

# Euler primes and the number of such primes less than the given integer q

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**ABSTRACT.** The first part of the paper considers a simple method of evaluation of all so-called Euler primes of form  $x^2 + x + 41$  less than  $10^{2n}$ . Another part describes some estimates for the number of such primes less than the given integer  $q$ .

## (1) INTRODUCTION

The famous formula  $x^2 + x + 41$  (also known as the Euler polynomial, since Leonard Euler is credited to be its inventor<sup>i</sup>) produce prime numbers  $p$  such that  $p = x^2 + x + 41$  for  $x \in [0; 39]$ , while  $x$  is integer. In the original text (1772), the author wrote<sup>ii</sup>:

Cette progression 41. 43. 47. 53. 61. 71. 83. 97. 113. 131 &c. dont le terme général est  $41 - x + xx$ , est d'autant plus remarquable que les 40 premiers termes sont tous des nombres premiers.

(French)// The common formula for this progression 41. 43. 47. 53. 61. 71. 83. 97. 113. 131, etc. is  $41 - x + xx$ , and the most remarkable fact is that the first 40 terms are prime numbers.  
(Translation)

According to the Baker-Stark-Heegner theorem, we could explain, why the first 40 terms of the above-mentioned sequence are primes<sup>iii</sup> (hence, we could establish that  $x^2 + x + y$  is prime  $\forall x \in \mathbb{Z}^{\geq}: x \leq y - 2$  if and only if  $y = \{2, 3, 5, 11, 17, 41\}$ ). The current paper proposes a simple and unusual way to sieve all prime numbers of form  $x^2 + x + 41$  (at any  $x \in \mathbb{N}$ ).

## 2. ABOUT THE NEW ALGORITHM OF FINDING ALL EULER PRIMES LESS THAN $10^{2n}$

### *Notation and Definitions*

Symbol	Explanation	Examples and notes
$\mathbb{Z}^{\geq}, \mathbb{Z}^{+}$	the sets of integers: $\mathbb{Z}^{\geq}$ means $\{0,1,2,3,\dots\}$ , $\mathbb{Z}^{+}$ means $\{1,2,3,\dots\}$	
$\mathbb{P}$	the set of prime numbers	
$E$	the set of Euler numbers of form $x^2 + x + 41$ where $x \in \mathbb{Z}^{\geq}$	
$C$	the set of all composite $m \in E$ ; a subset of $E$	$E \supset C$
$L$	the set of all “left” or “smaller” (the smaller of the two) divisors for all composite $m \in E$	see <b>Theorem 1</b>
$\mathcal{L}$	the set of all possible terms of $L$ , including the “false” ones; $\mathcal{L} \supset L$	$1468 \in \mathcal{L}$ , but $1468 \notin L$ (see <b>Table 6-II in Appendix A</b> )
$\forall$	for all; for any; for each	
$E(q)$	the sequence of all terms of the given set (e.g., $E$ ) not exceeding $q$	$\forall m \in E(10^4), m < 10^4$
$\#E$	the cardinality of the given set	$\#E(10^4) = 100$ (see <b>Lemma 1</b> )
$\notin$	is not an element of the given set	$41 \in E$ , but $41 \notin C$
$\not\supset$	$X \not\supset Y$ means $X$ is not a superset for the set $Y$	
$\exists$	there exists	
$\exists!$	there exists exactly one	
$\nexists$	there doesn't exist	
:	such that	$\forall m \in E: m < 1681, m \in \mathbb{P}$
$ $	$d   c$ means $d$ divides $c$	$41   1681$
$\cdot$	$4 \cdot 9$ means the multiplication of 4 by 9	
$\Leftrightarrow$	$X \Leftrightarrow Y$ means $X$ is defined to be logically equivalent to $Y$	$41   1681 \Leftrightarrow 1681 = 41^2$
$\Rightarrow$	implies ( $X \Rightarrow Y$ means if $X$ is true then $Y$ is also true, but in general not vice versa)	$m \in E \Rightarrow m \in L$ (see <b>Lemma 3</b> )
$\therefore$	therefore; hence	
$\wedge$	logical conjunction: $X \wedge Y$ is true if $X$ and $Y$ are both true; otherwise, it is false	$\forall m \in C: m > 1681 \wedge m = l \cdot r, l < r$ , while $l$ and $r$ are integers (see <b>Lemma 2</b> )
$\because$	because	$41 \notin C \because 41 \in \mathbb{P}$
Q.E.D.	marks the end of the proof	
$l(i, j, t)$	the 3-dimensional matrix with $i \in \mathbb{Z}^{+}$ , $j \in \mathbb{Z}$ , and $t \leq \frac{i}{2}$ , while $i$ is even, $t \leq \frac{(i+1)}{2}$ , while $i$ is odd	see <b>Proposition 2</b> and <b>Appendices A-B</b>
$r(i, j, t, u)$	the 4-dimensional matrix with $i \in \mathbb{Z}^{+}$ , $j \in \mathbb{Z}$ , $t \leq \frac{i}{2}$ , while $i$ is even, $t \leq \frac{(i+1)}{2}$ , while $i$ is odd, and $u \in \mathbb{Z}^{\geq}$	see <b>Lemma 9</b> and <b>Appendices A-B</b>
$l_{i,j,t}$	the $j$ -th term of the sequence of $l(i, j, t)$	$l_{1,-1,1} = 41$ (see <b>Table 1 in Appendix A</b> )
$r_{i,j,t,u}$	the $j$ -th term of the sequence of $r(i, j, t, u)$	$r_{1,-1,1,0} = 43$ (see <b>Table 1 in Appendix A</b> )
$\pi_{Euler}(q)$	the number of all Euler primes $p \in \mathbb{P}: p \in E \wedge p \leq q$	$\pi_{Euler}(100) = 8$

We use the upper bound  $10^{2n}$  because it's easy to establish the number of all terms in  $E(10^{2n})$  – prime and composite ones:

**Lemma 1.**  $\forall n \in \mathbb{N}: n \geq 2, \#E(10^{2n}) = 10^n$ :

$$(1) \quad \#E(10^{2n}) = 10^n \quad (n \in \mathbb{N}, n \geq 2).$$

**Proof.** Trivial.

Any composite number could be written as a product of at least two integers; therefore, we may write any composite  $m \in E(10^{2n})$  as  $m = l \cdot r$ , where  $l$  and  $r$  are integer numbers,  $41 \leq l \leq r$ .

In fact, there is only one integer square in  $E$ :

**Lemma 2.**  $\exists! q \in \mathbb{N}: q = \sqrt{m}, m \in E \Leftrightarrow \sqrt{m} \in \mathbb{N} \wedge m \in E \therefore m = 1681 = 41^2$ .

**Proof.** Let  $S = \{s: s = x^2 + x + y, x \in \mathbb{N}, y \in \mathbb{N}\}$ .

$$\forall s: s = x^2 + x + y,$$

$$(2) \quad \sqrt{s} = \begin{cases} z \in \mathbb{R}; & x < y \text{ (i)} \\ v \in \mathbb{R}, v \notin \mathbb{Q}; & x = y \text{ (ii)} \\ v \in \mathbb{R}, v \notin \mathbb{Q}; & x > y \text{ (iii)} \end{cases}$$

(trivial)

Hence,  $\forall \sqrt{s} \in \mathbb{N}, x < y$ .

