A note on A336266

Peter Bala, March 24 2024

Let u(n) = (2*n + 1)*(4*n^2 + 4*n + 3)/3 = A057813(n).

Using Maple's sum command we can verify the identity

A336266 = 3*Pi/16 = 1/2 + Sum_{k >= 0} (-1)^k/(u(k)*u(k+1)).

We show how the alternating series can be converted to the following continued fraction expansion:

3*Pi/16

= 1/(11 + 3/(12 + 15/(12 + ... + (4*n^2 - 1)/(12 + ...)))).

It turns out that this result is a particular case of a more general result described below.

First, we define two sequences {A(n): n >= 0} and {B(n): n >= 0} by

A(n) = B(n) * Sum_{k = 0..n} (-1)^k/(u(k)*u(k+1))

and

B(n) = (2*n + 1)!/(2^n * n!) * u(n+1).

It is easy to check that B(n) satisfies the 3-term recurrence

B(n) = 12*B(n-1) + (4*n^2 - 1)*B(n-2) ... (1)

for n > = 2, with initial conditions B(0) = 11, B(1) = 135.

With a little bit more work one can verify that A(n) satisfies the same recurrence

A(n) = 12*A(n-1) + (4*n^2 - 1)*A(n-2) ... (2)

for n >= 2, with initial conditions A(0) = 1, A(1) = 12.

By comparing (1) and (2) with the fundamental 3-term recurrences satisfied by the numerators and denominators of the convergents to a generalised continued fraction, we find that for n >= 1,

A(n)/B(n) = 1/(11 + 3/(12 + 15/(12 + ... + (4*n^2 - 1)/(12)))).

Letting n -> oo, we obtain

3*Pi/16 = 1/2 + Sum_{k >= 0} (-1)^k/(u(k)*u(k-1))

= 1/(11 + 3/(12 + 15/(12 + ... + (4*n^2 - 1)/(12 + ...)))).

This can be rearranged to give

(3*Pi - 8)/(3*Pi + 8) =

1/(12 + 3/(12 + 15/(12 + ... + (4*n^2 - 1)/(12 + ...)))).

Generalisation.

Note that u(n) = (2*n + 1)*(4*n^2 + 4*n + 3)/3 is the unique polynomial solution of degree 3 to the difference equation

(2*n + 1)*(u(n+1) - u(n-1)) = 12*u(n)

normalised so that u(0) = 1.

More generally, for r a positive integer, let u_r(n) denote the unique polynomial solution of degree r to the difference equation

(2*n + 1)*(u_r(n+1) - u_r(n-1)) = 4*r*u_r(n) ... (3)

normalised so that u_r(0) = 1.

The first few values are u_1(n) = 2*n + 1, u_2(n) = (2*n + 1)^2 and u_3(n) = (2*n + 1)*(4*n^2 + 4*n + 3)/3 (called u(n) above).

The polynomials u_r may be obtained from the coefficients of t^r in the power series expansion of

((1 + t)/(1 - t))^(x+1/2) = 1 + (2*x + 1)*t +

(2*x + 1)^2*t^2/2! + (8*x^3 + 12*x^2 + 10*x + 3)*t^3/3! +

(16*x^4 + 32*x^3 + 56*x^2 + 40*x + 9)*t^4/4! +

(32*x^5 + 80*x^4 + 240*x^3 + 280*x^2 + 178*x + 45)*t^5/5!

\+ ...

Thus u_r(x) is the Meixner polynomial of the first kind M_r(x + 1/2; 0, -1) and is given by the explicit formula u_r(x) =

(-1)^r*r!*Sum_{k = 0..r} binomial(x+1/2,k)*binomial(-x-1/2,n-k).

Define two sequences by

B_r(n) = (2*n + 1)!/(2^n * n!) * u_r(n+1)

A_r(n) = B_r(n) * Sum_{k >= 0} (-1)^k/(u_r(k)*u_r(k+1)).

It easily follows from (3) that B_r(n) satisfies the 3-term recurrence

B_r(n) = 4*r*B_r(n-1) + (4*n^2 - 1)*B_r(n-2)

for n >= 2.

Using this, one can show that A_r(n) satisfies the same recurrence

A_r(n) = 4*r*A_r(n-1) + (4*n^2 - 1)*A_r(n-2)

for n >= 2.

As in the above case r = 3, we can convert the series

Sum_{k >= 0} (-1)^k/(u_r(k)*u_r(k+1)) = lim_{n->oo} A_r(n)/B_r(n)

to a generalised continued fraction

1/(4*r +- 1 + 3/(4*r + 15/(4*r + ... + (4*n^2 - 1)/(4*r + ...))))

(where the +-1 choice depends on the parity of r).

The continued faction can be evaluated in terms of the constant Pi using results in Lorentzen and Waadeland, p. 586, equation 4.7.9 (for r even) and equation 4.7.10 (for r odd). We give some examples below.

Examples.

(i) r = 1: u_1(n) = (2*n + 1).

Sum_{k >= 0} (-1)^k/(u_1(k)*u_1(k+1))

= 1/(3 + 3/(4 + 15/(4 + ... + (4*n^2 - 1)/(4 + ...))))

= Pi/4 - 1/2,

which rearranges to

(Pi - 2)/(Pi + 2) =

1/(4 + 3/(4 + 15/(4 + ... + (4*n^2 - 1)/(4 + ...)))).

(ii) r = 2: u_2(n) = (2*n + 1)^2.

Sum_{k >= 0} (-1)^k/(u_2(k)*u_2(k+1))

= 1/(9 + 3/(8 + 15/(8 + ... + (4*n^2 - 1)/(8 + ...))))

= 1/2 - Pi/8,

which rearranges to

(4 - Pi)/(4 + Pi) =

1/(8 + 3/(8 + 15/(8 + ... + (4*n^2 - 1)/(8 + ...)))).

(iii) r = 3: u_3(n) = (2*n + 1)*(4*n^2 + 4*n + 3)/3.

Sum_{k >= 0} (-1)^k/(u_3(k)*u_3(k+1))

= 1/(11 + 3/(12 + 15/(12 + ... + (4*n^2 - 1)/(12 + ...))))

= 3*Pi/16 - 1/2,

which rearranges to

(3*Pi - 8)/(3*Pi + 8) =

1/(12 + 3/(12 + 15/(12 + ... + (4*n^2 - 1)/(12 + ...))))

(iv) r = 4: u_4(n) = (16*n^4 + 32*n^3 + 56*n^2 + 40*n + 9)/9.

Sum_{k >= 0} (-1)^k/(u_3(k)*u_3(k+1))

= 1/(17 + 3/(16 + 15/(16 + ... + (4*n^2 - 1)/(16 + ...))))

= 1/2 - 9*Pi/64,

which rearranges to

(32 - 9*Pi)/(32 + 9*Pi) =

1/(16 + 3/(16 + 15/(16 + ... + (4*n^2 - 1)/(16 + ...)))).

(v) r = 5:

u_5(n) = (32*n^5 + 80*n^4 + 240*n^3 + 280*n^2 + 178*n + 45)/45.

Sum_{k >= 0} (-1)^k/(u_5(k)*u_5(k+1))

= 1/(19 + 3/(20 + 15/(20 + ... + (4*n^2 - 1)/(20 + ...))))

= 45*Pi/256 - 1/2,

which rearranges to

(45*Pi - 128)/(45*Pi + 128) =

1/(20 + 3/(20 + 15/(20 + ... + (4*n^2 - 1)/(20 + ...)))).