1\}$
$\{1,4,6,4,1\}$
It probably is not a surprise that this is really just an application of the Binomial theorem
$\sum_{k=0}^{n-1}\binom{n-1}{k}=2^{n-1}$.
If we remove the Permutations from the above code we still don't get all of the partitions of a labeled set, in fact we obviously get fewer partitions, but it is still interesting so we consider this as well.
$\ln [0]=$ IntegerPartitions [5]
Out $[\cdot]=\{\{5\},\{4,1\},\{3,2\},\{3,1,1\},\{2,2,1\},\{2,1,1,1\},\{1,1,1,1,1\}\}$
$\ln [\cdot]=$ Block [\{n=6\}, FoldPairList [TakeDrop, Range[n], \#] \& /@ IntegerPartitions [n]]
Out $[0=\{\{\{1,2,3,4,5,6\}\},\{\{1,2,3,4,5\},\{6\}\},\{\{1,2,3,4\},\{5,6\}\}$,
$\{\{1,2,3,4\},\{5\},\{6\}\},\{\{1,2,3\},\{4,5,6\}\},\{\{1,2,3\},\{4,5\},\{6\}\}$,
$\{\{1,2,3\},\{4\},\{5\},\{6\}\},\{\{1,2\},\{3,4\},\{5,6\}\},\{\{1,2\},\{3,4\},\{5\},\{6\}\}$,
$\{\{1,2\},\{3\},\{4\},\{5\},\{6\}\},\{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\}\}$
We can again gather some data
$\ln [\rho]=$ Table[Length[IntegerPartitions[n]], $\{n, 1,10\}]$
Out $[0=\{1,2,3,5,7,11,15,22,30,42\}$

```
ln[v]:= FindSequenceFunction[%, n]
```

Out[-]= PartitionsP[n]

The number of ways to partition the integer $n$ into at most $n$ parts. We can do the same thing we did above and break up each of these numbers into parts so that the sum of the rows in the elements in the triangle gives us the above sequence.
$\ln [f]=$ Table[Length[FoldPairList[TakeDrop, Range[n], \#] \& /@ IntegerPartitions[n, \{k\}]], \{n, 1, 10\}, \{k, 1, n\}] // Column
\{1\}
$\{1,1\}$
$\{1,1,1\}$
$\{1,2,1,1\}$
$\{1,2,2,1,1\}$
$\{1,3,3,2,1,1\}$
$\{1,3,4,3,2,1,1\}$
$\{1,4,5,5,3,2,1,1\}$
$\{1,4,7,6,5,3,2,1,1\}$
$\{1,5,8,9,7,5,3,2,1,1\}$
Not as pretty as the binomial coefficients, but still interesting.

## Second Approach

The second bit of Wolfram code we found in our online search to partition a set was from Robert Dickau and is as follows

```
BellList[1] = {{{1}}};
(*the basic case*) BellList[n_Integer ? Positive] := BellList[n] = (*do some caching*)
    Flatten[Join[Map[ReplaceList[#, {S__} -> {S, {n}}] &, BellList[n-1]],
    Map[ReplaceList[#, {b___, {S__}, a___} -> {b, {S, n}, a}] &, BellList[n-1]]], 1]
```

This is some nice rule based code! Note also his nice use of argument type checking and dynamic programming. I need to do type checking more often. Who am I kidding? I never do this but I should. Let's test it.
m[ $[\mathrm{f}:=$ BellList [4] // Column
$\{\{1\},\{2\},\{3\},\{4\}\}$
$\{\{1,2\},\{3\},\{4\}\}$
$\{\{1,3\},\{2\},\{4\}\}$
$\{\{1\},\{2,3\},\{4\}\}$
$\{\{1,2,3\},\{4\}\}$
$\{\{1,4\},\{2\},\{3\}\}$
$\{\{1\},\{2,4\},\{3\}\}$
Out $[0]=\{\{1\},\{2\},\{3,4\}\}$
$\{\{1,2,4\},\{3\}\}$
$\{\{1,2\},\{3,4\}\}$
$\{\{1,3,4\},\{2\}\}$
$\{\{1,3\},\{2,4\}\}$
$\{\{1,4\},\{2,3\}\}$
$\{\{1\},\{2,3,4\}\}$
$\{\{1,2,3,4\}\}$

This is what we want: all of the partitions of a labeled set of $n$ distinguishable elements where we don't care about order. Since we don't care about order, the subsets are listed in increasing lexicographic order and the numbers inside the subsets are in increasing order.

We have another opportunity to gather some data
$\ln [0]:=$
Out[-]=
$\ln [\cdot]:=$
Out $[0]=$
The famous "Bell" numbers. In retrospect, I guess we shouldn't be surprised by this. We can also count partitions of $\{1,2,3, \ldots, n\}$ with varying numbers of parts as follows

Table[Length[Select[BellList[n], Length[\#] == k \& ] ], \{n, 1, 6\}, \{k, 1, n\}] // Column
\{1\}
$\{1,1\}$
$\{1,3,1\}$
$\{1,7,6,1\}$
$\{1,15,25,10,1\}$
$\{1,31,90,65,15,1\}$
These are the famous Stirling numbers of the second kind!
In[ $\mathrm{f}_{\mathrm{I}}=$ Table[StirlingS2[n, k], \{n, 1, 6\}, \{k, 1, n\}] // Column
\{1\}
$\{1,1\}$
$\{1,3,1\}$
$\{1,7,6,1\}$
$\{1,15,25,10,1\}$
$\{1,31,90,65,15,1\}$
If we count, for example, the number of partitions of $\{1,2,3,4\}$ with various numbers of parts we see the following: 1 part -- 1, 2 parts -- 7,3 parts --6 , and 4 parts --1 . These are just the Stirling numbers of the second kind which total to the above Bell number Bell(4)
$\ln [\cdot]:=$ Table[StirlingS2[4, k], \{k, 1, 4\}]
Out $[\cdot]=\{\mathbf{1}, \mathbf{7}, \mathbf{6}, \mathbf{1}\}$
$\ln [\rho]=\% / /$ Total
Out $[0=15$

In general, the sum of the Stirling numbers of the second kind are the Bell numbers: $B_{n}=\sum_{k=0}^{n}\left\{\begin{array}{l}n \\ k\end{array}\right\}$ where $\left\{\begin{array}{l}n \\ k\end{array}\right\}=S_{2}(n, k)$ and this is understood by Mathematica.
$\ln [\odot]:=\sum_{k=1}^{n} \operatorname{StirlingS} 2[n, k]$
Out[ $0=$ BellB[n]

This splitting idea has proven to be interesting so let's do it again on the Stirling numbers themselves. In particular, let's consider $S_{2}(4,2)=7$. We will now work a little harder and collect the partitions which have the same length patterns, regardless of the order. There are five different partitions of the number 4 , but only two have 2 parts, $\{3,1\}$ and $\{2,2\}$.

```
In[\sigma:= IntegerPartitions[4, {2}]
```

Out $[0]=\{\{3,1\},\{2,2\}\}$

This gives us 4 partitions of type $\{3,1\}$ and 3 partitions of type $\{2,2\}$ for the set $\{1,2,3,4\}$
$\{\{1\},\{2,3,4\}\}$
$\{\{1,2,3\},\{4\}\}$
$\{\{1,2,4\},\{3\}\}$
$\{\{1,3,4\},\{2\}\}$
$\{\{1,2\},\{3,4\}\}$
$\{\{1,3\},\{2,4\}\}$
$\{\{1,4\},\{2,3\}\}$
We can automate this accounting as follows
Tally [Map [Length, Select[BellList[4], Length[\#] =: 2 \&], \{2\}], Sort[\#1] == Sort[\#2] \&]
$\{\{\{3,1\}, 4\},\{\{2,2\}, 3\}\}$
We reverse sort so that it corresponds to the standard form for integer partitions with the largest numbers to the left, but in this case it is not necessary

```
|n[\rho]:= %/.{rr_, p_}:-> {ReverseSort[rr], p}
```

Out $[0=\{\{\{3,1\}, 4\},\{\{2,2\}, 3\}\}$

By taking Last, we get the count of the various types of set partitions
$\ln [\rho]=$ Map [Last, \%, \{1\}]
Out $[0]=$
$\{4,3\}$
As usual, we now roll up the above calculations into a function which counts the number of partitions by actually constructing the partitions using BellList and then counting the various types

```
unsortedPartitionsCount[n_, k_] := Module[{temp1, temp2}, temp1 = Tally[
            Map[Length, Select[BellList[n], Length[#] == k &], {2}], Sort[#1] == Sort[#2] &];
    temp2 = temp1 /. {rr_, p_}:-> {ReverseSort[rr], p};
    Map[Last, temp2, {1}]
]
```

