Notes on A326792

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Triangular array: $T(n,k)$ equals the number of small Schröder paths such that the area between the path and the x-axis contains n up-triangles and k down-triangles

The triangles in a triangle stack come in two types - *up-triangles* with vertices at the integer lattice points (x,y), $(x+1,y+1)$ and $(x+2,y)$ and *down-triangles* with vertices at the integer lattice points (x,y), $(x-1,y+1)$ and $(x+1,y+1)$. Both types of triangle have unit area.

There are two kinds of triangle stack. The first kind has a bottom row of contiguous up-triangles. We refer to this type of stack as a triangle stack of small Schröder type. The triangle stacks considered in A224704 are of small Schröder type. In the second kind of triangle stack, the bottom row of the stack consists of contiguous up-triangles. We refer to stacks of this sort as triangle stacks of large Schröder type. See, for example, A309400. In the case of A326792 we are dealing with triangle stacks of small Schröder type.

Define an (n,k,k_d) triangle stack to be a triangle stack of small Schröder type on n triangles with k contiguous up-triangles on the bottom row of the stack and k_d down-triangles in the stack. Let q (resp. b and d) mark the area of a triangle stack (resp. triangles in the bottom row of the stack and down-triangles in the stack). We attach the weight $q^n b^k d^{k_d}$ to an (n,k,k_d) stack. In [1, equation 8], we expressed the formal power series generating function for the number of (n,k,k_d) triangle stacks as a formal continued fraction:

$$\sum_{\substack{\text{all triangle}\\ \text{stacks}}} q^n b^k d^{k_d} = \frac{1}{1} - \frac{qb}{1-q^2bd} - \frac{q^3bd}{1-q^4bd^2} - \frac{q^5bd^2}{1-q^6bd^3} - \cdots . \tag{1}$$

Let now u mark up-triangles in a stack and let k_u denote the number of up-triangles in an (n,k,k_d) stack. Clearly, the number of up-triangles in a stack $= n -$ the number of down-triangles in the stack, i.e., $k_u = n - k_d$. In (1), make the change of variables $q \to qu$ and $d \to d/u$, to find

$$\sum_{\substack{\text{all triangle}\\ \text{stacks}}} q^n b^k u^{k_u} d^{k_d} = \frac{1}{1} - \frac{qbu}{1-q^2bud} - \frac{q^3bu^2d}{1-q^4bu^2d^2} - \frac{q^5bu^3d^2}{1-q^6bu^3d^3} - \cdots . \tag{2}$$

This is the generating function for the number of triangle stacks by the 4 parameters: area of a stack, the number of triangles in the bottom row of a stack and the numbers of up- and down-triangles in a stack.

Two other continued fraction representations for this generating function are found by making the change of variables $q \to qu$ and $d \to d/u$ in [1, equations (12) and (13)]:

$$\sum_{\substack{\text{all triangle}\\ \text{stacks}}} q^n b^k u^{k_u} d^{k_d} = \frac{1}{1 -} \frac{qbu}{1 -} \frac{\left(q^2ud + q^3u^2d\right)b}{1} - \frac{q^5bu^3d^2}{1} - \frac{\left(q^4u^2d^2 + q^7u^4d^3\right)b}{1} - \cdots \tag{3}$$

and

$$\sum_{\substack{\text{all triangle}\\ \text{stacks}}} q^n b^k u^{k_u} d^{k_d} = \frac{1}{1+b} - \frac{(1+qu)b}{1+b} - \frac{\left(1+q^3u^2d\right)b}{1+b} - \frac{\left(1+q^5u^3d^2\right)b}{1+b} - \cdots . \tag{4}$$

Succesively setting $q = 1$ and $b = 1$ in (2), (3) and (4) gives three continued fraction representations for the generating function for A326792 - the number of triangle stacks (of small Schröder type) by up- and down-triangles:

$$\sum_{\substack{\text{all triangle}\\ \text{stacks}}} u^{k_u} d^{k_d} = \frac{1}{1} - \frac{u}{1-ud} - \frac{u^2d}{1-u^2d^2} - \frac{u^3d^2}{1-u^3d^3} - \cdots \tag{5}$$

$$\sum_{\substack{\text{all triangle}\\ \text{stacks}}} u^{k_u} d^{k_d} = \frac{1}{1 -} \frac{u}{1 -} \frac{\left(ud + u^2d\right)}{1} - \frac{u^3d^2}{1} - \frac{\left(u^2d^2 + u^4d^3\right)}{1} - \cdots \tag{6}$$

and

$$\sum_{\substack{\text{all triangle}\\ \text{stacks}}} u^{k_u} d^{k_d} = \frac{1}{2 -} \frac{(1+u)}{2} - \frac{\left(1+u^2d\right)}{2} - \frac{\left(1+u^3d^2\right)}{2} - \cdots \tag{7}$$

The change of variables $q \to qu$ and $d \to d/u$ in [1, equations (14) and (15)] gives a representation for the generating function (2) as a ratio of q-series:

$$\sum_{\substack{\text{all triangle}\\\text{stacks}}} q^n b^k u^{k_u} d^{k_d} = \frac{N(q,b,u,d)}{D(q,b,u,d)},$$

where

$$N(q,b,u,d) = \sum_{n=0}^{\infty} \frac{(-1)^n b^n u^{n^2+n} d^{n^2} q^{2n^2+n}}{(1-udq^2)\cdots(1-u^n d^n q^{2n})(1-budq^2)\cdots(1-bu^n d^n q^{2n})} \tag{8}$$

and

$$D(q,b,u,d) = \sum_{n=0}^{\infty} \frac{(-1)^n b^n u^{n^2} d^{n^2-n} q^{2n^2-n}}{(1-udq^2)\cdots(1-u^n d^n q^{2n})(1-budq^2)\cdots(1-bu^n d^n q^{2n})}. \tag{9}$$

Setting $q = 1$ and $b = 1$ in (8) and (9) gives the generating function for A326792 as the ratio of q-series

$$\frac{N(u,d)}{D(u,d)},$$

where

$$N(u,d) = \sum_{n=0}^{\infty} \frac{(-1)^n u^{n^2+n} d^{n^2}}{(1-ud)^2\cdots(1-u^n d^n)^2} \tag{10}$$

and

$$D(u,d) = \sum_{n=0}^{\infty} \frac{(-1)^n u^{n^2} d^{n^2-n}}{(1-ud)^2\cdots(1-u^n d^n)^2}. \tag{11}$$

References

[1] P. Bala, The area beneath small Schröder paths: Notes on A224704, A326453 and A326454