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Dear Daniel,

Richard Stanley's problem in the February issue was too tantalizing to put down, so I enclose a solution.

Sincerely,

Don

Donald E. Knuth
Professor

DEK/pw

Enclosure: Solution to problem 10572

cc: Gessel, Sloane, Stanley, Wilf

Solution to problem 10572. The graphs in question can conveniently be called Stanley graphs. Let us say that a subspace *involves* coordinate k if it contains at least one vector such that the kth coordinate is nonzero; a subspace is called *full* if it involves all n coordinates. According to well-known principles of combinatorial enumeration [explained for example in *Combinatorial Species and Tree-like Structures* by Bergeron, Labelle, and Leroux (Cambridge University Press, 1997)], $e^{-x}\sum g(n)\,x^n/n!$ is the exponential generating function for the number of full subspaces. Therefore it suffices to define a one-to-one correspondence between Stanley graphs on $\{1,\ldots,n\}$ and full subspaces of $GF(2)^n$.

Every subspace of dimension k has a canonical basis

$$u_1=(u_{11},\ldots,u_{1n})\,,\ u_2=(u_{21},\ldots,u_{2n})\,,\ \ldots\,,u_k=(u_{k1},\ldots,u_{kn})$$

such that

$$u_{in_i}=1\,,\quad u_{ij}=0\quad\text{for } j>n_i\,,\quad u_{ln_i}=0\quad\text{for } l\neq i\,,$$

for $1\le i\le k$, where $n\ge n_1>n_2>\cdots>n_k\ge 1$. For example, if $n=8$, $k=4$, $n_1=8$, $n_2=5$, $n_3=3$, and $n_4=2$, the basis has the schematic pattern

$$\begin{aligned}
u_1&=(u_{11},0,0,u_{14},u_{16},u_{17},1)\,,\\
u_2&=(u_{21},0,0,u_{24},1,0,0,0)\,,\\
u_3&=(u_{31},0,1,0,0,0,0,0)\,,\\
u_4&=(u_{41},1,0,0,0,0,0,0)\,.
\end{aligned}$$

This basis, obtained by a familiar triangularization procedure, is uniquely defined by the subspace. [See D. E. Knuth, "Subspaces, subsets, and partitions," *J. Combinatorial Theory* **10** (1971), 178–180, for further elementary properties of the canonical basis.]

A subspace is full if and only if no component is zero in all of its canonical basis vectors. For example, the basis above defines a full subspace if and only if $u_{11}\vee u_{21}\vee u_{31}\vee u_{41}=u_{14}\vee u_{24}=u_{16}=u_{17}=1$. We define a Stanley graph corresponding to each canonical basis by making vertex j adjacent to vertex n_i when $u_{ij}=1$, for $1\le i\le k$. For example, the subspace spanned by the vectors

$$\begin{aligned}
u_1&=(1,0,0,0,0,1,1,1)\,,\\
u_2&=(0,0,0,1,1,0,0,0)\,,\\
u_3&=(0,0,1,0,0,0,0,0)\,,\\
u_4&=(1,1,0,0,0,0,0,0)
\end{aligned}$$

has edges $\{1,9\}$, $\{7,9\}$, $\{8,9\}$, $\{4,5\}$, and $\{1,2\}$. It is clear that this procedure defines a Stanley graph, since no vertex has neighbors both to the left and to the right. Furthermore, the construction is reversible: A Stanley graph with exactly $n-k$ vertices that have neighbors to the right corresponds to a full subspace of dimension k. Q.E.D.

Appendix. (The following remarks are mostly for my own records, but I include them here because I'm sending a copy of this letter to Ira Gessel, Richard Stanley, Herb Wilf, and Neil Sloane.) Stanley graphs are interesting combinatorial objects that do not seem to have been investigated before; at least, the number of such graphs

$$f(0), f(1), f(2), \ldots = 1, 1, 2, 6, 158, 1330, 15414, 245578, 5382862, \ldots$$

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is a sequence not yet in Sloane and Plouffe's *Encyclopedia of Integer Sequences*. Nor is the number of such graphs without isolated vertices,

$$1, 0, 1, 2, 11, 72, 677, 8686, 152191, 3632916, \ldots,$$

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obtained by multiplying the exponential generating function by e^{-x}; this corresponds to full subspaces in which the canonical basis vectors all have at least two nonzero components. Nor is the number of *connected* Stanley graphs,

$$0, 1, 1, 2, 8, 52, 502, 6824, 127166, 3205924, \ldots,$$

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obtained from $\ln \sum f(n)x^n/n!$. Nor is the number of Stanley *trees*,

$$0, 1, 1, 2, 7, 36, 246, 2140, 21652, 260720, 3598120, \ldots;$$ A7889

it is interesting to enumerate the latter (see below).