Now we can test this.
$\ln [\cdot]:=$ Table[unsortedPartitionsCount [n, k], \{n, 1, 7\}, \{k, 1, n\}] // Column
\{ \{1\} \}
$\{\{1\},\{1\}\}$
$\{\{1\},\{3\},\{1\}\}$
Out $0=\{\{1\},\{4,3\},\{6\},\{1\}\}$
$\{\{1\},\{5,10\},\{10,15\},\{10\},\{1\}\}$
$\{\{1\},\{6,15,10\},\{15,60,15\},\{20,45\},\{15\},\{1\}\}$
$\{\{1\},\{7,21,35\},\{21,105,70,105\},\{35,210,105\},\{35,105\},\{21\},\{1\}\}$
We can see that the sum of the numbers in the inner lists are just the Stirling numbers of the second kind. Again, $4+3=7=\left\{\begin{array}{l}4 \\ 2\end{array}\right\}$ because there are 2 partitions of 4 of size 2,4 of type $\{3,1\}$ and 3 of type $\{2,2\}$.
Let's now dig a bit deeper with the larger example of $n=6$ and ask how many ways we can partition the set $\{1,2,3,4,5,6\}$ into $k=4$ parts, keeping track of type, where we don't care about order, but without actually calculating all of the partitions.

This is the standard combinatorial problem of counting unordered partitions of a set which we can formulate in general as follows: Given the set $[n]=\{1,2,3, \ldots, n\}$, a set partition into $k$ parts is just a collection of $k$ disjoint subsets of $[n]$ of sizes $\left\{t_{1}, t_{2}, \ldots, t_{k}\right\}$ where $1 \leq t_{i} \leq n$, the $t_{i}$ are non increasing and $\sum_{i=1}^{k} t_{i}=n$. In other words, a set partition of the set [ $n$ ] gives us an integer partition of $n, P=\left\{t_{1}, t_{2}, \ldots, t_{k}\right\}$, which we call the "type" of the set partition of $[n]$ and say it has size $|P|=k$, the number of parts.
Following the Wolfram language we call the set of all such integer partitions of $n$
IntegerPartitions[ $n,\{k\}]$. To each such integer partition $P$ we can associate the unique "tallied" partition $\left.\left\{\left\{s_{1}, s_{2}, \ldots, s_{r}\right\},\left\{m_{1}, m_{2}, \ldots, m_{r}\right\}\right\}\right)$ where $s_{1}, s_{2}, \ldots, s_{r}$ are the distinct integers which appear as the sizes of the subsets of the partition and the $m_{i}$ are the number of subsets, or multiplicities, of the sets of size $s_{j}$. In other words, we just eliminate the duplicates in $\left\{t_{1}, t_{2}, \ldots, t_{k}\right\}$ but keep track of the multiplicities. Therefore we have $\sum_{i=1}^{r} s_{i} m_{i}=n$ and $\sum_{i=1}^{r} m_{i}=k$. If we now let $c(n, P)$ be the number, or count, of all unordered set partitions of $[n]$ with $k$ parts of type $P$ then standard counting principles give us

$$
c(n, P)=\frac{n!}{\left(m_{1}!s_{1}!^{m_{1}}\right)\left(m_{2}!s_{2}!^{m_{2}}\right) \ldots\left(m_{r}!s_{r}!^{m_{r}}\right)}
$$

Continuing our example with $n=6$ with $k=4$, there are only two types: $\{3,1,1,1\}$ and $\{2,2,1,1\}$ both with $r=2$ distinct subset sizes. Type $\{3,1,1,1\}$ gives $s_{1}=3$ and $s_{2}=1$ and $m_{1}=1$ and $m_{2}=3$ or tallied type $\{\{3,1\},\{1,3\}\}$. Type $\{2,2,1,1\}$ gives $s_{1}=2$ and $s_{2}=1$ and $m_{1}=2$ and $m_{2}=2$ or tallied type $\{\{2,1\},\{2,2\}\}$. This gives us the following values for $c(6,\{\{3,1\},\{1,3\}\})$ and $c(6,\{\{2,1\},\{2,2\}\})$
$\operatorname{mn}[\theta]=\left\{\frac{6!}{\left(1!3!^{1}\right)\left(3!1!^{3}\right)}, \frac{6!}{\left(2!2!^{2}\right)\left(2!1!^{2}\right)}\right\}$
out $f=\{20,45\}$
We can automate this process as follows.
$\operatorname{m[}[-]:=$ IntegerPartitions [6, \{4\}]
Out $[\rho=\{\{3,1,1,1\},\{2,2,1,1\}\}$

```
ln[v]:= #! & /@%
Out[-]= {{6, 1, 1, 1}, {2, 2, 1, 1}}
mn[ ]:= Tally/@%
Out[r = { {{6, 1}, {1, 3}}, {{2, 2}, {1, 2} }}
In[\rho]== Map[#[[2]]! #[[1]] #[[2]] &, %,{2}]
Out[\rho]={{6, 6}, {8, 2}}
ln[v]:= Times @@@%
Out[ 0]= {36, 16}
ln[0]= 6!/%
Out[f= = {20, 45}
```

Now we roll up the above sequence of calculations into a single calculation as follows

```
In[f]:= 6!/Times @@@ Map[#[[2]]! #[[1]]#[[2]] &,(Tally/@ (#! &/@ IntegerPartitions[6, {4}])),{2}]
```

Out $[0]=\{20,45\}$
which enables us to define a function which we call the unsortedPartitions.

```
ln[11]:=
```

    unsortedPartitions [n_, k_] := n!/Times @@@
    \(\operatorname{Map}\left[\#[[2]]!\#[[1]]^{\#[2]]} \&,(\right.\) Tally \(/ @(\#!\& / @\) IntegerPartitions \([n,\{k\}])\) ) \{2\}]
    $\ln \left[\sigma^{2}\right]=$ unsortedPartitions [4, 2]
Out [0] $=$
$\{4,3\}$
Table[unsortedPartitions [n, k], \{n, 1, 7\}, \{k, 1, n\}] // Column
\{ \{1 \} \}
\{\{1\}, $\{1\}\}$
$\{\{1\},\{3\},\{1\}\}$
Out[ $[\cdot]=\{\{\mathbf{1}\},\{\mathbf{4}, \mathbf{3}\},\{6\},\{1\}\}$
$\{\{1\},\{5,10\},\{10,15\},\{10\},\{1\}\}$
$\{\{1\},\{6,15,10\},\{15,60,15\},\{20,45\},\{15\},\{1\}\}$
$\{\{1\},\{7,21,35\},\{21,105,70,105\},\{35,210,105\},\{35,105\},\{21\},\{1\}\}$

Note: that what we have unsortedPartitions $[n, k]=\{c(n, P)\}_{P \in \operatorname{IntegerPartitions[n}\{k\}]}$. Moreover, the total number of elements in the $n$ sets is just PartitionsP[n] as the following shows
$\ln [\cdot]:=$ Length /@ Flatten /@ Table[unsortedPartitions [n, k], $\{\mathbf{n}, \mathbf{1}, \mathbf{2 0}\}$, \{k, 1, n\}]
Out $\cdot]=\{1,2,3,5,7,11,15,22,30,42,56,77,101,135,176,231,297,385,490,627\}$
$\ln [-]:=$ FindSequenceFunction [\%, n ]
Out[o]= PartitionsP [n]
Key point: unsortedPartitions is obviously the same function as unsortedPartitionsCount but, again, no explicit construction and counting of partitions. Finally, we can define a function as follows
myStirlingS2[n_, k_] := Total[unsortedPartitions [n, k]]
We can compare this to the built in Stirling function of the second kind as follows

```
m[f]:= Block[{n=6}, Table[StirlingS2[n, k], {k, 1, n}]]
Out[f]={1, 31, 90, 65, 15, 1}
In[f]:= Block[{n=6}, Table[myStirlingS2[n, k], {k, 1, n}]]
Out[\rho]={1, 31, 90, 65, 15, 1}
```

The same. Mathematically, we can write this as follows

$$
S_{2}(n, k)=\sum_{P \in \operatorname{IntegerPartitions}[n,\{k\}]} c(n, P)=\sum_{P \in \operatorname{IntegerPartitions}[n,\{k\}]} \frac{n!}{\left(m_{1}!s_{1}!^{m_{1}}\right)\left(m_{2}!s_{2}!^{m^{2}}\right) \ldots\left(m_{r}!s_{r}!^{m_{r}}\right)}
$$

Note: For this reason we could also call the function unsortedPartitions something like bifurcatedStirlingS2. In fact, this is what we actually called it initially.