There are at least two trivial cases such that  $x < y$  and  $\sqrt{s} \in \mathbb{N}$ :

$$(3) \quad \sqrt{s} = \begin{cases} z \in \mathbb{N}; & y = x + 1 \text{ (i)} \\ z \in \mathbb{N}; & y = 3(x + 2) - 2 \text{ (ii)} \end{cases}$$

In fact, (it may be referenced as a separate Theorem, though we have no need to do so in the current paper),

$\sqrt{s} \in \mathbb{Z}^+$  if  $\exists d \in \mathbb{Z}^+: w = 4d - 2, v = w \pm 1$ , and

$$y = \begin{cases} v(x + d) - (d - 1), & v = w - 1 \\ v(x + d) + d, & v = w + 1 \end{cases} \quad (\text{trivial})$$

At  $y = 41$ ,  $\exists! x < y: \sqrt{s} \in \mathbb{N} \Leftrightarrow \sqrt{s} \in \mathbb{N}: y = 41 \Rightarrow \sqrt{s} = 41 = y \wedge x = 40$  (trivial), Q.E.D.

**Theorem 1.**  $\forall m \in E: m = l \cdot r$  ( $l \in \mathbb{N}, r \in \mathbb{N}$ ,  $41 \leq l \leq r$ ,  $l = r$  if and only if  $r = 41$ ),  $\exists m' \in E: l = m' \cdot i^2 \pm k$ ,  $m' \leq \sqrt{m}$  ( $m' = l = \sqrt{m}$  if and only if  $r = 41$ ),  $i \in \mathbb{Z}^+$ ,  $k \in \mathbb{Z}^\geq$ .  $\forall p \in \mathbb{P}: \exists s' \in E: p \mid s'$ ,  $p \in L = \{l: \exists w \in E: w = l \cdot r\}$ .

**Lemma 3.**  $\forall s = m_j \cdot m_{j+1}: m_j$  and  $m_{j+1}$  are two consecutive terms of  $E$ ,  $s \in E$ .

*Proof.*  $\forall s = l \cdot r: l$  and  $r$  – two consecutive terms of  $E$  (e.g.,  $l = m_j = 43$  and  $r = m_{j+1} = 47$ ),  $s \in E$  (trivial),  $l = m' \cdot 1 + 0$ , where  $m' \in E$ ,  $m' < \sqrt{s}$  ( $m' = \sqrt{s}$  only at  $r = 41$ , as follows from **Lemma 2**). Thus,  $l = 41 = 41 \cdot 1 + 0$ ,  $l = 43 = 43 \cdot 1 + 0$ ,  $l = 47 = 47 \cdot 1 + 0$ , etc.:

$$(4) \quad l_j = m_j, \text{ while } i = 1$$

$\forall j \geq 0: i = 1$  (here,  $s = l \cdot r_u$ ,  $s \in E$ ,  $l = m_j$ ;  $r_{-1} = 41$  and is defined only at  $l = 41$ , otherwise,  $u \in \mathbb{Z}^\geq$ ),

$$(5) \quad r_0 = m_{j+1} = 41 + (j+1)(j+2);$$

$$(6) \quad r_u = m_j \left( \frac{u+3}{2} \right)^2 + \left( \frac{u+1}{-2} \right) - \left( \frac{u+3}{2} \right) \cdot 2j, \text{ while } u \text{ is odd, } u > 0;$$

$$(7) \quad r_u = m_j \left( \frac{u+2}{2} \right)^2 + \left( \frac{u}{2} + 2 \right) + \left( \frac{u+2}{2} \right) \cdot 2j, \text{ while } u \text{ is even, } u > 0$$

(see Table 1, for details).

**Lemma 4.**  $\forall s = l_{2,j,1} \cdot r_{2,j,1,u}, s \in E$ .

*Proof.* Trivial, since,  $\forall j \geq -1: i = 2$  (here,  $s = l \cdot r_u$ ,  $s \in E$ ,  $m_{-1} = 41$ , whereas  $m_0 = 41$ ,  $m_1 = 43$ ,  $m_2 = 47$ , etc.;  $u \in \mathbb{Z}^\geq$ ),

$$(8) \quad l_j = m_j \cdot 2^2 + 3 + 2^2 j, \text{ while } i = 2;$$

$$(9) \quad r_u = m_j (u+3)^2 - \left( \left( \frac{u}{2} + 1 \right) \left( \frac{u}{2} + 2 \right) - (u+2)(u+3)(j+1) \right), \text{ while } u \text{ is even, } u \geq 0, i = 2;$$

$$(10) \quad r_u = m_j (u+2)^2 - \left( \left( \frac{(u-1)}{2} + 1 \right) \left( \frac{(u-1)}{2} + 2 \right) - (u+2)(u+3)(j+1) \right), \text{ while } i = 2, u \text{ is odd, } u \geq 1 \text{ (thus, if } u \text{ is even, at } j = -1, r_u = r_{u+1});$$

(see Table 2, for details).

**Proposition 1.** (see **Lemma 9**)

However, we don't need to find  $r_u$ , assuming that  $\forall s \in E: s = l \cdot r, r \in L$ .

**Proposition 2.** (see **Lemma 7**)

Let  $i \geq 3$ . We could define a number of ‘dimensions’ (or rather ‘tablets’; see below for details) for any  $i$ : denote any tablet with  $t$ -index;  $t \in \mathbb{N}$ ,  $t \geq 1$ ,  $T$  – the set of all tablets for the given  $i$ . Suppose that, for any  $i$ , we have exactly  $\frac{i+1}{2}$  tablets, while  $i$  is odd, and  $\frac{i}{2}$  tablets, while  $i$  is even (so, we have two tablets at  $i = 3$ :  $t = 1$  and  $t = 2$ ):

$$(11) \quad \#T = \frac{i+1}{2}, \text{ while } i \text{ is odd} ;$$

$$(12) \quad \#T = \frac{i}{2}, \text{ while } i \text{ is even} ;$$

Any tablet we could define as a unique set of  $l$ -factors for the given  $i$ . Though, we have to note that, at any  $i$ , the first tablet contains all  $l$ -factors from the last one and vice versa; so,  $l(4, j, 1) = l(4, j, 2)$ ,  $l(5, j, 1) = l(5, j, 3)$ , etc. (see Appendix A for details)

At  $t = 1$ , we have the following  $j$ -indices of  $l$ -factors (as well as  $m$ -terms), while  $i \geq 3$ :  $-1; 1; -2; 2; -3; 3$ ; etc., and the following  $m'$ -terms (or initial  $m$ -terms):  $41; 41; 43; 43; 47; 47$ ; etc.:

(13) Sample of  $j$ -indices at  $t = 1$  for  $i \geq 3$

$j$	$m'$
-1	41
1	41
-2	43
2	43
-3	47
3	47

(see Table 3-I, for details)

At  $t > 1$ , we have the following  $j$ -indices of  $l$ -factors (as well as  $m$ -terms), while  $i \geq 3$ :  $-1; 0; -2; 1; -3; 2$ ; etc., and the following  $m'$ -terms (or initial  $m$ -terms):  $41; 41; 43; 43; 47; 47$ ; etc.:

(14) Sample of  $j$ -indices at  $t > 1$  for  $i \geq 3$

$j$	$m'$
-1	41
0	41
-2	43
1	43
-3	47
2	47

(see Table 3-II, for details)

(Of course, the order could be changed; I used such indices to make the below-mentioned formulas and tables more convenient).

