A325148 - Squares m^2 = n * Rev(n) (File)
The different ways

Definition: The reversal of a number n is the number which is obtained when n is written backwards (see A004086).

Convention: "Leading zeros (after the reversal has taken place) are omitted" (NJAS).

Notation in OEIS: Rev(n) or rev(n) or R(n)

The numbers n such that n * Rev(n) is square are in A306273. Now, the consideration of the squares N = m^2 which are equal to the product of an integer and its reversal in one or several ways, give many interesting properties of these particular squares.

I) Squares N equal to the product of an integer and its reversal... in at least one way

These squares N = m^2 form the sequence A325148.
Some squares are equal to the product of an integer and its reversal in exactly one, two, or three ways. No square equal to the product of an integer and its reversal in four ways or more is known today. These different cases are examined below.
A076750 is a subsequence of squares which are the product of a non-palindrome and its reversal, where trailing zeros are not allowed.
A014186, with the squares of palindromes is another subsequence.

II) Squares equal to the product of an integer and its reversal in only one way

These squares N = m^2 form the sequence A325149.

2.1 N = m^2 = n * Rev(n) with n that doesn't end with 0

There are two families of integers which satisfy this condition.

2.1.1 The integer m is palindrome and m^2 = n * Rev(n) implies n = m, therefore, these palindromes m are in A002113 but not in A117281 (palindromes ERPN in III). These squares m^2 form a subsequence of A014186 (squares of palindromes).
Examples:
1089 = 33^2 = 33 * 33 and 58564 = 242^2 = 242 * 242 are terms, but,
3504 = 252^2 = 252 * 252 = 144 * 411 and 7683984 = 2772^2 = 1584 * 4851 are no terms.

2.1.2. The integer m is non-palindrome and the Diophantine equation N = m^2 = n * Rev(n) has only one solution, with n palindrome which doesn't end with zero. These numbers m form the sequence A325151 {403, 504, 816, 3267, ...}, subsequence of A207373 and the corresponding squares m^2 : {162409, 254016, 665856, ...} form a subsequence of A076750.
Examples:
162409 = 403^2 = 169 * 961 and 665856 = 816^2 = 768 * 867 are terms.
23282265528900 = 4825170^2 = 4020975 * 5790204 is an other term (N ends with two zeros but n and Rev(n) end respectively with 975 and 204).

2.2 $N = m^2 = n * Rev(n)$ with n ends with 0

There is only one family generated by palindromes m of 2.1.1 with $m^2 = m * Rev(m)$, and for $k >= 1$, $N = (10^k * m)^2$ is a term, because
$N = (10^k * m)^2 = (10^{2k} * m) * m = (10^{2k} * m) * (Rev(10^{2k} * m))$.
Example: $110^2 = 12100 = 1100 * 11$.

III) Squares equal to the product of an integer and its reversal in at least two ways

The numbers, no necesarily squares, which are equal to the product of an integer and its reversal in at least two different ways are called in OEIS: EPRN = Equal Product of Reversible Numbers, with reference to Shyam Sunder Gupta website. These integers form the sequence A066531.

When these EPRN are squares, they form the sequence A083408, and the square roots are either in A117281 when the square roots are palindromes, or in A206642 when the square roots are non-palindromes.

IV) Squares equal to the product of an integer and its reversal in exactly two ways

These squares $N = m^2$ form the sequence A325150.

4.1. Squares don't end with 0

These square numbers N are exactly square of palindromes m which are the terms of A117281 (palindromes whose squares are EPRNs).

$$N = m^2 = m * Rev(m) = n * Rev(n) \text{ with n no palindrome.}$$

This smallest such square is 63504 = 252 * 252 = 144 * 441, with 144 in A035090.
The second one is 7683984 = 2772 * 2772 = 1584 * 4851 with 1584 in A082994.
These even squares without trailing zeros are exactly in A083408.
The first such odd square is 1239016098321 = 1113111 * 1113111 = 1022121 * 1212201, it's the fourteenth term of this family. These odd squares are exactly in A083407.