## Balanced Partitions

Let's come back to our favorite example, all the partitions of $\{1,2,3,4\}$.
|n[o]:= BellList[4] // Column
$\{\{1\},\{2\},\{3\},\{4\}\}$
$\{\{1,2\},\{3\},\{4\}\}$
$\{\{1,3\},\{2\},\{4\}\}$
$\{\{1\},\{2,3\},\{4\}\}$
$\{\{1,2,3\},\{4\}\}$
$\{\{1,4\},\{2\},\{3\}\}$
$\{\{1\},\{2,4\},\{3\}\}$
Out $[0]=\{\{1\},\{2\},\{3,4\}\}$
$\{\{1,2,4\},\{3\}\}$
$\{\{1,2\},\{3,4\}\}$
$\{\{1,3,4\},\{2\}\}$
$\{\{1,3\},\{2,4\}\}$
$\{\{1,4\},\{2,3\}\}$
$\{\{1\},\{2,3,4\}\}$
$\{\{1,2,3,4\}\}$
In the course of some previous mathematical ruminations we became interested in those partitions of $\{1,2,3, \ldots, n\}$ which we called "balanced", which just means that the subsets all have the same average. In our current case the only two partitions which are balanced are $\{\{1,4\},\{2,3\}\}$ and $\{\{1,2,3,4\}\}$ for which the subsets have average $\frac{5}{2}$. We can determine how many of our partitions are balanced for a given $n$ using the following code.
$\ln [\sigma]=$ Table[Length[Select[Table[
Union [Mean /@partitions], \{partitions, BellList[n]\}], \# ==\{ $\left.\left.\left.\left.\frac{n+1}{2}\right\} \&\right]\right],\{n, 1,10\}\right]$
Out $0=\{1,1,2,2,5,5,18,16,75,64\}$
This is a weird sequence. FindSequenceFunction doesn't help so we try the OEIS and we get sequence A326512 from Gus Wiseman which is the number of set partitions of $\{1, \ldots, n\}$ where every block has the
same average. I don't think an explicit formula is known.
As we can see, there are many ways to get balanced partitions, especially for larger $n$. The simplest example that comes to mind is $\{\{1,2, \ldots, n\}\}$ which only has 1 part and the average is $\frac{1}{n} \frac{n(n+1)}{2}=\frac{n+1}{2}$. Another example is provided by $\left\{\{1, n\}, \ldots,\{k, n-k+1\}, \ldots,\left\{\left\lfloor\frac{n+1}{2}\right\rfloor, n-\left\lfloor\frac{n+1}{2}\right\rfloor+1\right\}\right\}$ which has $\left\lfloor\frac{n+1}{2}\right\rfloor$ parts with each part having average $\frac{n+1}{2}$. From the perspective of the number of parts these are the extreme cases since it is clear that if the average is of each part is $\frac{n+1}{2}$ then the smallest the sets can be is size 2 possibly with the exception of 1 singleton set if $n$ is odd.

## The Orbits for Subgroups of $S_{n}$

First we will explain our original motivations for this investigation and do some foreshadowing. We were originally interested in determining all the subgroups of $S_{n}$ whose orbits produced balanced partitions of $\{1,2,3, \ldots, n\}$. Then we became interested in the distribution of all possible orbits of all possible subgroups of $S_{n}$. At some point we figured out that this was too ambitious and settled for solving the initial problem of figuring out how many subgroups had the simplest type of balanced orbit, namely, $\{\{1,2,3, \ldots, n\}\}$. These are the transitive subgroups under the canonical action.

Clearly, in order to carry out this program we first need to get a handle on the set of all possible subgroups of $S_{n}$. The number of all subgroups of $S_{n}$ is the sequence A005432 at the OEIS and is defined as the "number of permutation groups of degree $n$ (or, number of distinct subgroups of symmetric group $S_{n}$, counting conjugates as distinct". Starting at $n=1$, the 18 terms of A005432 are as follows
$\ln [13]:=$

```
a={1, 2, 6, 30, 156, 1455, 11 300, 151 221, 1694 723, 29 594446,
    404126 228, 10594925 360, 175 238 308 453, 5651774693 595, 117 }053117995 400,
    5320744503742 316, 125 889 331236297 288, 7598016157515 302 757};
```

These numbers can be determined directly from GAP, but one should not take for granted just how formidable these calculations are. In fact, the rapid rate of growth of this sequence can be quantified by a result from L. Pyber that says $2\left(\frac{1}{16}+o(1)\right) n^{2} \leq a_{n} \leq 24\left(\frac{1}{6}+o(1)\right) n^{2}$. The lower bound is conjectured to be sharp incidentally. If we only count conjugacy classes of subgroups we get much smaller numbers, and this is sequence A000638 the "number of conjugacy classes of subgroups of [the] symmetric group $S_{n}$ " Starting at $n=1$, the first 13 terms are as follows

```
{1, 2, 4, 11, 19, 56, 96, 296, 554, 1593, 3094, 10723, 20832, 75 154}
```

Much smaller numbers. When you calculate all of the subgroups of $S_{n}$, not just conjugacy classes of subgroups, GAP will warn you that you are using lots of memory and suggest that you should just consider the conjugacy classes of subgroups, since any representative of a class is just a relabeling of any element of the class and therefore has the same group theoretic properties. Unfortunately, we are interested in the orbits of the natural actions of these subgroups so we can't simply consider the conjugacy classes of our subgroups.

The first key simple idea: is that group orbits are just partitions so we will be able to use our understanding of counting partitions to count orbits. Note that, after sorting, this is the same as what we obtain by applying the GroupOrbits function to our subgroups union to remove duplicates since some subgroups have the same orbits.

In[ $]$ := \{BellList[4] // Sort // Column, GroupOrbits [\#, Range[4]] \& /@allSubgroupsOfS4 // Union // Column\}

```
        {{1, 2, 3, 4}} {{1, 2, 3, 4}}
        {{1}, {2, 3, 4}} {{1}, {2, 3, 4}
        {{1, 2},{3,4}} {{1, 2},{3,4}}
        {{1,3},{2,4}} {{1,3},{2,4}}
        {{1,4},{2,3}} {{1,4},{2,3}}
        {{1,2,3},{4}}}{{1,2,3},{4}
        {{1,2,4}, {3}} {{1, 2, 4}, {3}}
Out[0]=
    {{1,3,4}, {2}} , {{1,3,4}, {2}}
        {{1}, {2}, {3, 4}} {{1}, {2}, {3, 4}}
        {{1},{2,3},{4}} {{1},{2, 3}, {4}}
        {{1},{2, 4}, {3}} {{1},{2, 4}, {3}}
        {{1, 2}, {3}, {4}} {{1, 2}, {3}, {4}}
        {{1,3},{2}, {4}} {{1,3}, {2}, {4}}
        {{1,4},{2},{3}} {{1, 4},{2},{3}}
        {{1},{2},{3},{4}} {{1}, {2}, {3}, {4}}
In[-]:= (BellList[4] // Sort) == (GroupOrbits[#, Range[4]] & /@ allSubgroupsOfS4 // Union)
Out[0]= True
```


## Detailed calculations of the orbits

Now we want to consider the orbits of all of the subgroups of $S_{n}$ in detail. We will use $S_{4}$ as our initial example. Here are all possible orbits.
$\ln [14]:=$

```
data4 = Block[{n=4},
```