**Lemma 5.**  $\forall i \geq 3$ , any  $l_{i,j,t} \in \mathcal{L}$  could be written as a function of  $m_j \in E$ .

**Proof.** Trivial, since, at  $t = 1$  and  $i \geq 3$ , any  $l_j$  could be evaluated as follows:

$$(15) \quad l_j = m_j \cdot i^2 + (2i(j+1) - (i-1)), \text{ while } j < 0 ;$$

$$(16) \quad l_j = m_j \cdot i^2 + i + 1 + 2i(j-1), \text{ while } j > 0 ;$$

At  $t > 1$  and  $i \geq 3$ , any  $l_j$  could be evaluated as follows:

$$(17) \quad l_j = m_j \cdot i^2 + b + ja, \text{ where } a = (i^2 - 2i) - 2i(t-2),$$

$$b = a - \left( \frac{i^2-4}{4} - ((t-2)^2 + 2(t-2)) \right), \text{ while } i \text{ is even};$$

$$(18) \quad l_j = m_j \cdot i^2 + b + ja, \text{ where } a = (i^2 - i) - 2i(t-2),$$

$$b = a - \left( \frac{i^2-1}{4} - ((t-2)^2 + (t-2)) \right), \text{ while } i \text{ is odd}.$$

**Lemma 6.**  $\forall s \in E: s = l \cdot r$ , both  $l$  and  $r$  are squarefree.

**Proof.** As follows from Lemmas 2-5 and their Proofs,

$$(19) \quad \forall a: l_j \in \mathcal{L} \wedge l_j = m_j \cdot i^2 + b + ja, \nexists d \in \mathbb{Z}^+: d > 1 \wedge d \mid (m_j \cdot i^2) \wedge d \mid b, \text{ Q.E.D.}$$

**Lemma 7.**  $\forall i: l(i,j,t) \subset \mathcal{L}, t \leq \frac{i}{2}$ , while  $i$  is even,  $t \leq \frac{i+1}{2}$ , while  $i$  is odd;  $t \in \mathbb{Z}^+$ .

**Proof.** Let  $k_j = b + ja$  (see Proof of Lemma 5). Then

$$\forall i > 2: t = 1 \text{ and } j = -1, l_{-1} = m_{-1} \cdot i^2 - (i-1) \therefore k_{-1} = (i-1);$$

$$\forall i > 2: t = 1 \text{ and } j = 1, k_1 = 2i - k_{-1}, l_1 = m_1 \cdot i^2 + k_1;$$

$$\forall i > 2: t = 1 \text{ and } j < -1, k_j = k_{j+1} + 2i, l_j = m_j \cdot i^2 - k_j;$$

$$\forall i > 2: t = 1 \text{ and } j > 1, k_j = k_{j-1} + 2i, l_j = m_j \cdot i^2 + k_j;$$

$\forall i > 2: t = 2, a = i^2 - 2i$ , while  $i$  is even;  $a = i^2 - i$ , while  $i$  is odd (see formulas (17), (18));

$$\forall i > 2: t = 3, a = i^2 - 4i, \text{ while } i \text{ is even, } a = i^2 - 3i, \text{ while } i \text{ is odd;}$$

$$\forall i > 2: t = 4, a = i^2 - 6i, \text{ while } i \text{ is even, } a = i^2 - 5i, \text{ while } i \text{ is odd; etc. etc.}$$

Let  $i = 3$ . Suppose, that we have at least 3 tablets at  $i = 3$ , with following indices:  $t = 1, t = 2$ , and  $t = 3$ . At  $t = 2$ ,  $a = 6$  and  $b = 4 \Leftrightarrow k_{-1} = 4 + (-1 \cdot 6) = 2$ ; but the same values ( $k_{-1} = 3 - 1 = 2$  and  $a = 2i = 2 \cdot 3 = 6$ ) we have at  $t = 1$ .

And at  $t = 3$ ,  $a = (3^2 - 3) - 2 \cdot 3(3 - 2) = 0$ ,  $b = 0 - (2 - (1^2 + 1)) = 0 \therefore$   
 $\forall \text{ odd } i \geq 3: t = \frac{i+1}{2} + 1, a = 0 \text{ and } b = 0 \Leftrightarrow s_j \notin E$  (see also **Lemmas 6 and 8**)

$\forall \text{ even } i \geq 4: t = \frac{i}{2} + 1, a = 0 \text{ and } b = 0 \Leftrightarrow s_j \notin E$

Thus,  $\nexists i: t = \frac{i+1}{2} + 1 \wedge l(i, j, t) \in L$ , while  $i$  is odd, and  $\nexists i: t = \frac{i}{2} + 1 \wedge l(i, j, t) \in L$ , while  $i$  is even, Q.E.D.

**Lemma 8.**  $\forall p \in \mathbb{P}: \exists s \in E \wedge p \mid s, p \in L$ .

$\forall i > 0$ , at any  $t$ , while  $\exists d \in \mathbb{N}: d > 1 \wedge d \mid i \wedge d \mid a \wedge d \mid k_{-1}$  (see formulas (17) and (18), where  $ja = k_j$ ) and  $\nexists m' \in E: d \mid m' \wedge m' < s_{-1}$ , for the given  $i$  and  $t$ ,  $\nexists l \in L$  and  $\nexists s \in E$ . Hence,  $\forall p \in \mathbb{P}: \exists s' \in E \wedge p \mid s', p \in L$ :

(20)  $\forall t: \exists d \in \mathbb{N} (d > 1; d \mid i \wedge d \mid a \wedge d \mid k_{-1}; \nexists m' \in E: d \mid m' \wedge m' < s_{-1}),$   
 $\nexists l \in L$  and  $\nexists s \in E \therefore \forall p \in \mathbb{P}: \exists s' \in E \wedge p \mid s', p \in L$  (trivial)

In other words, while the given tablet  $l(i, j, t) \subset \mathcal{L}$ ,  $\forall p \in \mathbb{P}: p \mid i \wedge p \notin L$ ,  
 $L \not\supset l(i, j, t)$  if and only if

$$t = v \bmod p, \text{ where } v = \begin{cases} \frac{(3+p)}{2}, & i \text{ and } p \text{ are odd, } i \geq 3 \text{ (i)} \\ \frac{(2p+2)}{2}, & i \text{ is even, } i \geq 4, p > 2 \text{ (ii)} \\ 3, & i \equiv 0 \pmod{4}, i \geq 4, p = 2 \text{ (iii)} \\ 0, & i \equiv 2 \pmod{4}, i \geq 4, p = 2 \text{ (iv)} \end{cases}$$

(21) (trivial; see Table 6-II and Appendix B for details)

According to the formulas mentioned in the current Proof of **Theorem 1**, we could establish that  $\forall s \in E: s = l \cdot r$ ,  $r \in L$  (here we assume that, for any set  $l(i, j, t)$  and for any set  $r(i, j, t, u)$ , we use the same  $j$ -indices, e.g.: -1,0,1,2,3,4,5 etc.):