There is an infinite number of terms in this sequence A117281, therefore, also in A325150. For instance, 27x72 is a term where x is a string of k repeated digits 9 where k >= 0, with: $2772^2 = 1584 * 4851$, $27972^2 = 15984 * 48951$, $279972^2 = 159984 * 489951$, ..., $(27x72)^2 = 15x84 * 48x51$ (Shyam Sunder Gupta website).

4.2. Squares with trailing zeros

There are two possibilities to create such squares with trailing zeros.

4.2.1. We consider now square numbers N of 2.1.2, when the equation $N = m^2 = n * Rev(n)$ has only one solution, with m and n non-palindromes and n which doesn't end with zero. The integers m form the sequence A325151; then, squares N generate squares N' in A32515 with, for k >=1,
$N' = 10^{2k} * N = (10^k * m)^2 = (10^{2k} * n) * Rev(10^{2k} * n) = (10^{2k} * Rev(n)) * Rev(10^{2k} * Rev(n))$
which is simplified in:

$$N' = 10^{2k} * N = (10^k * m)^2 = (10^{2k} * n) * Rev(n) = (10^{2k} * Rev(n)) * n.$$

The smallest example comes from N = 162409 = 403^2 = 169 * 961 which gives the fourth term of this sequence A325150: N' = 16240900 = 4030^2 = 16900 * 961 = 96100 * 169.

These numbers m are also in A207373 with other integers {660, 660660, 660660660, ...} which form the next family 4.2.2.

4.2.2. We consider non-palindromes q, q end in one zero, with q/10 is palindrome, and such that the square q^2 satisfy also N = q^2 = n * Rev(n) with n no palindrome and no ending in 0.

As q/10 = Rev(q) = Rev(10*q), we can write for this particular squares N:

$$N = q^2 = (10*q) * (q/10) = (10*q) * Rev(10*q) = n * Rev(n) \text{ which simplifies in}$$

$$N = q^2 = (10*q) * Rev(q) = n * Rev(n).$$

These non palindromes q are terms of the sequence A207373.

The smallest example in this case is 435600 = 660^2 = 6600 * 66 = 528 * 825, when the two ending zeros of 435600 come from the multiplication of 28 by 25, it's the second term of A325150. The next one is 436471635600 = 660660^2 = 6606600 * 66066 = 528528 * 825825.

There is only one such family q until $6*10^{15}$: {660, 660660, 660660660, 660660660660, ...}.

V) Squares equal to the product of an integer and its reversal in exactly 3 ways

These squares N = m^2 form the sequence A307019.

5.1. Remark: Why do all these terms end with an even number of zeros?
Is it possible to find a term which does not end with zeros?
If such a term N = m^2 without trailing zeros exists, the number m must satisfy the Diophantine equation

$$m^2 = a * Rev(a) = b * Rev(b) = c * Rev(c) \text{ with } a < b < c.$$

No solution (m,a,b,c) with m that does not end with zero is known, until $6*10^{15}$.

5.2. Squares end with an even number of zeros.

5.2.1. From the Diophantine equation in 4.1, N = m^2 = m * Rev(m) = n * Rev(n) with m palindrome in A117281, and n not palindrome, each solution (N,m,n) generates terms N' = 10^{2k} * N of the sequence A307019 with the relations for k >= 1,

$$N' = 10^{2k} * N = (10^k * m)^2 = (10^{2k} * m) * m = (10^{2k} * n) * Rev(n) = (10^{2k} * Rev(n)) * n.$$

The first such term a(1) is 6350400 = 2520^2 = 25200 * 252 = 14400 * 441 = 44100 * 144.
So, 6350400 is also the smallest square which is the product of a number and its reversal in three different ways.

