# TRIANGLE STACKS OF LARGE SCHRÖDER TYPE 

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## 1. INTRODUCTION

Sequence A224704 gives the number of triangle stacks on $n$ triangles. The triangles in a triangle stack come in two types - up-triangles with vertices at the integer lattice points $(x, y),(x+1, y+1)$ and $(x+2, y)$ and downtriangles with vertices at the integer lattice points $(x, y),(x-1, y+1)$ and $(x+1, y+1)$. Both types of triangle have unit area. The construction of a triangle stack of the type considered in A224704 begins with a bottom row of contiguous up-triangles. Down-triangles may be placed in the gaps between adjacent up-triangles. Further up-triangles may then be placed on these down-triangles and the process continued.

In these notes we vary the construction of a triangle stack by starting with the bottom row of the stack consisting of contiguous down-triangles. We define an $\left(n, k, k_{u}, k_{d}\right)$ triangle stack of large Schröder type to be a triangle stack of $n$ triangles with $k$ contiguous down-triangles in the bottom row of the stack, $k_{u}$ up-triangles and $k_{d}$ down-triangles in the stack. Clearly, $k_{u}+k_{d}=n$. As an example, Figure 1 shows two triangle stacks of large Schröder type - a $(17,8,11,6)$ stack and a $(24,7,13,11)$ stack (in what we might call the standard position, where the $x$-axis rests on the bottom row of down-triangles of the stack and the leftmost vertex of the stack is at the origin).


Figure 1. Examples of triangle stacks of large Schröder type

Two triangle stacks are considered the same if one is mapped onto the other by some translation of the plane.

Each triangle stack of large Schröder type is associated in a obvious way with a large Schröder path (shown highlighted in black in the examples in Figure 1). There is a bijection between large Schröder paths and triangle stacks of large Schröder type (hence the choice of name).

We associate the weight $q^{n} b^{k} u^{k_{u}} d^{k_{d}}$ to an $\left(n, k, k_{u}, k_{d}\right)$ triangle stack; thus $q$ marks the area of the stack, $b$ the triangles in the bottom row of the stack, $u$ marks the up-triangles and $d$ marks the down-triangles in the stack. The empty triangle stack $(n=0)$ is assigned a weight of 1 . Our interest is in determining the generating function for the number $f\left(n, k, k_{u}, k_{d}\right)$ of ( $n, k, k_{u}, k_{d}$ ) triangle stacks

$$
\begin{aligned}
F(q, b, u, d)= & \sum_{\substack{\text { all }\left(n, k, k_{u}, k_{d}\right) \\
\text { triangle stacks }}} q^{n} b^{k} u^{k_{u}} d^{k_{d}} \\
= & \sum_{n, k, k_{u}, k_{d}} f\left(n, k, k_{u}, k_{d}\right) q^{n} b^{k} u^{k_{u}} d^{k_{d}} .
\end{aligned}
$$

Our approach is standard: we describe a decomposition of triangle stacks, which leads to a functional equation satisfied by the generating function $F(q, b, u, d)$. This functional equation can be solved to give a representation for $F(q, b, u, d)$ as a continued fraction. In Section 5 we use a result from Ramanujan's last notebook to find alternative representations for the generating function $F(q, b, u, d)$.

## 2. PRIMITIVE STACKS

We say an $\left(n, k, k_{u}, k_{d}\right)$ triangle stack, with $k \geq 1$, is a primitive triangle stack if its next-to-bottom row has $k$ up-triangles resting on the $k$ downtriangles in the bottom row, and has no empty positions (i.e., contains $k-1$ down-triangles).


Figure 2. A primitive $(14,4,8,6)$ stack

A primitive stack corresponds to a large Schröder path that begins with an upstep and makes its first returns to the $x$-axis at the end of the stack.

Let $g\left(n, k, k_{u}, k_{d}\right)$ denote the number of primitive ( $n, k, k_{u}, k_{d}$ ) triangle stacks and let $G(q, b, u, d)=\sum g\left(n, k, k_{u}, k_{d}\right) q^{n} b^{k} u^{k_{u}} d^{k_{d}}$, where the sum is taken over all primitive stacks, denote the generating function for the number of weighted primitive triangle stacks. Removing the bottom row of $k$ downtriangles from a primitive $\left(n, k, k_{u}, k_{d}\right)$ stack, together with the $k$ up-triangles resting on these base triangles, results in a (not-necessarily- primitive) triangle stack on $n-2 k$ triangles, having $k-1$ triangles in its bottom row and containing $k_{u}-k$ up-triangles and $k_{d}-k$ down-triangles. In Figure 3, for instance, this process applied to the primitive $(14,4,8,6)$ stack of Figure 2 produces a (non-primitive) $(6,3,4,2)$ triangle stack.


Figure 3.