**Lemma 9.**  $\forall s \in E: s = l \cdot r \wedge l \in L, r \in L$ .

**Proof.** Let  $r'(i', j, t', u')$  be the  $r$ -tablet such that the  $x$ -index of its first term (or  $x'$ )  $x'(r'_{i', -1, t', u'}) \geq 122 \therefore 122^2 + 122 + 41 = 15047 \wedge l(3, j, 1) = r'(1, j, 1, 3) \cup r'(1, j, 1, 4)$  (see Table 1, Appendix A; Table II, Appendix B). Then  $\exists l(i, j, t): i \geq 3, x(l_{i, -1, t}) = x' + r'_{i', -1, t', u'} \wedge \exists r''(i'', j, t'', u''): x''(r''_{i'', -1, t'', u''}) = x - 2x' - 1 \wedge r'_{i', -1, t', u'} = r''_{i'', -1, t'', u''} = l_{i, -1, t}$  (here,  $x = x(l_{i, -1, t})$ ,  $x' = x'(r'_{i', -1, t', u'})$ ,  $x'' = x''(r''_{i'', -1, t'', u''})$ ), and  $i' < i'' < i$ :

$$(22) \quad x = x' + r'_{i', -1, t', u'} ;$$

- $$(23) \quad x'' = x - 2x' - 1; x = 2x' + x'' + 1;$$
- $$(24) \quad r'_{i',-1,t',u'} = r''_{i'',-1,t'',u''} = l_{i,-1,t};$$
- $$(25) \quad i' < i'' < i$$

(trivial)

And, as follows from the formulas (22)-(25),  $\forall l(i,j,t): i \geq 3, \exists r'(i',j,t',u') \wedge \exists r''(i'',j,t'',u''): l(i,j,t) = r'(i',j,t',u') = r''(i'',j,t'',u'')$ , Q.E.D.  
(for details, see Table 1, Appendix A, and Table II, Appendix B)

According to Lemmas 1-9, Theorem 1 is proved.

We could rewrite our formulas for  $l$ -factors and  $j$ -indices (for any integer  $i > 2$ ) as follows (use sample (14) for  $j$ -indices):

$$l_{(i,t+1,j)} = m_j \cdot i^2 + b + ja, \text{ where } i, t, \text{ and } j \text{ are integers (}j \text{ could be negative), } i > 0; a = (i^2 - 2i) - 2i(t-1), b = a - \left( \frac{i^2-4}{4} - ((t-1)^2 + 2(t-1)) \right), 0 < t < (\frac{i}{2}), \text{ while } i \text{ is even; } a = (i^2 - i) - 2i(t-1), b = a - \left( \frac{i^2-1}{4} - ((t-1)^2 + (t-1)) \right), 0 < t < (\frac{i+1}{2}), \text{ while } i \text{ is odd (Note: } l_{(i,1,j)} = l_{(i,\frac{t}{2},j)}, \text{ while } i \text{ is even; } l_{(i,1,j)} = l_{(i,\frac{t+1}{2},j)}, \text{ while } i \text{ is odd).}$$

(26)

### 3. SOME QUANTITATIVE RESULTS

By the above-mentioned method of sieve, we could establish that, for  $E(10^6)$ , we have exactly 419 composite terms and 581 primes:

$$\pi_{\text{Euler}}(10^6) = \#E(10^6) - \#C(10^6) = 10^3 - 419 = 581, \text{ since}$$

$$(27) \quad \pi_{\text{Euler}}(10^{2n}) = \#E(10^{2n}) - \#C(10^{2n}) \quad (\text{trivial})$$

and  $\#C(10^{2n})$  could be evaluated with formulas (4), (8), (17), and (18).

For  $E(10^8)$ , we have 5851 composite terms and 4149 primes:

$$\pi_{\text{Euler}}(10^8) = \#E(10^8) - \#C(10^8) = 10^4 - 5851 = 4149.$$

For  $E(10^{10})$ , we have 68015 composite terms and 31985 primes:

$$\pi_{\text{Euler}}(10^{10}) = \#E(10^{10}) - \#C(10^{10}) = 10^5 - 68015 = 31985.$$

For  $E(10^{12})$ , we have 738919 composite terms and 261081 primes:

$$\pi_{\text{Euler}}(10^{12}) = \#E(10^{12}) - \#C(10^{12}) = 10^6 - 738919 = 261081.$$

Within the above-mentioned method of ‘sieve’ we could *use* – or could *not use* – a small amount of primes (e.g., for  $E(10^{10})$ ), to slightly improve the algorithm, we need not more than six first primes: 2, 3, 5, 7, 11, 13, since, below  $i = 50$ , while  $p$  is prime,  $p > 13$ , there is no  $3p < i$ ; see formulas (11), (12), and (21)), and, for  $i \geq 50$ , there is no  $l_{i,j,t} < 10^5$ ; thus, we could remove all  $l(i,j,t)$ :  $l(i,j,t) \subset \mathcal{L} \wedge L \not\ni l(i,j,t)$  (see **Lemmas 7-8**).

Another improvement could be made by removing all composite distinct  $l$ -terms from  $L(10^n)$ ; we could do it, since we know all regular primes  $p < \text{ceiling}(\sqrt{10^n})$  ( $p \in \mathbb{P} \wedge p \notin L$ ; of course, there is no reason to apply the last way to enormously large upper bounds).

Compare the number of Euler primes less than the given integer  $q$  ( $q = 10^{2n}$ ), or  $\pi_{\text{Euler}}(q)$ , with  $2^{3n}$ :

$n$	$q = 10^{2n}$	$\pi_{\text{Euler}}(q)$	$3n$	$2^{3n}$	$\frac{\pi_{\text{Euler}}(q)}{2^{3n}}$
1	$10^2$	8	3	8	1
2	$10^4$	86	6	64	1.34375
3	$10^6$	581	9	512	$\approx 1.1348$
4	$10^8$	4149	12	4096	$\approx 1.0129$
5	$10^{10}$	31985	15	32768	$\approx 0.9761$
6	$10^{12}$	261251	18	262144	$\approx 0.9966$
7	$10^{14}$	2208197	21	2097152	$\approx 1.053$

While  $1 < n < 8$ ,  $\pi_{\text{Euler}}(10^{2n}) \approx 2^{3n}$  (at  $n = 1$ ,  $\pi_{\text{Euler}}(10^{2n}) = 2^{3n}$ ).

However, there is no evidence that the current equation holds for any  $n > 9$  (or even for infinitely many  $n > 9$ ). Some authors propose another formula<sup>iv</sup>:

$$\forall n \in \mathbb{Z}^+, \pi_{\text{Euler}}(10^n) \sim 2 \cdot S \cdot \left(\frac{10^2}{n} \cdot \log 10\right) \text{ where } S = 3.319773 \dots , S \text{ is the Hardy-Littlewood constant for } x^2 + x + 41.$$

## 4. INSTEAD OF CONCLUSION

The above-mentioned method of ‘sieve’ has several distinct and unusual traits (unlike many other algorithms for prime number search):

- 1) We don’t need any pre-evaluated primes at all;
- 2) Instead of one prime, you receive a number of primes;
- 3) The received set
- 4) is a strict sequence of Euler primes of form  $p_1, p_2, \dots, p_n$  (no missed terms below an upper bound,  $10^{2n}$ ).

Unfortunately, the current method couldn’t satisfy all wishes: e.g., it couldn’t use different lower bounds, or  $\mathbf{LB}$  (unless you know all  $l$ -terms of  $L(10^n)$  where any  $l$  could be written as  $41 \leq l < 10^n$ ); its  $\mathbf{LB}$  is rather a constant:  $\mathbf{LB} = 41$ .

I would be glad to hear some critical response or advice, or notes on any troubles (or mistakes, errors) in the current paper.