    Tally[Table[GroupOrbits[group, Range[n]], \{group, allSubgroupsArray[[n]]\}]]];
    In [ $\sigma]=$ data4 // Column
$\{\{\{1\},\{2\},\{3\},\{4\}\}, 1\}$
$\{\{\{1,2\},\{3\},\{4\}\}, 1\}$
$\{\{\{1,3\},\{2\},\{4\}\}, 1\}$
$\{\{\{1,4\},\{2\},\{3\}\}, 1\}$
$\{\{\{1\},\{2,3\},\{4\}\}, 1\}$
$\{\{\{1\},\{2,4\},\{3\}\}, 1\}$
$\{\{\{1\},\{2\},\{3,4\}\}, 1\}$
Out $[-]=\{\{\{1,2\},\{3,4\}\}, 2\}$
$\{\{\{1,3\},\{2,4\}\}, 2\}$
$\{\{\{1,4\},\{2,3\}\}, 2\}$
$\{\{\{1,2,3\},\{4\}\}, 2\}$
$\{\{\{1,2,4\},\{3\}\}, 2\}$
$\{\{\{1,3,4\},\{2\}\}, 2\}$
$\{\{\{1\},\{2,3,4\}\}, 2\}$
$\{\{\{1,2,3,4\}\}, 9\}$
Note that the 30 groups yield 15 distinct orbits and this is just the Bell number.
$\ln [\rho]:=$ \{Length[data4], BellB [4] \}
Out $0=$ = $\{15,15\}$
Of course we still get 30 groups by adding the frequencies

```
In[॰]:= Last/@ data4
```

Out $[\cdot]=\{1,1,1,1,1,1,1,2,2,2,2,2,2,2,9\}$
Total [\%]
30

If we count the number of parts we see the following: 1 orbit with 1 part, 7 orbits with 2 parts, 6 orbits with 3 parts, and 1 orbit with 4 parts. These are just the Stirling numbers of the second kind which total to the above Bell number

In[f]:= Table[StirlingS2[4, k], \{k, 1, 4\}]
Out[ $[\cdot]=\{1,7,6,1\}$

In[॰]:= \% / / Total
Out[o]= 15
We will now work a little harder and collect the orbits which have the same length patterns, or type, regardless of the order. This gives us the five different partitions of the number 4.
$\ln [\cdot]:=$ PartitionsP [4]
Out $[0]=5$
$\ln [\rho]:=$ Tally[Transpose[ \{Map[Length, First/@ data4, \{2\}], Last/@ data4\}], Sort[\#1[[1]]] == Sort[\#2[[1]]] \&]
Out $[\cdot]=\{\{\{\{1,1,1,1\}, 1\}, 1\},\{\{\{2,1,1\}, 1\}, 6\}$, $\{\{\{2,2\}, 2\}, 3\},\{\{\{3,1\}, 2\}, 4\},\{\{\{4\}, 9\}, 1\}\}$

We simplify the above by removing the extra braces and reverse sorting. This is so it corresponds to the standard form for integer partitions with the largest numbers to the left.
$\ln [\cdot]:=\% / \cdot\left\{\left\{r r_{-}, p_{-}\right\}, \mathbf{q}_{-}\right\}: \rightarrow\{\operatorname{ReverseSort}[r r], p, q\}$
Out[ $]=\{\{\{1,1,1,1\}, 1,1\},\{\{2,1,1\}, 1,6\},\{\{2,2\}, 2,3\},\{\{3,1\}, 2,4\},\{\{4\}, 9,1\}\}$
Then we sort the above list in the order of increasing numbers of parts.
$\mid n[\rho]=$ Sort [\%, Length[\#1[[1]]] $\leq$ Length[\#2[[1]]] \&]
Out[ $]=\{\{\{4\}, 9,1\},\{\{2,2\}, 2,3\},\{\{3,1\}, 2,4\},\{\{2,1,1\}, 1,6\},\{\{1,1,1,1\}, 1,1\}\}$
Now we collect the partitions that have the same number of parts.
$\ln [\cdot]:=$ temp = Gather [\%, Length [\#1[[1] ]] == Length [\#2[[1] ]] \&]
Out $[f=\{\{\{\{4\}, 9,1\}\},\{\{\{2,2\}, 2,3\},\{\{3,1\}, 2,4\}\},\{\{\{2,1,1\}, 1,6\}\},\{\{\{1,1,1,1\}, 1,1\}\}\}$
By taking First, we isolate the partitions of 4.
$\ln [\cdot]:=$ Map[First, \%, \{2\}]
Out $[\rho]=\{\{\{4\}\},\{\{2,2\},\{3,1\}\},\{\{2,1,1\}\},\{\{1,1,1,1\}\}\}$
Here is the standard way we would generate the partitions of 4.
$\ln [-]$ ]: Table[IntegerPartitions [4, \{k\}], \{k, 1, 4\}]
Out[ $\cdot 0=\{\{\{4\}\},\{\{3,1\},\{2,2\}\},\{\{2,1,1\}\},\{\{1,1,1,1\}\}\}$
Note that the ordering of the partitions we found is not quite the same as the standard order from IntegerPartitions for the second element. We now record this deviation using FindPermutation
$\ln [\cdot]:=\operatorname{MapThread}[$ FindPermutation, $\{\{\{\{4\}\},\{\{2,2\},\{3,1\}\},\{\{2,1,1\}\},\{\{1,1,1,1\}\}\}$, $\{\{\{4\}\},\{\{3,1\},\{2,2\}\},\{\{2,1,1\}\},\{\{1,1,1,1\}\}\}\}]$
Out[0]= \{Cycles[\{\}], Cycles[\{\{1, 2\}\}], Cycles[\{\}], Cycles[\{\}]\}
We can now rearrange our list using this list of permutations
$\mid n[\sigma \mid=$ MapThread [Permute, \{temp, \%\}]
Out $[=]=\{\{\{\{4\}, 9,1\}\},\{\{\{3,1\}, 2,4\},\{\{2,2\}, 2,3\}\},\{\{\{2,1,1\}, 1,6\}\},\{\{\{1,1,1,1\}, 1,1\}\}\}$

Finally, we can extract the 2nd and 3rd components of each of the above parts as follows.
The 3rd component $q$ of each tuple $\{P, r, q\}$ represents the number of distinct orbits of integer partition type $P$ in $S_{n}$, grouped by the number of parts, but this has nothing to do with $S_{n}$ and is just the number of partitions of $n$ of type $P$, so $c(n, P)$.

In[ 0 : $=$ Map [\#[ [3]] \&, \%, \{2\}]
Out $0=\{\{1\},\{4,3\},\{6\},\{1\}\}$

The 2nd component $r$ of each tuple $\{P, r, q\}$ represents the multiplicity of each orbit of integer partition type $P$ in $S_{n}$, again, grouped by the number of parts which we can call $m(n, P)$
$\operatorname{In}[$ [ $]=$ Map [\#[ [2] ] \&, \%\%, \{2\}]
Out[-] $=\{\{9\},\{2,2\},\{1\},\{1\}\}$

## The Key Mathematica Code

The code below is inspired in part by our unsortedPartitionsCount code.
As usual, we now roll up the above calculations into a function called orbitSpectrum and test it. The idea is that the function encapsulates all of the orbit information for all of the subgroups of $S_{n}$ but resolved by orbit type which is why we decided to call this quantity a "spectrum". Also, I just like the word spectrum. Since we have only calculated all subgroups of $S_{n}$ for $n \leq 7$ the code is limited however. The code is generic for $n \geq 3$ using the above prototyping but $n=1,2$ are just hard coded. Please note that, inspired and reminded by the above Dickau code, I even used pattern matching to do some type checking!
$\ln [15]:=$

```
orbitSpectrum[n_Integer /; 1\leqn\leq 7] :=
    Module[{temp0, temp1, temp2, temp3, temp4, temp5, permTemp1, permTemp2, permTemp3},
        temp0 = Tally[Table[GroupOrbits[group, Range[n]], {group, allSubgroupsArray[[n]]}]];
        Which[n == 1, permTemp3 = {{{{1}, 1, 1}}};,
            n== 2, permTemp3 = {{{{2}, 1, 1}}, {{{1, 1}, 1, 1}}};, n> 2,
            temp1 = Tally[Transpose[{Map[Length, First /@ temp0, {2}], Last /@ temp0}],
            Sort[#1[[1]]] == Sort[#2[[1]]] &];
        temp2 = temp1 /. {{rr_, p_}, q_} :-> {ReverseSort[rr], p, q};
        temp3 = Sort[temp2, Length[#1[[1]]] \leq Length[#2[[1]]] &];
        temp4 = Gather[temp3, Length[#1[[1]]] == Length[#2[[1]]] &];
        permTemp1 = Map[First, temp4, {2}];
        permTemp2 = MapThread[FindPermutation,
            {permTemp1, Table[IntegerPartitions[n, {k}], {k, 1, n}]}];
        permTemp3 = MapThread[Permute, {temp4, permTemp2}];];
        {Map[#[[2]] &, permTemp3, {2}], Map[#[[3]] &, permTemp3, {2}]}
    ]
```

$\ln [\theta]:=$ orbitSpectrum [4]
Out $[0=\{\{\{9\},\{2,2\},\{1\},\{1\}\},\{\{1\},\{4,3\},\{6\},\{1\}\}\}$

The output is two sequences of sequences. We will call $\{\{9\},\{2,2\},\{1\},\{1\}\}$ the first component and $\{\{1\},\{4,3\},\{6\},\{1\}\}$ the second component. The 9 in the first sequence corresponds with the 1 in the second sequence and indicates that there are 9 sequences with the 1 part $\{1,2,3,4\}$. On the other hand the $\{2,2\}$ in the first sequence corresponds to the $\{4,3\}$ in the second sequence and indicates that there are 4 sequences of type $\{3,1\}$ and 3 sequences of type $\{2,2\}$ and both of these have 2 parts. We now extract and separate all of the first and second components of the above orbit calculations for $n=1,2, \ldots, 7$.