Thus we see that
$g\left(n, k, k_{u}, k_{d}\right)=f\left(n-2 k, k-1, k_{u}-k, k_{d}-k\right), \quad n \geq 2 k, k \geq 1, k_{u} \geq k, k_{d} \geq k$.
This is equivalent to the relation between generating functions

$$
\begin{equation*}
G(q, b, u, d)=q^{2} b u d F\left(q, q^{2} b u d, u, d\right) \tag{1}
\end{equation*}
$$

## 3. DERIVING THE GENERATING FUNCTION $F$

Let now $\mathcal{L}$ be a non-empty triangle stack of large Schröder type. Two cases can occur.

Case 1) As in the first example in Figure 1, the Schröder path associated with the stack $\mathcal{L}$ may begin with a flat step. In this case $\mathcal{L}$ begins with a down-triangle (of weight $q b d$ ) with no up-triangle resting on it, followed by an arbitrary triangle stack of large Schröder type. Therefore, the generating function for the number of weighted stacks of this type is equal to $q b d F(q, b, u, d)$.

Case 2) The other possibility is that the Schröder path associated with the triangle stack $\mathcal{L}$ begins with an upstep. In this case $\mathcal{L}$ decomposes uniquely into an initial non-empty primitive stack (which extends to where the large Schröder path associated with $\mathcal{L}$ first returns to the $x$-axis) followed by an arbitrary triangle stack of large Schröder type. The second example in Figure 1 is of this type, beginning with an initial primitive stack on a base of three down-triangles.

This decomposition shows that the generating function for the number of stacks of this second type factorises as $G F$.

In total then, the generating function $F$ decomposes as

$$
F=1+q b d F+G F
$$

leading to

$$
F=\frac{1}{1-q b d-G}
$$

Hence, by (1), we have

$$
F=\frac{1}{1-q b d-q^{2} b u d F\left(q, q^{2} b u d, u, d\right)}
$$

Succesive iterations of this identity lead to a formal continued fraction expansion for the generating function of the number of weighted triangle stacks of large Schröder type:

$$
\begin{equation*}
F(q, b, u, d)=\frac{1}{1-q b d}-\frac{q^{2} b u d}{1-q^{3} b u d^{2}}-\frac{q^{4} b u^{2} d^{2}}{1-q^{5} b u^{2} d^{3}}-\frac{q^{6} b u^{3} d^{3}}{1-q^{7} b u^{3} d^{4}}-\ldots \tag{2}
\end{equation*}
$$

## 4. SPECIALISATIONS

1) Setting the variables $q, u$ and $d$ in (2) equal to 1 gives the generating function for the number of triangle stacks of large Schröder type with $n$ down-triangles in the bottom row as the continued fraction

$$
\begin{aligned}
& \frac{1}{1-b}-\frac{b}{1-b}-\frac{b}{1-b}-\cdots \\
= & 1+2 b+6 b^{2}+22 b^{3}+90 b^{4}+394 b^{5}+\cdots .
\end{aligned}
$$

This is a known representation for the generating function of the sequence of large Schröder numbers A006318, and simply reflects the bijection between large Schröder paths of semilength $n$ and triangle stacks of large Schröder type with a bottom row of $n$ down-triangles.
2) Setting the variables $b, u$ and $d$ in (2) equal to 1 gives the generating function for the number of triangle stacks of large Schröder type on $n$ triangles as the continued fraction

$$
\begin{aligned}
& \frac{1}{1-q}-\frac{q^{2}}{1-q^{3}}-\frac{q^{4}}{1-q^{5}}-\frac{q^{6}}{1-q^{7}}-\cdots \\
= & 1+q+2 q^{2}+3 q^{3}+5 q^{4}+9 q^{5}+16 q^{6}+28 q^{7}+\cdots .
\end{aligned}
$$

The coefficient sequence $[1,1,2,3,5,9,16,28, \ldots]$ is already in the OEIS as entry A088352, but without a combinatorial interpretation.
3) Setting $q=b=1$ in (2) gives the generating function for the number of triangle stacks of large Schröder type on $n$ down-triangles and $k$ up-triangles as

$$
\frac{1}{1-d}-\frac{u d}{1-u d^{2}}-\frac{u^{2} d^{2}}{1-u^{2} d^{3}}-\frac{u^{3} d^{3}}{1-u^{3} d^{4}}-\ldots
$$

The associated coefficient array begins

| $n \backslash k$ | 1 | $u$ | $u^{2}$ | $u^{3}$ | $u^{4}$ | $u^{5}$ | $u^{6}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 |  |  |  |  |  |  |
| $d$ | 1 | 1 |  |  |  |  |  |
| $d^{2}$ | 1 | 2 | 1 |  |  |  |  |
| $d^{3}$ | 1 | 3 | 4 | 2 |  |  |  |
| $d^{4}$ | 1 | 4 | 8 | 8 | 3 |  |  |
| $d^{5}$ | 1 | 5 | 13 | 20 | 16 | 5 |  |
| $d^{6}$ | 1 | 6 | 19 | 38 | 46 | 31 | 9 |

The sequence of row sums of the array $[1,2,4,10,24,60,150, \ldots]$ is A088354.