**Appendix A. Tables for  $i = \{1; 6\}$  ( $u \leq 4, j \leq 9$ , while  $j$  is non-negative;  $j \leq -9$ , while  $j$  is negative)**

**Table 1.** Here  $i = 1, {}_u^j s = l_j \cdot {}_u^j r, s \in E$  (see formulas (4), (5), (6), and (7))

j	$m_j$	$l_j$	$u = -1$		$u = 0$		$u = 1$		$u = 2$		$u = 3$	
			${}_u^j r$	${}_u^j s$								
0	41	41	41	1681	43	1763	163	6683	167	6847	367	15047
1	43	43	No values		47	2021	167	7181	179	7697	379	16297
2	47	47			53	2491	179	8413	199	9353	409	19223
3	53	53			61	3233	199	10547	227	12031	457	24221
4	61	61			71	4331	227	13847	263	16043	523	31903
5	71	71			83	5893	263	18673	307	21797	607	43097
6	83	83			97	8051	307	25481	359	29797	709	58847
7	97	97			113	10961	359	34823	419	40643	829	80413
8	113	113			131	14803	419	47347	487	55031	967	109271
9	131	131			151	19781	487	63797	563	73753	1123	147113

**Table 2.** Here  $i = 2, {}_u^j s = l_j \cdot {}_u^j r, s \in E$  (see formulas (8), (9), (10))

J	$m_j$	$l_j$	$u = 0$		$u = 1$		$u = 2$		$u = 3$	
			${}_u^j r$	${}_u^j s$						
-1	41	163	367	59821	367	59821	1019	166097	1019	166097
0	41	167	373	62291	379	63293	1039	173513	1049	175183
1	43	179	397	71063	409	73211	1109	198511	1129	202091
2	47	199	439	87361	457	90943	1229	244571	1259	250541
3	53	227	499	113273	523	118721	1399	317573	1439	326653
4	61	263	577	151751	607	159641	1619	425797	1669	438947
5	71	307	673	206611	709	217663	1889	579923	1949	598343
6	83	359	787	282533	829	297611	2209	793031	2279	818161
7	97	419	919	385061	967	405173	2579	1080601	2659	1114121
8	113	487	1069	520603	1123	546901	2999	1460513	3089	1504343

**Table 3-I.** Here  $i = 3, t = 1, {}_u^j s = l_j \cdot {}_u^j r, s \in E$  (see formulas (11)-(21))

J	$m_j$	$l_j$	Formula for $l_j$	$u = 0$			$u = 1$		
				${}_u^j r$	Formula for ${}_u^j r$	${}_u^j s$	${}_u^j r$	Formula for ${}_u^j r$	${}_u^j s$
-1	41	367	(41 · 3 <sup>2</sup> - 2)	653	(41 · 4 <sup>2</sup> - 3)	239651	1019	(41 · 5 <sup>2</sup> - 6)	373973
1	41	373	(41 · 3 <sup>2</sup> + 4)	661	(41 · 4 <sup>2</sup> + 5)	246553	1039	(41 · 5 <sup>2</sup> + 14)	387547
-2	43	379	(43 · 3 <sup>2</sup> - 8)	677	(43 · 4 <sup>2</sup> - 11)	256583	1049	(43 · 5 <sup>2</sup> - 26)	397571
2	43	397	(43 · 3 <sup>2</sup> + 10)	701	(43 · 4 <sup>2</sup> + 13)	278297	1109	(43 · 5 <sup>2</sup> + 34)	440273
-3	47	409	(47 · 3 <sup>2</sup> - 14)	733	(47 · 4 <sup>2</sup> - 19)	299797	1129	(47 · 5 <sup>2</sup> - 46)	461761
3	47	439	(47 · 3 <sup>2</sup> + 16)	773	(47 · 4 <sup>2</sup> + 21)	339347	1229	(47 · 5 <sup>2</sup> + 54)	539531
-4	53	457	(53 · 3 <sup>2</sup> - 20)	821	(53 · 4 <sup>2</sup> - 27)	375197	1259	(53 · 5 <sup>2</sup> - 66)	575363
4	53	499	(53 · 3 <sup>2</sup> + 22)	877	(53 · 4 <sup>2</sup> + 29)	437623	1399	(53 · 5 <sup>2</sup> + 74)	698101



**Table 3-II.** Here  $i = 3$ ,  $t = 2$ ,  ${}_u^j s = l_j \cdot {}_u^j r$ ,  $s \in E$  (see formulas (11)-(20))

$j$	$m_j$	$l_j$	<i>Formula for</i> $l_j$	$u = 0$			$u = 1$			$u = 2$		
				${}_u^j r$	<i>Formula for</i> ${}_u^j r$	${}_u^j s$	${}_u^j r$	<i>Formula for</i> ${}_u^j r$	${}_u^j s$	${}_u^j r$	<i>Formula for</i> ${}_u^j r$	${}_u^j s$
-1	41	367	$(41 \cdot 3^2 - 2)$	653	$(41 \cdot 4^2 - 3)$	239651	1019	$(41 \cdot 5^2 - 6)$	373973	1999	$(41 \cdot 7^2 - 10)$	733633
0	41	373	$(41 \cdot 3^2 + 4)$	661	$(41 \cdot 4^2 + 5)$	246553	1039	$(41 \cdot 5^2 + 14)$	387547	2027	$(41 \cdot 7^2 + 18)$	756071
-2	43	379	$(43 \cdot 3^2 - 8)$	677	$(43 \cdot 4^2 - 11)$	256583	1049	$(43 \cdot 5^2 - 26)$	397571	2069	$(43 \cdot 7^2 - 38)$	784151
1	43	397	$(43 \cdot 3^2 + 10)$	701	$(43 \cdot 4^2 + 13)$	278297	1109	$(43 \cdot 5^2 + 34)$	440273	2153	$(43 \cdot 7^2 + 46)$	854741
-3	47	409	$(47 \cdot 3^2 - 14)$	733	$(47 \cdot 4^2 - 19)$	299797	1129	$(47 \cdot 5^2 - 46)$	461761	2237	$(47 \cdot 7^2 - 66)$	914933
2	47	439	$(47 \cdot 3^2 + 16)$	773	$(47 \cdot 4^2 + 21)$	339347	1229	$(47 \cdot 5^2 + 54)$	539531	2377	$(47 \cdot 7^2 + 74)$	1043503
-4	53	457	$(53 \cdot 3^2 - 20)$	821	$(53 \cdot 4^2 - 27)$	375197	1259	$(53 \cdot 5^2 - 66)$	575363	2503	$(53 \cdot 7^2 - 94)$	1143871
3	53	499	$(53 \cdot 3^2 + 22)$	877	$(53 \cdot 4^2 + 29)$	437623	1399	$(53 \cdot 5^2 + 74)$	698101	2699	$(53 \cdot 7^2 + 102)$	1346801

**Table 4-I/II.** Here  $i = 4$ ,  $t = 1$  and  $t = 2$ ,  ${}_u^j s = l_j \cdot {}_u^j r$ ,  $s \in E$  (see formulas (11)-(21))