In[f]:= Table[orbitSpectrum[n], \{n, 1, 7\}][[All, 1]] // Column
\{ \{1\} \}
$\{\{1\},\{1\}\}$
$\{\{2\},\{1\},\{1\}\}$
Out -$]=\{\{9\},\{2,2\},\{1\},\{1\}\}$
$\{\{20\},\{9,3\},\{2,2\},\{1\},\{1\}\}$
$\{\{279\},\{20,25,13\},\{9,3,6\},\{2,2\},\{1\},\{1\}\}$
$\{\{512\},\{279,33,42\},\{20,25,13,8\},\{9,3,6\},\{2,2\},\{1\},\{1\}\}$
In[ $]$ ]: Table[orbitSpectrum[n], \{n, 1, 7\}][[All, 2]] // Column
\{ \{1 \} \}
$\{\{1\},\{1\}\}$
$\{\{1\},\{3\},\{1\}\}$
Out $\cdot=\{\{1\},\{4,3\},\{6\},\{1\}\}$
$\{\{1\},\{5,10\},\{10,15\},\{10\},\{1\}\}$
$\{\{1\},\{6,15,10\},\{15,60,15\},\{20,45\},\{15\},\{1\}\}$
$\{\{1\},\{7,21,35\},\{21,105,70,105\},\{35,210,105\},\{35,105\},\{21\},\{1\}\}$
By construction, the second components are just our unsortedPartitions or our bifurcated Stirling functions of the second kind where the total of each part just gives us the usual Stirling function of the second kind.
|n[f]:= Table[unsortedPartitions[n, k], \{n, 1, 7\}, \{k, 1, n\}] // Column
\{ \{1\} \}
$\{\{1\},\{1\}\}$
$\{\{1\},\{3\},\{1\}\}$
Out $0 \cdot=\{\{1\},\{4,3\},\{6\},\{1\}\}$
$\{\{1\},\{5,10\},\{10,15\},\{10\},\{1\}\}$
$\{\{1\},\{6,15,10\},\{15,60,15\},\{20,45\},\{15\},\{1\}\}$
$\{\{1\},\{7,21,35\},\{21,105,70,105\},\{35,210,105\},\{35,105\},\{21\},\{1\}\}$
Note that we can again recover the total number of subgroups of $S_{n}, a(n)$, by a dot product calculation (once we flatten) where we multiply the number of types times the number of orbits of each type and adding.
$\ln [$ ] $:=$ Table[Dot @@ Flatten /@ orbitSpectrum [n], \{n, 1, 7\}]
Out $[=\{=\{1,2,6,30,156,1455,11300\}$

The same as
$\ln [\rho]=\mathbf{a}[\mathbf{1 ; ~ ; ~ 7 ] ~ ] ~}$
Out $[0=\{1,2,6,30,156,1455,11300\}$
The second key simple idea: We have de-coupled the problem of counting the subgroups of $S_{n}$ by separating the orbit types and the multiplicity of the orbit types. The orbits types are just the integer partitions of $n$, but the multiplicity of those orbit types depends on $S_{n}$. As is often the case, the recoupling calculation amounts to a dot product or more generally a matrix product.

We can formulate the above calculation as a theorem as follows
Theorem: If $a(n)$ is the number of subgroups of $S_{n}$, OEIS sequence A005432 then we have

$$
a(n)=\sum_{k=1}^{n} \sum_{P \in \operatorname{IntegerPartitions}[n,\{k\}]} m(n, P) c(n, P)
$$

Pf: Immediate by construction.

## Defining our sequence and computing it four different ways

First we make our key definition.
Definition: The sequence $t(n)$ is defined to be the number of distinct transitive subgroups of $S_{n}$ where we regard conjugate subgroups as distinct.

Remark: The transitive subgroups are precisely those with the simplest type of balanced orbit:
$\{\{1, \ldots, n\}\}$.
We can compute 7 terms easily directly from the definition as follows just by selecting out the groups whose orbit is the set $\{\{1,2, \ldots, n\}\}$ and counting, but this is very inefficient and uses too much memory to go any further.

```
In[v]:= Table[Length@Select[allSubgroupsArray[[n]], GroupOrbits[#, Range[n]] == {Range[n]} &],
```

    \{n, 1, 7\}]
    Out $[0]=\{1,1,2,9,20,279,512\}$

Getting back to the two components in the output of orbitSpectrum, we fully understand the second component and are therefore really interested in understanding the numbers in the first component which appears, at first glance, to be a hot mess. Upon further review however we see that this list of lists in the first component are partially recursive. Moreover the initial numbers of these lists, $\{1,1,2,9,20,279,512\}$ look familiar and are particularly special in that the are the only numbers which occur in all subsequent rows. Although it is not central, we make this precise with the following

Theorem: $m(n+1,\{n, 1\})=m(n,\{n\})$
Pf: $m(n+1,\{n, 1\})$ is the number of subgroups of $S_{n+1}$ with orbits $\{\{1,2,3, \ldots, \hat{j}, \ldots, n+1\},\{j\}\}$ where the "carat" above the $j$ means we omit this number so $j$ is fixed by the subgroup. By symmetry it is clear that these numbers are all the same and equal to the number of subgroups of $S_{n}$ with orbit equal to $\{1,2,3, \ldots, n\}$ which is just $m(n,\{n\})$ so we're done.

On the other hand, the following simple result is quite central
Theorem: $t(n)=m(n,\{n\})$
Pf: $m(n,\{n\})$ is the number of subgroups whose orbit has a set partition of only 1 part, namely
$\{1, \ldots, n\}$, which is just $t(n)$, the number of transitive subgroups of $S_{n}$, where we consider conjugate subgroups distinct.
This gives us another way to compute the first 7 terms of $t(n)$ using the above theorem and our orbitSpectrum code as follows

Table[(First /@orbitSpectrum [n] [[1]]) [[1]], \{n, 1, 7\}]
Out[ 0$]=\{1,1,2,9,20,279,512\}$
This method also suffers from inefficiency and excessive memory usage, but it makes up for it conceptually. The sequence $t(n)$ has just been added as A337015 at the OEIS using the fourth calculation listed below but we will continue to refer to it as $t(n)$ in what follows. On the other hand, if we only count subgroups "up to conjugacy" then the number of transitive subgroups of $S_{n}$ is the sequence A002106 at the OEIS, the unlabeled version of A337015. The first 13 terms starting with the first term $n=1$ are $\{1,1,2,5,5,16,7,50,34,45,8,301,9\}$. The relatively late addition of A337015 to the OEIS may result from the fact that groups theorists are often interested in groups up to conjugation, and GAP even tells that you might want to just look conjugacy classes when you ask for all subgroups due to memory constraints.

Which brings us to a computational problem: we would like more than 7 terms of $t(n)$. Unfortunately, as amazing and versatile as Mathematica is, we now again require the more specialized GAP software, and not just to produce lists of subgroups. The situation is even worse than we imagined and merely using GAP won't save us. As we mentioned if you try to consider all the subgroups of $S_{n}$, GAP will start throwing warning messages that you are using lot's of memory, and at $S_{9}$ I did run out of memory so we need a workaround.

