Integer solutions of special periodic continued fractions

Statement:

The continued fraction

$\sqrt{r} = \left[m; [c, c, \ldots, c, 2m]\right]$ $with\ m > 0, c > 0\ and\ period\ p$ either has no integer solution for r or an infinite number of such solutions which are terms of a quadratic sequence.

Proof in 3 steps:

1) Introduction

Let r be a rational number and $x = \sqrt{r}$ and $m = \lfloor x \rfloor$.

Generally, x has a continued fraction with period p.

Example p=3: $x = [m; [c, c, 2m]]$:

$$x - m = \cfrac{1}{c + \cfrac{1}{c + \cfrac{1}{2m + \cfrac{1}{c + \cfrac{1}{c + \cfrac{1}{2m + \cdots}}}}}}$$

Shortcut for this continued fraction:

$$x - m = \cfrac{1}{c + \cfrac{1}{c + \cfrac{1}{2m + x - m}}}$$

or $x - m = f(p, c, x + m)\ with\ f(p, c, t) = \frac{1}{c + \frac{1}{c + \frac{1}{t}}}$

2) Recurrence for $p > 1$ and integer solutions for $p = 3$

$$f(1, c, t) = \frac{1}{t}, f(n, c, t) = \frac{1}{c + f(n-1, c, t)}, n \le p$$

In particular, for $p = 3$:

$$f(2, c, t) = \frac{t}{ct + 1} \qquad f(3, c, t) = \frac{ct + 1}{(c^2 + 1)t + c}$$

This yields

$$x - m = \frac{c(x + m) + 1}{(c^2 + 1)(x + m) + c}$$

which is a simple quadratic equation:

$$(c^2 + 1) \cdot (x^2 - m^2) - 2mc - 1 = 0$$

with the solution

$$(2.1)\ r = x^2 = m^2 + R(p,c,m)\ with\ R(p,c,m) = \frac{2mc+1}{c^2+1}, p = 3$$

We only accept solutions such that $R(p,c,m)$ is an integer.
If c is odd, there is no integer solution because c^2+1 is even and $2mc+1$ is odd. No integer has a square root with a continued fraction $[m;[1,1,2m]]$.

$c = 2$ with $R(3,2,m) = \frac{4m+1}{5}$ yields an integer solution for $m = 5k+1$ with $R(3,2,m) = 4k+1$ and $r = (5k+1)^2 + 4k + 1$.
This means that the integer solutions for $p = 3\ and\ c = 2$ are concentrated on a quadratic sequence.
$c = 4$ with $R(3,4,m) = \frac{8m+1}{17}$ yields $r = (17k+2)^2 + 8k + 1$.
This way we can generate quadratic sequences for p=3 and even values of c.

3) Integer solutions for any p>2

Lemma:
$f(p,c,t)$ has the general form $f(p,c,t) = \frac{a_p \cdot t + b_p}{e_p \cdot t + a_p}$ with $e_p = b_p + c \cdot a_p$.
Proof by induction
Base case: $f(2,c,t) = \frac{t}{ct+1}$ with $p = 2$: $a_2 = 1, b_2 = 0, e_p = c$
Induction step:

$$f(p+1,c,t) = \frac{1}{c + \frac{a_p \cdot t + b_p}{e_p \cdot t + a_p}} = \frac{e_p \cdot t + a_p}{\left((c^2+1)a_p + c \cdot b_p\right) \cdot t + e_p}$$

This is true for

$$a_{p+1} = e_p; \quad b_{p+1} = a_p;\ e_{p+1} = (c^2+1)a_p + c \cdot b_p = b_{p+1} + c \cdot a_{p+1}$$

General equation: $x - m = f(p,c,x+m) = \frac{a_p \cdot (x+m) + b_p}{e_p \cdot (x+m) + a_p}$

$$\text{or } e_p \cdot (x^2 - m^2) - 2m \cdot a_p - b_p = 0$$

with the solution

$$r = x^2 = m^2 + R(p,c,m);\ \ R(p,c,m) = \frac{2m \cdot a_p + b_p}{e_p}$$

If $q = \gcd(2a_p, e_p)$ is no divisor of b_p, then numerator and denominator are different modulo q for any m and there is no integer solution for r, see (2.1), example $R(3,1,m)$. Otherwise, according to the rules of elementary number theory, there is an infinite number of integer solutions for r like those described in the preceding chapter.
Herewith, the statement is proved.