$j$ at $t = 1$ // $j$ at $t = 2$	$m_j$	$l_j$	<i>Formula for</i> $l_j$	$u = 0$			$u = 1$		
				${}_u^j r$	<i>Formula for</i> ${}_u^j r$	${}_u^j s$	${}_u^j r$	<i>Formula for</i> ${}_u^j r$	${}_u^j s$
-1	41	653	$(41 \cdot 4^2 - 3)$	1021	$(41 \cdot 5^2 - 4)$	666713	1999	$(41 \cdot 7^2 - 10)$	1305347
1 // 0	41	661	$(41 \cdot 4^2 + 5)$	1031	$(41 \cdot 5^2 + 6)$	681491	2027	$(41 \cdot 7^2 + 18)$	1339847
-2	43	677	$(43 \cdot 4^2 - 11)$	1061	$(43 \cdot 5^2 - 14)$	718297	2069	$(43 \cdot 7^2 - 38)$	1400713
2 // 1	43	701	$(43 \cdot 4^2 + 13)$	1091	$(43 \cdot 5^2 + 16)$	764791	2153	$(43 \cdot 7^2 + 46)$	1509253
-3	47	733	$(47 \cdot 4^2 - 19)$	1151	$(47 \cdot 5^2 - 24)$	843683	2237	$(47 \cdot 7^2 - 66)$	1639721
3 // 2	47	773	$(47 \cdot 4^2 + 21)$	1201	$(47 \cdot 5^2 + 26)$	928373	2377	$(47 \cdot 7^2 + 74)$	1837421
-4	53	821	$(53 \cdot 4^2 - 27)$	1291	$(53 \cdot 5^2 - 34)$	1059911	2503	$(53 \cdot 7^2 - 94)$	2054963
4 // 3	53	877	$(53 \cdot 4^2 + 29)$	1361	$(53 \cdot 5^2 + 36)$	1193597	2699	$(53 \cdot 7^2 + 102)$	2367023



**Table 5-I/III.** Here  $i = 5$ ,  $t = 1$  and  $t = 3$ ,  ${}_u^j s = l_j \cdot {}_u^j r$ ,  $s \in E$  (see formulas (11)-(21))

$j$ at $t = 1$ // $j$ at $t = 3$	$m_j$	$l_j$	Formula for $l_j$	$u = 0$			$u = 1$		
				${}_u^j r$	Formula for ${}_u^j r$	${}_u^j s$	${}_u^j r$	Formula for ${}_u^j r$	${}_u^j s$
-1	41	1021	(41 · 5 <sup>2</sup> - 4)	1471	(41 · 6 <sup>2</sup> - 5)	1501891	3307	(41 · 9 <sup>2</sup> - 14)	3376447
1 // 0	41	1031	(41 · 5 <sup>2</sup> + 6)	1483	(41 · 6 <sup>2</sup> + 7)	1528973	3343	(41 · 9 <sup>2</sup> + 22)	3446633
-2	43	1061	(43 · 5 <sup>2</sup> - 14)	1531	(43 · 6 <sup>2</sup> - 17)	1624391	3433	(43 · 9 <sup>2</sup> - 50)	3642413
2 // 1	43	1091	(43 · 5 <sup>2</sup> + 16)	1567	(43 · 6 <sup>2</sup> + 19)	1709597	3541	(43 · 9 <sup>2</sup> + 58)	3863231
-3	47	1151	(47 · 5 <sup>2</sup> - 24)	1663	(47 · 6 <sup>2</sup> - 29)	1914113	3721	(47 · 9 <sup>2</sup> - 86)	4282871
3 // 2	47	1201	(47 · 5 <sup>2</sup> + 26)	1723	(47 · 6 <sup>2</sup> + 31)	2069323	3901	(47 · 9 <sup>2</sup> + 94)	4685101
-4	53	1291	(53 · 5 <sup>2</sup> - 34)	1867	(53 · 6 <sup>2</sup> - 41)	2410297	4171	(53 · 9 <sup>2</sup> - 122)	5384761
4 // 3	53	1361	(53 · 5 <sup>2</sup> + 36)	1951	(53 · 6 <sup>2</sup> + 43)	2655311	4423	(53 · 9 <sup>2</sup> + 130)	6019703

**Table 5-II.** Here  $i = 5$ ,  $t = 2$ ,  ${}_u^j s = l_j \cdot {}_u^j r$ ,  $s \in E$  (see formulas (11)-(21))

$J$	$m_j$	$l_j$	Formula for $l_j$	$u = 0$			$u = 1$		
				${}_u^j r$	Formula for ${}_u^j r$	${}_u^j s$	${}_u^j r$	Formula for ${}_u^j r$	${}_u^j s$
-1	41	1019	(41 · 5 <sup>2</sup> - 6)	1997	(41 · 7 <sup>2</sup> - 12)	2034943	2609	(41 · 8 <sup>2</sup> - 15)	2658571
0	41	1039	(41 · 5 <sup>2</sup> + 14)	2039	(41 · 7 <sup>2</sup> + 30)	2118521	2657	(41 · 8 <sup>2</sup> + 33)	2760623
-2	43	1049	(43 · 5 <sup>2</sup> - 26)	2053	(43 · 7 <sup>2</sup> - 54)	2153597	2689	(43 · 8 <sup>2</sup> - 63)	2820761
1	43	1109	(43 · 5 <sup>2</sup> + 34)	2179	(43 · 7 <sup>2</sup> + 72)	2416511	2833	(43 · 8 <sup>2</sup> + 81)	3141797
-3	47	1129	(47 · 5 <sup>2</sup> - 46)	2207	(47 · 7 <sup>2</sup> - 96)	2491703	2897	(47 · 8 <sup>2</sup> - 111)	3270713
2	47	1229	(47 · 5 <sup>2</sup> + 54)	2417	(47 · 7 <sup>2</sup> + 114)	2970493	3137	(47 · 8 <sup>2</sup> + 129)	3855373
-4	53	1259	(53 · 5 <sup>2</sup> - 66)	2459	(53 · 7 <sup>2</sup> - 138)	3095881	3233	(53 · 8 <sup>2</sup> - 159)	4070347
3	53	1399	(53 · 5 <sup>2</sup> + 74)	2753	(53 · 7 <sup>2</sup> + 156)	3851447	3569	(53 · 8 <sup>2</sup> + 177)	4993031

**Table 6-I/III.** Here  $i = 6$ ,  $t = 1$  and  $t = 3$ ,  $\frac{j}{u}s = l_j \cdot \frac{j}{u}r, s \in E$  (see formulas (11)-(21))

$j$ at $t = 1$ // $j$ at $t = 3$	$m_j$	$l_j$	Formula for $l_j$	$u = 0$			$u = 1$		
				$\frac{j}{u}r$	Formula for $\frac{j}{u}r$	$\frac{j}{u}s$	$\frac{j}{u}r$	Formula for $\frac{j}{u}r$	$\frac{j}{u}s$
-1	41	1471	(41 · 6 <sup>2</sup> - 5)	2003	(41 · 7 <sup>2</sup> - 6)	2946413	4943	(41 · 11 <sup>2</sup> - 18)	7271153
1 // 0	41	1483	(41 · 6 <sup>2</sup> + 7)	2017	(41 · 7 <sup>2</sup> + 8)	2991211	4987	(41 · 11 <sup>2</sup> + 26)	7395721
-2	43	1531	(43 · 6 <sup>2</sup> - 17)	2087	(43 · 7 <sup>2</sup> - 20)	3195197	5141	(43 · 11 <sup>2</sup> - 62)	7870871
2 // 1	43	1567	(43 · 6 <sup>2</sup> + 19)	2129	(43 · 7 <sup>2</sup> + 22)	3336143	5273	(43 · 11 <sup>2</sup> + 70)	8262791
-3	47	1663	(47 · 6 <sup>2</sup> - 29)	2269	(47 · 7 <sup>2</sup> - 34)	3773347	5581	(47 · 11 <sup>2</sup> - 106)	9281203
3 // 2	47	1723	(47 · 6 <sup>2</sup> + 31)	2339	(47 · 7 <sup>2</sup> + 36)	4030097	5801	(47 · 11 <sup>2</sup> + 114)	9995123
-4	53	1867	(53 · 6 <sup>2</sup> - 41)	2549	(53 · 7 <sup>2</sup> - 48)	4758983	6263	(53 · 11 <sup>2</sup> - 150)	11693021
4 // 3	53	1951	(53 · 6 <sup>2</sup> + 43)	2647	(53 · 7 <sup>2</sup> + 50)	5164297	6571	(53 · 11 <sup>2</sup> + 158)	12820021