The third key simple idea: Since we are interested in the transitive subgroups of $S_{n}$, and the property of being a transitive is invariant under conjugation, we really only need to consider conjugation classes
of subgroups and we implement this idea in the following GAP program

```
transitiveSubgroupsCount := function(n)
    local numTransitiveSubgroups,G,cc,class;
    numTransitiveSubgroups := 0;
    G := SymmetricGroup(n);
    cc:=ConjugacyClassesSubgroups(G);
    for class in cc do
        if IsTransitive(Representative(class),[1..n]) then
                numTransitiveSubgroups := numTransitiveSubgroups + Size(class);
            fi;
    od;
return numTransitiveSubgroups;
end;
```

The code just goes up to each class, chooses a representative to check if it's a transitive subgroup, and, if so, just adds the size of the class to an accumulator variable numTransitiveSubgroups. Unfortunately, this code only gave us 13 values for $t(n)=$ transitiveSubgroupsCount $(n)$. At $n=14$ my computer threw out a memory error. So we need a better algorithm.

The fourth key simple idea: Get help from an expert! Fortunately, Alexander Hulpke sent me the following GAP code which gives us our fourth and by far the best way to calculate $t$ and I was able to get up to $n \leq 31$ using it but we will only use the first 18 values since this is all we know about A005432 which has become the new bottleneck.

```
NrTransSubSn:=function(n)
local s,cnt,i,u,no;
    s:=SymmetricGroup(n);
cnt:=0;
for i in [1..NrTransitiveGroups(n)] do
    u:=TransitiveGroup(n,i);
    no:=Normalizer(s,u);
    cnt:=cnt+IndexNC(s,no);
    Print("Class ",i,", found ",IndexNC(s,no)," new, total: ",cnt,"\n");
od;
return cnt;
end;
```

```
t = {1, 1, 2, 9, 20, 279, 512, 19 087, 71 602, 636 365, 1 517 042, 321965 982, 240 609 602,
    8809543 877, 144729615032, 26 818608209 252, 6603755558402, 2737593592637477} ;
```

This agrees with the output from transitiveSubgroupsCount and with what we found above using orbitSpectrum at the smaller values. Now we can use the above theorem to connect $a(n)$ and $t(n)$.

Corollary: If $a(n)$ is the number of subgroups of $S_{n}$, OEIS sequence A005432, and $t(n)$ is the number of transitive subgroups of $S_{n}$ then we have

$$
a(n)=t(n)+\sum_{k=2}^{n-1} \sum_{P \in \operatorname{IntegerPartitions[n,\{ k\} ]}} m(n, P) c(n, P)+1
$$

Pf: First we apply the above theorem

$$
\begin{gathered}
a(n) \\
\sum_{k=1}^{n} \sum_{P \in \text { IntegerPartitions[n,\{k\}]}} m(n, P) c(m, P)
\end{gathered}
$$

Now we peel off the first and last terms.

$$
\begin{aligned}
& \sum_{P \in \operatorname{IntegerPartitions}[p,\{1\}]} m(n, P) c(n, P)+ \\
& \quad \sum_{k=2}^{n-1} \sum_{P \in \operatorname{IntegerPartitions}[p,\{k\}]} m(n, P) c(n, P)+\sum_{P \in \operatorname{lntegerPartitions[n,\{ n\} ]}} m(n, P) c(n, P)
\end{aligned}
$$

The first and last terms simplify since there is only one integer partition type $P$ with 1 part, namely $\{n\}$, and only one integer partition type $P$ with $n$ parts, namely $\{1,1, \ldots, 1\}$.

$$
m(n,\{n\}) c(n,\{n\})+\sum_{k=2}^{n-1} \sum_{P \in \operatorname{IntegerPartitions}[n,\{k\}]} m(n, P) c(p, P)+m(n,\{1,1, \ldots, 1\}) c(n,\{1,1, \ldots, 1\})
$$

The first term simplifies further since $m(n,\{n\})=t(n)$ is the number of subgroups whose orbit has a set partition of only 1 part, namely $\{1, \ldots, n\}$, which is just $t(n)$, the number of transitive subgroups of $S_{n}$ and $c(n,\{n\})=1$, the number of set partitions of $[n]$ with 1 part.
Similarly, the last term simplifies further since $m(n,\{1,1, \ldots, 1\})=1$ is the number of subgroups whose orbit is a set partition with $n$ parts, namely $\{\{1\}, \ldots,\{n\}\}$ which is just the identity group and $c(n,\{1,1, \ldots, 1\})=1$, the number of set partitions of $[n]$ with $n$ parts.

$$
t(n)+\sum_{k=2}^{n-1} \sum_{P \in \operatorname{IntegerPartitions[n,\{ k\} ]}} m(n, P) c(n, P)+1
$$

## Focus on Primes

Let's step back for a moment and consider the second component, containing the $c(n, P)$ variables above. One can't help but notice that, with the exception of the 1 's, the prime rows are divisible by the prime in question. This was only the first 7 rows, so the only prime rows were $2,3,5$, and 7 which is a bit paltry and it is the reason that we worked hard to gather more data. Unfortunately, the primes are infrequent as somewhat quantified by the prime number theorem and we only get three more primes, 11,13 , and 17 for all of our effort :-( but that's better than nothing

```
In[f]:= PrimePi[17]
```

Out [0] $=7$
$\ln [-\rho:=$ Prime [Range[7]]
Out[ 0$]=\{2,3,5,7,11,13,17\}$
We now consider $a(p) \bmod p$ and $t(p) \bmod p$ for the sequences $a$ and $t$ occurring at prime positions we notice the following
$\ln \left[{ }^{2}\right]=$ MapThread[Mod[\#1, \#2] \&, \{a[[Prime[Range[7]]]], Prime[Range[7]]\}]
Out $[0=\{0,0,1,2,0,0,14\}$
$\operatorname{In}[\cdot]:=$ MapThread[Mod[\#1, \#2] \&, \{t[[Prime[Range[7]]]] + 1, Prime[Range[7]]\}]
Out $[0=\{0,0,1,2,0,0,14\}$
Despite only having 7 examples, it's seems worth pursuing and rewrite the above computation as follows
$\ln [\rho]=$ Table [Mod[a[[Prime[n]]]-t[[Prime[n]]], Prime[n]], \{n, Range[7]\}]
Out $[=0=\{1,1,1,1,1,1,1\}$
This leads us to "conjecture" the following
Corollary: If $a(n)$ is the number of subgroups of $S_{n}$, OEIS sequence A005432, and $t(n)$ is the number of transitive subgroups of $S_{n}$, then if $n=p$ is prime we have $a(p)-t(p) \equiv 1 \bmod p$

Pf: By the above corollary we have

$$
a(p)
$$

$$
t(p)+\sum_{k=2}^{p-1} \sum_{P \in \text { IntegerPartitions }[p,\{k]]} m(p, P) c(p, P)+1
$$

But we know that

$$
c(p, P)=\frac{p!}{\left(m_{1}!s_{1} m^{m_{1}}\right)\left(m_{2}!s_{2}!^{m_{2}}\right) \ldots\left(m_{r}!s_{!}!m^{\left(m_{1}\right)}\right.}
$$

where extracting out the unique integers from the integer partition $P=\left\{t_{1}, t_{2}, \ldots, t_{k}\right\}$ yields the set of distinct element $\left\{s_{1}, s_{2}, \ldots, s_{r}\right\}$ and we have $\sum_{i=1}^{r} s_{i} m_{i}=p$ and $\sum_{i=1}^{r} m_{i}=k$. But $2 \leq k \leq p-1$ implies that $s_{i}$ and $m_{i}<p$ which implies that $\left(m_{1}!s_{1}!^{m_{1}}\right)\left(m_{2}!s_{2}!^{m_{2}}\right) \ldots\left(m_{r}!s_{r}!^{m_{r}}\right)$ can't have a factor of $p$ and therefore $p \mid c(p, P)$. Therefore we have $a(p)-t(p) \equiv 1 \bmod p$ and we're done.
Note: This is essentially the same proof as the result $(a+b)^{p} \equiv a+b \bmod p$ "modulo" some details which is the standard $\bmod p$ result for the binomial coefficient triangle

Note: We can think of this result as a crude invariant for symmetric groups of prime degree or as a frankly insane necessity test for primality if we want.

