

Integer sequences from noncommuting multivariate polynomials

Noncommutative multivariate polynomials

We consider multivariate polynomials with non-commuting indeterminates, as in the `freealg` package. For example:

```
library("freealg")
p <- as.freealg("1+x+y")
p^2
```

```
## free algebra element algebraically equal to
## + 1 + 2*x + 1*xx + 1*xy + 2*y + 1*yx + 1*yy
```

See how terms `xy` and `yx` are retained: variables are not assumed to commute. We can follow Haiman (1993) and consider the expression

$$E(p) = (x + y + x^{-1} + y^{-1})^p, \quad p \geq 0$$

On the understanding that the variables do not commute, Haiman asks what the constant term of $E(p)$ is. The package answers that easily (package idiom for x^{-1} is uppercase `X`):

```
f <- function(p){constant(as.freealg("x+y+X+Y")^p)}
sapply(0:9,f)
```

```
## [1] 1 0 4 0 28 0 232 0 2092 0
```

It's clear in hindsight that only even p will have nonzero constant:

```
sapply(2*(0:5),f)
```

```
## [1] 1 4 28 232 2092 19864
```

This is Sloane's sequence A035610, <http://oeis.org/A035610>. We can ask the same question but for different expressions.

```
g <- function(p,string){constant(as.freealg(string)^p)}
sapply(0:7,g,"1+x+y+X+Y")
```

```
## [1] 1 1 5 13 53 181 713 2689
```

This sequence is not recorded on OEIS. We might also wonder about other expressions:

```
sapply(0:7,g,"x+y+XY")
```

```
## [1] 1 0 0 3 0 0 21 0
```

this is only nonzero when $p = 0 \pmod 3$, duh:

```
sapply(3*(0:4),g,"x+y+XY")
```

```
## [1] 1 3 21 183 1773
```

again not in OEIS. Or even:

```
sapply(3*(0:4),g,"x+y+XY+YX")
```

```
## [1] 1 6 84 1464 28368
```

another new sequence.