5.2.2. We consider the same non-palindromes q of 4.2.2, q ends in one zero, with q/10 is palindrome, and such that the square numbers q^2 satisfy also N = q^2 = n * Rev(n) with n no palindrome and no ending in 0.
As q/10 = Rev(q) = Rev(10*q), we can write for this particular squares N:

$$N = q^2 = (10*q) * (q/10) = (10*q) * Rev(10*q) = n * Rev(n) \text{ which simplifies in}$$

$N = q^2 = (10*q) * Rev(q) = n * Rev(n)$.

Then, we generate new terms of A307019 with, for k >= 1:
$N' = 10^{2k} * N = (10^k * q)^2 = (10^{2k+1} * q) * Rev(q) = (10^{2k} * n) * Rev(n) = (10^{2k} *Rev(n)) * n$.

The first such term a(2) comes from q = 660 with
a(2) = 43560000 = 6600^2 = 660000 * 66 = 52800 * 825 = 82500 * 528.
There is only one such family q until $6*10^{15}$: {660, 660660, 660660660, 660660660660, ...}.

VI) Squares equal to the product of an integer and its reversal in four ways (or more).

If the Diophantine equation $m^2 = a * Rev(a) = b * Rev(b)$ with a <> b and a and b not palindromes has a solution, then it's possible to get integers q equal $(10*m)^2$ which can be expressed as the produt of two reversible numbers in exactly four different ways which are:

$q = (10*m)^2 = (100*a) * Rev(100*a) = (100*Rev(a)) * [Rev(100*Rev(a))] =$
$(100*b) * Rev(100*b) = (100*Rev(b)) * [Rev(100*Rev(b))]$ which can be simplified as
$q = (10*m)^2 = (100*a) * Rev(a) = (100*Rev(a)) * a = (100*b) * Rev(b) = (100*Rev(b)) * b$.

We don't know if such a solution exists, and there is not such square until $6*10^{15}$.

However, there exist numbers, but no squares, which are equal to the product of an integer and its reversal in four ways, they are in A066598 and the smaller example is:
1446480 = 1680 * 861 = 8610 * 168 = 2940 * 492 = 4920 * 294.

Cross-references in page 5.

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Sequences in OEIS

EPORNs = Numbers equal to the product of an integer and its reversal in at least 2 ways.

A066531 = EPORN (squares and no squares)
A066590 = EPORN squares and no squares, in at least 3 ways
A117282 = EPORN squares and no squares, not ending in 0.
A077760 = EPORN, no squares at least two ways
A066598 = EPORN in exactly four different ways (no square is known)

The squares

A014186 = Squares of palindromes
A325148 = Squares which are the product of an integer and its reversal (in at least one way).
A325149 = Squares which are the product of an integer and its reversal in only 1 way.
A083408 = Squares equal to the product of an integer and its reversal in at least 2 ways.
A083406 = ... in at least 2 ways, when these squares are even.
A083407 = ... in at least 2 ways, when these squares are odd, in fact, it's exactly in 2 ways.
A083408 = A083406 Union A083407 with empty intersection.
A325150 = Squares which are the product of an integer and its reversal in exactly 2 ways.
A307019 = Squares which are the product of an integer and its reversal in exactly 3ways
A076750 = Squares which are the product of a non-palindrome without ending zero and its reversal (or, where leading zeros are not allowed).

The square roots:

A117281 = palindromes m whose square is the product of an integer and its reversal in 2 ways, $m^2 = m * Rev(m) = n * Rev(n)$ (generate the terms of A325150 which don't end with zero).
A206642 = Non-palindromes whose square is the product of an integer and its reversal in at least 2 ways, all these numbers end with 0.
Squares of terms in A117281 and A206642 are exactly terms of A083408.
A207373 = Numbers whose square is the product of an integer without ending zero and its reversal in at least one way.
A325151 = A207373\A117281 = Non-palindromes numbers whose square is the product of an integer without ending zero and its reversal in exactly one way; these integers generate the terms of A325150 which end with 0s.
A062917 = Nonpalindromic numbers n such that n is not divisible by 10 and n*Rev(n) is square.