Note: We know that the subgroup structure is much richer for symmetric groups of composite degree, but we may as well calculate this quantity for $1 \leq n \leq 18$
$\operatorname{In}[\rho]:=$ Table[Mod[a[[n]] -t[[n]], n], $\{n, 1,18\}]$
Out $[\cdot]=\{0,1,1,1,1,0,1,6,7,1,1,2,1,8,3,8,1,16\}$
This shows that the converse of the above theorem is certainly not true.
Note: Of course, all of this leads one to generalize and ask if given a group $G<S_{n}$ one should consider a quantity like the following or perhaps something even more general

$$
|\{H \mid H<G\}|-\mid\{H \mid H<G \text { and } H \text { is transitive }\} \mid \bmod n
$$

Note: We have to concede that the "conjecture" aspect is a bit phoney. The calculations we have presented are just what one could have done to arrive at this result. In fact, the result is obvious once you have written the number of all subgroups as a summation.

## A Problem/Conjecture

Recall that we mentioned the Pyber inequality earlier which implies that $\ln (a(n))=\Theta\left(n^{2}\right)$. We can roughly visualize this result as follows.
$\ln [\cdot]:=$ ListPlot[Log[a]]


It does indeed look approximately and slightly parabolic. This is purely heuristic, but following the above result, we can try and fit the data with a translated parabola as follows
$\ln [\cdot]:=\operatorname{Module}\left[\{p, q, r\}, p n^{2}+q n+r / . \operatorname{FindFit}\left[\log [a], p n^{2}+q n+r,\{p, q, r\}, n\right]\right]$
Out[ []$=-1.85601+1.07041 n+0.08108 n^{2}$
The coefficient of $n^{2}$ is roughly consistent with the Pyber inequality result since it is included in the following interval
$\ln [\theta]:=\log \left[\left\{2^{\frac{1}{16}}, 24^{\frac{1}{6}}\right\}\right] / / N$
Out[ []$=\{0.0433217,0.529676\}$

This begs the following
Question/Problem: What can be said about the asymptotics of $t$ and the relationship between $t$ and $a$ ?
First we consider $\ln (t)$ and plot
$\ln [\rho]:=\log [\mathrm{t}] / / \mathrm{N}$
Out[ $\}=\{0 ., 0 ., 0.693147,2.19722,2.99573,5.63121,6.23832,9.85676,11.1789$, $13.3635,14.2323,19.59,19.2987,22.8991,25.6981,30.9201,29.5187,35.5459\}$


Not quite as nice. Inspired by our theorem connecting $a$ and $t$ we consider the following

```
\(\ln [\cdot]:=\operatorname{ListPlot}\left[\log \left[\frac{\mathbf{a}}{\mathbf{t}}\right]\right.\), PlotRange \(\rightarrow\) All \(]\)
```



Again, a bit messy. Since $a$ is monotonic, and $t$ is not monotonic, neither of the above two graphs is monotonic. This second graph suggests however that for composite degree $n$, a higher proportion of the subgroups of $S_{n}$ are transitive. Like our previous theorem, we therefore expect that the behavior of the sequence at the primes is simpler.

## Focus on Primes Again

We see that focusing on prime degree leads to monotonically increasing growth for $t$.

In[f]:= ListPlot[Transpose[\{Prime[Range[7]], Log[t[[Prime[Range[7]]]]]\}]]


This also looks slightly parabolic so we can try and fit it as well.
$\ln [-]:=\operatorname{Module}\left[\{p, q, r\}, p \mathbf{n}^{2}+\mathbf{q n}+\mathbf{r} /\right.$. FindFit $[$
Transpose[\{Prime[Range[7]], $\log \left[t\left[\right.\right.$ Prime[Range[7]]]]]\}], $\left.\left.p n^{2}+q n+r,\{p, q, r\}, n\right]\right]$
Out $[0]=-2.10576+0.760588 n+0.06547 n^{2}$
We therefore expect that the difference of $\ln (a(p))$ and $\ln (t(p))$ should also be slightly parabolic looking. Of course, due to the rules of logarithms, this is the same as considering $\ln \left(\frac{a(p)}{t(p)}\right)$ as follows
$\ln \left[{ }^{2}\right]=\operatorname{data}=\operatorname{Transpose}[\{\operatorname{Prime}[\operatorname{Range[7]],\operatorname {Log}[a[[Prime[Range[7]]]]/t[[Prime[Range[7]]]]]//N\} ]}$
Out $[\cdot]=\{\{2,0.693147\},\{3,1.09861\},\{5,2.05412\}$, $\{7,3.09423\},\{11,5.58496\},\{13,6.59073\},\{17,9.85552\}\}$
$\operatorname{In}[\odot]:=$ ListPlot [data, PlotRange $\rightarrow$ All]


We expected monotonicity, but this almost looks like it is approximately asymptotically linear until $n=17!$ ? Let's do a fit.
$\ln [\rho]=\operatorname{Module}[\{\mathbf{q}, \mathbf{r}\}, \mathbf{q} \mathbf{p}+\mathbf{r} /$. FindFit[data, $\mathbf{q p + r , \{ q , r \} , p ] ]}$
Out $[0=-0.831103+0.599811 p$
Nothing jumps out about this. Since least squares regression is sensitive to outliers, and we assume
there is noise anyway, let's drop the last point which seemed to disrupt our trend.
Module[\{q, r\}, qp+r/. FindFit[data[[1; 6]] $, q p+r,\{q, r\}, p]]$
$-0.561316+0.548383 p$
This is suspiciously close to $\frac{1}{2} p-\frac{1}{2}$ which is much nicer to look at. Let's plot $\frac{1}{2} n-\frac{1}{2}$ along side the full dataset.
$\ln [\cdot]=\operatorname{Show}\left[\operatorname{Plot}\left[\frac{n-1}{2},\{n, 0,18\}, \operatorname{PlotStyle} \rightarrow \operatorname{Red}\right], \operatorname{ListPlot}\left[\log \left[\frac{a}{t}\right]\right]\right.$, PlotRange $\rightarrow$ All $]$


Note: the decision to consider $\frac{n-1}{2}$ really was driven largely by aesthetics and the fact that we liked the fact that $n=1$ gives 0 .

Ignoring $n=1$, the line separates the data into two types. Upon further perusal one can't help but notice that the the primes are above the line and composites are below the line!!?? We can confirm this as follows
$\operatorname{Table}\left[\left\{n, \operatorname{If}\left[\operatorname{PrimeQ}[n], \log \left[\frac{a[[n]]}{t[n]]}\right]>\frac{n-1}{2}, \log \left[\frac{a[[n]]}{t[[n]]}\right]<\frac{n-1}{2}\right]\right\},\{n, 2,18\}\right]$
Out $0=\{\{2$, True $\},\{3$, True $\},\{4$, True $\},\{5$, True $\},\{6$, True $\}$, \{7, True, , 8, True $\},\{9, \operatorname{True}\},\{10$, True $\},\{11$, True, , 12, True $\}$, \{13, True, , 14, True, , 15, True $, ~\{16, ~ T r u e\}, ~\{17, ~ T r u e ~\}, ~\{18, ~ T r u e ~\} ~\} ~$

In any case, all of the above suggests the following
Problem/Conjecture: Show that

$$
\ln \left(\frac{a(n)}{t(n)}\right)>\frac{n-1}{2} \text { if } n \text { is prime and } \ln \left(\frac{a(n)}{t(n)}\right)<\frac{n-1}{2} \text { if } n \text { is composite }
$$

Therefore the problem is to compute more terms of $a$ ( $t$ is more tractable) and find a counterexample, or provide a proof, or revise the conjecture to something you can prove in light of the additional data. I see no intuitive reason why this statement should be true but I am not an expert in this area.

Note: The above was actually my original conjecture but then, listening to the action-movie inspired drivel that rattles around my brain I said to myself something like "Fortune favors the bold!" and conjectured that

$$
\left\lfloor\ln \left(\frac{a(n)}{t(n)}\right)\right\rfloor \leq \frac{n-1}{2}
$$

with equality occurring for primes. This was based on $n \leq 13$. I hemmed and hawed, mentioned the need for more data and the concerning deviations above line for $p=11$ and 13 but I thought it could be true. Hindsight is 20-20 so, of course, in retrospect, this was a little nuts. Of course, the problem was that I didn't have enough data until A. Hulpke sent me his code and I confirmed that $\left\lfloor\ln \left(\frac{a(17)}{t(17)}\right)\right\rfloor==9 \neq 8$. Which brings us to a fascinating point: If I had had more data and only had the linear fit $-0.831103+0.599811 p$ to look at, I strongly suspect I would never have considered the line $\frac{1}{2} p-\frac{1}{2}$. In other words, I am highlighting the role of perseverance combined with sheer dumb luck in any potential discovery, original or not. Is this on par with "Oh wait, why is there no bacteria next to that mold?" No, of course not, but this kind of experiment and testing of hypotheses detective work happens in mathematics too and this should be made clear to students to offset the whole "math is a purely deductive science" point of view.
Note: Math is a tough game. 90\% of the conjectures you make will be wrong (unless you're Euler, Gauss, etc..), and $90 \%$ of the conjectures you make that are correct will have been discovered already. I made up those numbers up obviously. The lack of priority doesn't concern us however since we are more interested in trying to illustrate the process. In fact, as I mentioned before, it helps that I don't know this area for my objective as it is very hard to make realistic mistakes when you already know the answer and very easy otherwise.