A graph on the vertices $\{1, 2, \ldots, n\}$ is a Stanley graph if and only if no vertex has both left and right neighbors. Thus every Stanley graph is bipartite: Every edge connects a vertex with right neighbors to a vertex with left neighbors. Conversely, the vertices of every bipartite graph can be labeled $\{1, 2, \ldots, n\}$ in such a way that we obtain a Stanley graph, simply by assigning the smallest labels to the vertices of one part and the largest labels to the vertices of the other.

Let $\left|{n \atop k}\right|$ be the number of Stanley trees on n vertices such that k of the vertices have right neighbors and $n-k$ have left neighbors. Let

$$\begin{aligned} s(x,y) &= \sum_{k,n} \left|{n \atop k}\right| \frac{x^k y^{n-k}}{n!} \\ &= \frac{xy}{2} + \frac{x^2y + xy^2}{6} + \frac{x^3y + 5x^2y^2 + xy^3}{24} + \cdots. \end{aligned}$$

Then by standard methods of combinatorial enumeration we find

$$y + \vartheta\, s(x,y) = y \exp\bigl(x + \vartheta_x\, s(x,y)\bigr)\,,$$

where ϑ_x is the differential operator $x\,\frac{\partial}{\partial x}$ and $\vartheta = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y}$. It follows that we have the recurrence

$$\genfrac{|}{|}{0pt}{}{n+1}{k} = \delta_{n1}\delta_{k1} + k\genfrac{|}{|}{0pt}{}{n}{k} + \genfrac{|}{|}{0pt}{}{n}{k-1} + \sum_{m,j}\binom{n-1}{m-1}\genfrac{|}{|}{0pt}{}{m}{j}\genfrac{|}{|}{0pt}{}{n+1-m}{k \mathbin{\dot-} j}\, j\,;$$

these numbers $\genfrac{|}{|}{0pt}{1}{n}{k}$ form the "Gessel-Stanley triangle"

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$$\begin{array}{c}
0 \\
0 \quad 0 \\
0 \quad 1 \quad 0 \\
0 \quad 1 \quad 1 \quad 0 \\
0 \quad 1 \quad 5 \quad 1 \quad 0 \\
0 \quad 1 \quad 17 \quad 17 \quad 1 \quad 0 \\
0 \quad 1 \quad 49 \quad 146 \quad 49 \quad 1 \quad 0 \\
0 \quad 1 \quad 129 \quad 922 \quad 922 \quad 129 \quad 1 \quad 0 \\
0 \quad 1 \quad 321 \quad 4887 \quad 11234 \quad 4887 \quad 321 \quad 1 \quad 0
\end{array}$$

By the correspondence above, $\genfrac{|}{|}{0pt}{1}{n}{k}$ is the number of ways to put 0s and 1s into a tableau shape (Ferrers diagram) with k rows and $n-k$ columns in such a way that exactly $n-1$ entries are equal to 1, and there is no way to partition the rows and columns into two parts $R_1 \cup R_2$ and $C_1 \cup C_2$ such that the entries of $R_1 \cap C_1$ and $R_2 \cap C_2$ are entirely zero. For example, the 17 such arrays when $n=5$ and $k=2$ are

1 1 1	1 1 1	1 1 1	1 1 1	1 1 1	1 1 1	1 1 0	1 1 0
1	1 0	1 0 0	0 1	0 1 0	0 0 1	1 0 1	0 1 1

1 0 1	1 0 1	1 0 1	0 1 1	0 1 1	0 1 1	0 0 1	0 1 0	1 0 0
1 1	1 1 0	0 1 1	1 1	1 1 0	1 0 1	1 1 1	1 1 1	1 1 1.

We have $\sum \genfrac{|}{|}{0pt}{1}{n}{2} x^n = x^3/(1-x)(1-2x)^2$, according to Sloane and Plouffe.

After writing the above, I happened to see Ira Gessel's paper in *Electronic J. Combinatorics* **3** (2) (1996), #R8 [*The Foata Festschrift*, 197–201], where the quantity $\genfrac{|}{|}{0pt}{1}{n}{k}$ is called $u_{n-1,k-1,n-k-1}$; Gessel's $u_{n,i,j}$ is the number of forests on vertices $\{1,2,\ldots,n\}$ having i descents and $j+1$ leaves. So Stanley trees have implicitly been considered before!