A note on spanning trees

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Abstract

1 Introduction

Let $\tau_3(n)$ be the number of spanning trees in the 3rd power of a cycle of length $n$. We give two expressions for $\tau_3(n)$ in Section 1 and 2.

2 [1]

By [1, Theorem 1], we have the following:

Theorem 2.1 ([1, Theorem 1]). Let

$$ T(n, z) := \cos(n \arccos(z)) $$

$$ z_1 := \frac{-3 + \sqrt{-7}}{4}, z_2 := \frac{-3 - \sqrt{-7}}{4}. $$

Then we have the following:

$$ \tau_3(n) := \frac{2n}{7}(T(n, z_1) - 1)(T(n, z_2) - 1). $$

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Let $T(n)$ be the numbers in A005822. Then we have

**Theorem 3.1 ([2]).**

\[
\begin{align*}
\tau_3(n) &= 2nT(n)^2 \text{ if } n \text{ is even}, \\
\tau_3(n) &= nT(n)^2 \text{ if } n \text{ is odd}.
\end{align*}
\]

where

\[ T(n + 8) = 4T(n + 6) + T(n + 4) + 4T(n + 2) - T(n). \]

**Proof.** The proof is similar to the discussion in [2, p.347 Theorem 9]. Let

\[ f = 1 + 3x + 6x^2 + 3x^3 + x^4. \]

We denote by

\[ a_1, a_2 \]

its roots up to conjugate. Let

\[ a(n) := \frac{(1 - a_1^n)(1 - a_2^n)}{\sqrt{14}(a_1a_2)^n} \]

Then

\[ \tau_3(n) = na(n)^2 \]

and we have $a(n)$

\[ a(n + 4) = \sqrt{2}a(n + 3) + a(n + 2) + \sqrt{2}a(n + 1) - a(n) \]

Then we obtain the following:

\[ a(n + 8) = 4a(n + 6) + a(n + 4) + 4a(n + 2) - a(n). \]

It is easy to check that

\[
\begin{align*}
T(n) &= \sqrt{2}a(n) \text{ if } n \text{ is even}, \\
T(n) &= a(n) \text{ if } n \text{ is odd}.
\end{align*}
\]

\[ \square \]
Acknowledgments

The authors are supported by JSPS KAKENHI (18K03217).

References