## Possible Consequences?

In a previous draft of this paper, I threw all caution to the wind and proved some "consequences" based on the above faulty conjecture. I'm not a complete amateur so I was careful of course to say "if this conjecture is true then..." but I'm not dumb enough to do that again. This situation is sort of similar to the history of non-euclidean geometry when Saccheri rejected the 5th postulate and derived what he believed were contradictions when in fact he was just proving theorems in non-euclidean geometry. I on the other hand was deriving theorems and then (probably) just ignoring the contradictions which would have told me that my assumption, that the conjecture was true, was false. In other words "similar" was being used in the sense of "exactly the opposite."

Despite what I just wrote, here we go again! The following could be described "low hanging fruit" but it's still fun. My favorite and aesthetically pleasing possible consequence are as follows:

Theorem: If the above conjecture is true then $n$ is prime iff $\frac{t(n)}{a(n)}<\mathbb{e}^{\frac{1-n}{2}}$
Pf: Just algebra.
Note that we can interpret $\frac{t(n)}{a(n)}$ as the probability of a subgroup of $S_{n}$ being transitive.
Theorem: If $f(n)=\frac{t(n)}{a(n)} e^{\frac{n-1}{2}}$ AND the above conjecture is true we have

$$
f(n)<1 \text { if } n \text { is prime, } f(1)=1 \text {, and } f(n)>1 \text { if } n \text { is composite }
$$

Pf: Just algebra.
We like the way $n=1$ nicely fits in this modest reformulation. As apparently simple as the number 1 is, it is "rather slippery" to quote Dr. Lecter. Is it prime or composite? Neither. Yet this was a point of some controversy in the past. In fact, the 1995 print edition of the OEIS says that a few sequences had to be changed from the 1973 edition since 1 was no longer regarded as prime! It is kind of amazing that this got settled so relatively recently.

We can also visualize this possible consequence as follows
Show[Plot [1, \{n, 0, 18.1\}, PlotStyle $\rightarrow$ Red, PlotRange $\rightarrow$ All], ListPlot $\left[\right.$ Table $\left[\left\{n, \frac{t[[n]]}{a[[n]]} e^{\frac{n-1}{2}}\right\},\{n, 1,18\}\right]$, PlotRange $\left.\left.\rightarrow\{\{0,18.1\},\{-.1, .6\}\}\right]\right]$


This nicely illustrates how the output $f$ for the primes seem to be below 1 , for the composites above 1 , and 1 is the unique point on the line. However, one can't help but notice a possible hint of collinearity for $n=4,8$, and 16 ??! Not only that, but when we ran the Hulpke code, the increasing powers of 2 were increasingly time consuming. Normally, I would say, "more data required" but A. Hulpke already told me that $n=32$ is awful so I know my laptop isn't up to it. I have no idea how the underlying functions work in his code, but primes go quickly, and powers of 2 are bad.

## What about a Proof?

Initially, I was hopeful, since $S_{p}$ for prime $p$ is quite special. For example, if I have understood things correctly, it turns out that the transitive subgroups of $S_{p}$ are the ones whose order is divisible by $p$ as the following calculation illustrates

Table[\{Length@Select[allSubgroupsArray[[n]], GroupOrbits[\#, Range[n]] == \{Range[n]\} \&], Length@Select[allSubgroupsArray[[n]], Divisible[GroupOrder[\#], n] \&]\}, \{n, 1, 7\}]

Out $[\cdot=\{\{1,1\},\{1,1\},\{2,2\},\{9,12\},\{20,20\},\{279,666\},\{512,512\}\}$
One might think that a proof to the problem/conjecture would follow readily. Incidentally, the divisibility result follows from the orbit stabilizer theorem and Cauchy's theorem and I found it in an example from some extremely well written free online notes from Keith Conrad about transitive group actions.

Also the number of Sylow $p$ - subgroups of $S_{p}$ is $(p-2)$ !. This follows from a simple counting argument I found on Math stackexchange in an answer from Derek Holt. Unfortunately, or perhaps fortunately, I then made the mistake of doing some additional googling to see what is known about counting subgroups and finally downloaded the papers by L. Pyber, L. Pyber and A. Shalev, and A. Lubotzky. After looking at them -- I won't even call it reading -- I just said to myself, "What the heck is all of this?!", and decided to pass the problem of computing additional terms and looking into the above problem/conjecture on to the mathematical community writ large. It was quite liberating since I am usually quite stubborn. Anyway, I think my wife was relieved.

## Addendum

I couldn't let this go entirely after all. Perhaps we can go at this "bare handed" as follows
Perhaps we can go at this "bare handed" as follows

$$
\begin{gathered}
a(n)=\sum_{k=1}^{n} \sum_{P \in \operatorname{lntegerPartitions[n,\{ k\} ]}} m(n, P) c(n, P) \\
a(n)=t(n)+\sum_{k=2}^{n-1} \sum_{P \in \operatorname{IntegerPartitions}[n,\{k]]} m(n, P) c(n, P)+1 \\
\frac{a(n)}{t(n)}=1+\frac{1}{t(n)} \sum_{k=2}^{n-1} \sum_{P \in \ln \text { negerPartitions }[n,\{k]]} m(n, P) c(n, P)+\frac{1}{t(n)}
\end{gathered}
$$

It is clear that the number of partitions of a number is related to the "compositeness" of the number, but I suspect that a more profitable approach might involve some as yet unbeknownst to me transform methods? It is not clear how to deal with $m(n, P)$.

## A Final Comment

This paper is actually a draft of the last chapter of my somewhat odd "magnum opus" book I have been writing. One reviewer wrote, "Erickson has created a genuine masterpiece in the re-integration of mathematics and math education!". That was me obviously. The book contains plenty of mathematics, but the primary point was actually just to make the case for a discovery, experimental, conjecture, and even recreational driven approach to learning and teaching mathematics along the lines of Polya, Ross, and even Moore (minus all of his racism obviously). In any case, the precise definitions, theorems, proofs, and conjectures were rather incidental. The idea for writing the book was initially a reaction to various trends in math education which I find extremely problematic and troubling, but I confess it also just became a way to keep busy during the pandemic. I often amused myself by saying that "This situation is a lot like when Sir Isaac left for the countryside to avoid the plague and write the Principia if you ignore the fact that I'm not Sir Issac and this ain't the Principia!" Also, I never left Chicago.

## Acknowledgements

Finally, I think it is important to acknowledge our inspirations, sources, and our all important tools. My interest in integer sequences and their patterns goes all the way back to my high school teacher Steve

Viktora. He had us do one of his famous home made "take home tests" with all sorts of integer sequences, series, and induction problems in what used to be called a "college algebra" course. It really captured my imagination and seeing the essence of mathematical patterns in the form of integer sequences has been a favorite topic ever since. This makes it all the more amusing that I didn't learn about the OEIS until 2018! Upon learning of its existence, my first reaction was to feel kind of cheated, but l've been making up for lost time. I would be remiss if I didn't mention the Ross program as another pre-college positive influence on my mathematical education. I learned to love group theory from Bhama Srinivasan as an undergraduate, but l'm obviously not an expert. Regarding tools, this work relies extremely heavily on Mathematica, GAP, and the OEIS. In particular, the OEIS reference to sequence A005432 contained the Pyber inequality for sequence A005432 which made us wonder what could be said about the asymptotics of $t(n)$. At some point, due to lack of sufficient data, the investigation ground to halt. Out of the blue, I received an email from Alexander Hulpke which enabled me to compute more terms of $t$ and get back to it. I am quite grateful. Finally, I also want to thank Jean-Paul Allouche for helpful comments on an earlier draft.

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