**Table 6-II.** Here  $i = 6$ ,  $t = 2$ ,  $\frac{j}{u}s = l_j \cdot \frac{j}{u}r, s \in E$  (see formulas (11)-(21))

$J$	$m_j$	$l_j$	Formula for $l_j$	$u = 0$			$u = 1$		
				$\frac{j}{u}r$	Formula for $\frac{j}{u}r$	$\frac{j}{u}s$	$\frac{j}{u}r$	Formula for $\frac{j}{u}r$	$\frac{j}{u}s$
-1	41	1468	(41 · 6 <sup>2</sup> - 8)						
0	41	1492	(41 · 6 <sup>2</sup> + 16)						
-2	43	1516	(43 · 6 <sup>2</sup> - 32)						
1	43	1588	(43 · 6 <sup>2</sup> + 40)						
-3	47	1636	(47 · 6 <sup>2</sup> - 56)						
2	47	1756	(47 · 6 <sup>2</sup> + 64)						
-4	53	1828	(53 · 6 <sup>2</sup> - 80)						
3	53	1996	(53 · 6 <sup>2</sup> + 88)						

No values  
(n/v)

For Table 6-II,  $a = 24, b = 16$  (see (2n)); thus,  $k_{-1} = 8$ , there is a prime  $d \in \mathbb{P}: d \notin L \wedge d \mid i \wedge d \mid a \wedge d \mid k_{-1}$ ; therefore, according to the Proof of **Theorem 1** (see **Lemma 8** and formulas (19)-(21)), there is no  $l$ -terms in  $l(6, j, 2)$  such that could be written as an  $l$ -factor of a composite  $m \in E$ .

**Appendix B. Table I, for values of some  $x$ , while  $3 \leq i \leq 7$**

$x'$	$i'$	$l'$	$r'$	$s'$	$x = x' + r'$	$i = i' + i''$	$l = l' + r'$	$r$	$s = x^2 + x + 41$	$x'' = x - 2x' - 1$	$i''$	$l''$	$r'' = l$	$s''$
122	1	41	367	15047	489	3	367	653	239651	244	2	163	367	59821
123	1	41	373	15293	496	3	373	661	246553	249	2	167	373	62291
127	1	43	379	16297	506	3	379	677	256583	251	2	167	379	63293
163	1	41	653	26773	816	4	653	1021	666713	489	3	367	653	239651
164	1	41	661	27101	825	4	661	1031	681491	496	3	373	661	246553
170	1	43	677	29111	847	4	677	1061	718297	506	3	379	677	256583
204	1	41	1021	41861	1225	5	1021	1471	1501891	816	4	653	1021	666713
205	1	41	1031	42271	1236	5	1031	1483	1528973	825	4	661	1031	681491
213	1	43	1061	45623	1274	5	1061	1531	1624391	847	4	677	1061	718297
407	2	163	1019	166097	1426	5	1019	1997	2034943	611	3	367	1019	373973
416	2	167	1039	173513	1455	5	1039	2039	2118521	622	3	373	1039	387547
418	2	167	1049	175183	1467	5	1049	2053	2153597	630	3	379	1049	397571
245	1	41	1471	60311	1716	6	1471	2003	2946413	1225	5	1021	1471	1501891
246	1	41	1483	60803	1729	6	1483	2017	2991211	1236	5	1031	1483	1528973
256	1	43	1531	65833	1787	6	1531	2087	3195197	1274	5	1061	1531	1624391
286	1	41	2003	82123	2289	7	2003	2617	5241851	1716	6	1471	2003	2946413
287	1	41	2017	82697	2304	7	2017	2633	5310761	1729	6	1483	2017	2991211
299	1	43	2087	89741	2386	7	2087	2729	5695423	1787	6	1531	2087	3195197
570	2	163	1997	325511	2567	7	1997	3301	6592097	1426	5	1019	1997	2034943
583	2	167	2039	340513	2622	7	2039	3373	6877547	1455	5	1039	2039	2118521
585	2	167	2053	342851	2638	7	2053	3391	6961723	1467	5	1049	2053	2153597
856	3	367	1999	733633	2855	7	1999	4079	8153921	1142	4	653	1999	1305347
869	3	373	2027	756071	2896	7	2027	4139	8389753	1157	4	661	2027	1339847
885	3	379	2069	784151	2954	7	2069	4219	8729111	1183	4	677	2069	1400713

**Table II, for values of  $l(i, j, t)$ , while  $3 \leq i \leq 15$** 

**Here,  $l(i, j, t) = r'(i', j, t', u') = r''(i'', j, t'', u'')$ ,  $i = i' + i''$   
(see formulas (22)–(25) and Appendix A)**