## 5. ALTERNATIVE REPRESENTATIONS FOR $F(q, b, u, d)$

We use a result from Ramanujan's lost notebook to find other representations for the generating function $F(q, b, u, d)$ of the number of weighted triangle stacks of large Schröder type. Define the $q$-series

$$
g(b ; \lambda)=\sum_{n=0}^{\infty} \frac{\lambda^{n} q^{n^{2}}}{(1-q) \cdots\left(1-q^{n}\right)(1+b q) \cdots\left(1+b q^{n}\right)} .
$$

Entry 6.3 .1 in Ramanujan's lost notebook (see [1, p. 159]) gives three formal continued fraction expressions for the $q$-series ratio $g(b ; \lambda q) / g(b ; \lambda)$ :

$$
\begin{align*}
\frac{g(b ; \lambda q)}{g(b ; \lambda)} & =\frac{1}{1}+\frac{\lambda q}{1}+\frac{\lambda q^{2}+b q}{1}+\frac{\lambda q^{3}}{1}+\frac{\lambda q^{4}+b q^{2}}{1}+\cdots  \tag{3}\\
& =\frac{1}{1}+\frac{\lambda q}{1+b q}+\frac{\lambda q^{2}}{1+b q^{2}}+\frac{\lambda q^{3}}{1+b q^{3}}+\cdots  \tag{4}\\
& =\frac{1}{1-b}+\frac{b+\lambda q}{1-b}+\frac{b+\lambda q^{2}}{1-b}+\frac{b+\lambda q^{3}}{1-b}+\cdots \tag{5}
\end{align*}
$$

It follows from (4) that

$$
\begin{equation*}
\frac{1}{\lambda q}\left(\frac{g(b ; \lambda)}{g(b ; \lambda q)}-1\right)=\frac{1}{1+b q}+\frac{\lambda q^{2}}{1+b q^{2}}+\frac{\lambda q^{3}}{1+b q^{3}}+\cdots \tag{6}
\end{equation*}
$$

Making the parameter replacements $b \rightarrow-\frac{b}{q u}, \lambda \rightarrow-\frac{-b}{q^{2} u d}, q \rightarrow q^{2} u d$, we find that the right-hand side of (6) becomes

$$
\frac{1}{1-q b d}-\frac{q^{2} b u d}{1-q^{3} b u d^{2}}-\frac{q^{4} b u^{2} d^{2}}{1-q^{5} b u^{2} d^{3}}-\frac{q^{6} b u^{3} d^{3}}{1-q^{7} b u^{3} d^{4}}-\ldots,
$$

which is the continued fraction representation for the generating function $F(q, b, u, d)$ given in (2).

The left-hand side of (6) simplifies to the $q$-series ratio

$$
\frac{N(q, b, u, d)}{D(q, b, u, d)}
$$

where

$$
\begin{equation*}
N(q, b, u, d)=\sum_{n=0}^{\infty} \frac{(-1)^{n} q^{2 n^{2}+2 n} b^{n} u^{n^{2}+n} d^{n^{2}+n}}{\prod_{k=1}^{n}\left(1-q^{2 k} u^{k} d^{k}\right) \prod_{k=1}^{n+1}\left(1-q^{2 k-1} b u^{k-1} d^{k}\right)} \tag{7}
\end{equation*}
$$

and

$$
\begin{equation*}
D(q, b, u, d)=\sum_{n=0}^{\infty} \frac{(-1)^{n} q^{2 n^{2}} b^{n} u^{n^{2}} d^{n^{2}}}{\prod_{k=1}^{n}\left(1-q^{2 k} u^{k} d^{k}\right) \prod_{k=1}^{n}\left(1-q^{2 k-1} b u^{k-1} d^{k}\right)} \tag{8}
\end{equation*}
$$

Thus we have a representation for the generating function $F$ as a $q$-series ratio:

$$
\begin{equation*}
F(q, b, u, d)=\frac{N(q, b, u, d)}{D(q, b, u, d)} \tag{9}
\end{equation*}
$$

where the $q$-series $N$ and $D$ are given by (7) and (8).
Another continued fraction representation for the generating function can be obtained from (3):

$$
\begin{gather*}
F(q, b, u, d)=\frac{1}{1}-\frac{q b d+q^{2} b u d}{1}-\frac{q^{4} b u^{2} d^{2}}{1}-\frac{q^{3} b u d^{2}+q^{6} b u^{3} d^{3}}{1}-\frac{q^{8} b u^{4} d^{4}}{1} \\
 \tag{10}\\
-\frac{q^{5} b u^{2} d^{3}+q^{10} b u^{5} d^{5}}{1}-\frac{q^{12} b u^{6} d^{6}}{1}-\cdots
\end{gather*}
$$

## References

[1] G. E. Andrews and B. C. Berndt

Ramanujan's Lost Notebook, Part 1, Springer 2005