$i$	$t$	$l(i, j, t)$	$l'(i', j, t')$	$r'(i', j, t', u')$	$l''(i'', j, t'')$	$r''(i'', j, t'', u'')$	value of $f$ , $f = l_{i-1,t} =$ $r'_{i'-1,t',u'} =$ $r''_{i''-1,t'',u''}$
3	1	$l(3, j, 1)$	$l'(1, j, 1)$	$r'(1, j, 1, 3) \cup r'(1, j, 1, 4)$	$l''(2, j, 1)$	$r''(2, j, 1, 0) \cup r''(2, j, 1, 1)$	367
4	1	$l(4, j, 1)$	$l'(1, j, 1)$	$r'(1, j, 1, 5) \cup r'(1, j, 1, 6)$	$l''(3, j, 1)$	$r''(3, j, 1, 0)$	653
5	1	$l(5, j, 1)$	$l'(1, j, 1)$	$r'(1, j, 1, 7) \cup r'(1, j, 1, 8)$	$l''(4, j, 1)$	$r''(4, j, 1, 0)$	1021
5	2	$l(5, j, 2)$	$l'(2, j, 1)$	$r'(2, j, 1, 2) \cup r'(2, j, 1, 3)$	$l''(3, j, 2)$	$r''(3, j, 2, 1)$	1019
6	1	$l(6, j, 1)$	$l'(1, j, 1)$	$r'(1, j, 1, 9) \cup r'(1, j, 1, 10)$	$l''(5, j, 1)$	$r''(5, j, 1, 0)$	1471
7	1	$l(7, j, 1)$	$l'(1, j, 1)$	$r'(1, j, 1, 11) \cup r'(1, j, 1, 12)$	$l''(6, j, 1)$	$r''(6, j, 1, 0)$	2003
7	2	$l(7, j, 2)$	$l'(2, j, 1)$	$r'(2, j, 1, 4) \cup r'(2, j, 1, 5)$	$l''(5, j, 2)$	$r''(5, j, 2, 0)$	1997
7	3	$l(7, j, 3)$	$l'(3, j, 2)$	$r'(3, j, 2, 2)$	$l''(4, j, 2)$	$r''(4, j, 2, 1)$	1999
8	1	$l(8, j, 1)$	$l'(1, j, 1)$	$r'(1, j, 1, 13) \cup r'(1, j, 1, 14)$	$l''(7, j, 1)$	$r''(7, j, 1, 0)$	2617
8	2	$l(8, j, 2)$	$l'(3, j, 2)$	$r'(3, j, 2, 3)$	$l''(5, j, 2)$	$r''(5, j, 2, 1)$	2609
9	1	$l(9, j, 1)$	$l'(1, j, 1)$	$r'(1, j, 1, 15) \cup r'(1, j, 1, 16)$	$l''(8, j, 1)$	$r''(8, j, 1, 0)$	3313
9	2	$l(9, j, 2)$	$l'(2, j, 1)$	$r'(2, j, 1, 6) \cup r'(2, j, 1, 7)$	$l''(7, j, 2)$	$r''(7, j, 2, 0)$	3301
9	4	$l(9, j, 4)$	$l'(4, j, 1)$	$r'(4, j, 1, 2)$	$l''(5, j, 1)$	$r''(5, j, 1, 1)$	3307
10	1	$l(10, j, 1)$	$l'(1, j, 1)$	$r'(1, j, 1, 17) \cup r'(1, j, 1, 18)$	$l''(9, j, 1)$	$r''(9, j, 1, 0)$	4091
10	3	$l(10, j, 3)$	$l'(3, j, 1)$	$r'(3, j, 1, 4)$	$l''(7, j, 3)$	$r''(7, j, 3, 0)$	4079
11	1	$l(11, j, 1)$	$l'(1, j, 1)$	$r'(1, j, 1, 19) \cup r'(1, j, 1, 20)$	$l''(10, j, 1)$	$r''(10, j, 1, 0)$	4951
11	2	$l(11, j, 2)$	$l'(2, j, 1)$	$r'(2, j, 1, 8) \cup r'(2, j, 1, 9)$	$l''(9, j, 2)$	$r''(9, j, 2, 0)$	4931
11	3	$l(11, j, 3)$	$l'(3, j, 1)$	$r'(3, j, 1, 5)$	$l''(8, j, 2)$	$r''(8, j, 2, 0)$	4933
11	4	$l(11, j, 4)$	$l'(4, j, 1)$	$r'(4, j, 1, 3)$	$l''(7, j, 3)$	$r''(7, j, 3, 1)$	4937
11	5	$l(11, j, 5)$	$l'(5, j, 1)$	$r'(5, j, 1, 2)$	$l''(6, j, 1)$	$r''(6, j, 1, 1)$	4943
12	1	$l(12, j, 1)$	$l'(1, j, 1)$	$r'(1, j, 1, 20) \cup r'(1, j, 1, 21)$	$l''(11, j, 1)$	$r''(11, j, 1, 0)$	5893
12	2	$l(12, j, 2)$	$l'(5, j, 2)$	$r'(5, j, 2, 2)$	$l''(7, j, 2)$	$r''(7, j, 2, 1)$	5869
13	1	$l(13, j, 1)$	$l'(1, j, 1)$	$r'(1, j, 1, 22) \cup r'(1, j, 1, 23)$	$l''(12, j, 1)$	$r''(12, j, 1, 0)$	6917
13	2	$l(13, j, 2)$	$l'(2, j, 1)$	$r'(2, j, 1, 10) \cup r'(2, j, 1, 11)$	$l''(11, j, 2)$	$r''(11, j, 2, 0)$	6887
13	3	$l(13, j, 3)$	$l'(5, j, 2)$	$r'(5, j, 2, 3)$	$l''(8, j, 2)$	$r''(8, j, 2, 1)$	6889
13	4	$l(13, j, 4)$	$l'(3, j, 1)$	$r'(3, j, 1, 6)$	$l''(10, j, 3)$	$r''(10, j, 3, 0)$	6893
13	5	$l(13, j, 5)$	$l'(4, j, 1)$	$r'(4, j, 1, 4)$	$l''(9, j, 4)$	$r''(9, j, 4, 0)$	6899
13	6	$l(13, j, 1)$	$l'(6, j, 1)$	$r'(6, j, 1, 2)$	$l''(7, j, 1)$	$r''(7, j, 1, 1)$	6907
14	1	$l(14, j, 1)$	$l'(1, j, 1)$	$r'(1, j, 1, 24) \cup r'(1, j, 1, 25)$	$l''(13, j, 1)$	$r''(13, j, 1, 0)$	8023
14	3	$l(14, j, 3)$	$l'(3, j, 1)$	$r'(3, j, 1, 7)$	$l''(11, j, 3)$	$r''(11, j, 3, 0)$	7991

14	5	$l(14, j, 5)$	$l'(5, j, 1)$	$r'(5, j, 1, 3)$	$l''(9, j, 4)$	$r''(9, j, 4, 1)$	8003
15	1	$l(15, j, 1)$	$l'(1, j, 1)$	$r'(1, j, 1, 26) \cup r'(1, j, 1, 27)$	$l''(14, j, 1)$	$r''(14, j, 1, 0)$	9211
15	2	$l(15, j, 2)$	$l'(2, j, 1)$	$r'(2, j, 1, 12) \cup r'(2, j, 1, 13)$	$l''(13, j, 2)$	$r''(13, j, 2, 0)$	9169
15	5	$l(15, j, 5)$	$l'(4, j, 1)$	$r'(4, j, 1, 5)$	$l''(11, j, 4)$	$r''(11, j, 4, 0)$	9181
15	7	$l(15, j, 7)$	$l'(7, j, 1)$	$r'(7, j, 1, 2)$	$l''(8, j, 1)$	$r''(8, j, 1, 1)$	9199

Some useful and simple formulas here:

At  $t = 1$ ,  $l(i, j, t) = r'(1, j, 1, 2i - 3) \cup r'(1, j, 1, 2i - 2) = r''(i - 1, j, 1, 0)$ ;  
at  $t = 2$ ,  $l(i, j, t) = r'(2, j, 1, i - 3) \cup r'(2, j, 1, i - 2) = r''(i - 2, j, 2, 0)$ , while  $i$  is odd,  $i > 5$  (at  $i = 1$ , we consider here the union of two adjacent sets as one tablet, as well as at  $i = 2$ );

at  $t = 2$ ,  $l(i, j, t) = r''\left(\frac{i}{2} + 1, j, 2, 1\right)$ , while  $i = 0 \pmod{4}$ ,  $i > 6$ ;

at  $t = \frac{i-1}{2}$ ,  $l(i, j, t) = r'\left(\frac{i-1}{2}, j, 1, 2\right) = r''\left(\frac{i+1}{2}, j, 1, 1\right)$ , while  $i$  is odd,  $i > 5$ ;

$\forall i > 3, i'' \geq \frac{i+1}{2} \wedge i' \leq \frac{i-1}{2}$ , while  $i$  is odd.

## References

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<sup>ii</sup> [Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres \(Année 1772\), p. 36. Berlin, 1774](#) (French).

<sup>iii</sup> [H. Koch. Algebraic Number Theory, vol. 62, pp. 227–228](#). Springer, 1997.

<sup>iv</sup> [Number of prime numbers of the form  \$k^2+k+41\$  below  \$10^n\$  \(A319906\)](#), OEIS.