April 29, 1997

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Dear Daniel,

Richard Stanley’s problem in the February issue was too tantalizing to put down, so I enclose a solution.

Sincerely,

Donald E. Knuth
Professor

DEK/pw

Enclosure: Solution to problem 10572
cc: Gessel, Sloane, Stanley, Wilf
Solution to problem 10572. The graphs in question can conveniently be called Stanley graphs. Let us say that a subspace involves coordinate $k$ if it contains at least one vector such that the $k$th coordinate is nonzero; a subspace is called full if it involves all $n$ coordinates. According to well-known principles of combinatorial enumeration [explained for example in Combinatorial Species and Tree-like Structures by Bergeron, Labelle, and Leroux (Cambridge University Press, 1997)], $e^{-x} \sum g(n) x^n/n!$ is the exponential generating function for the number of full subspaces. Therefore it suffices to define a one-to-one correspondence between Stanley graphs on $\{1, \ldots, n\}$ and full subspaces of $GF(2)^n$.

Every subspace of dimension $k$ has a canonical basis

$$u_1 = (u_{11}, \ldots, u_{1n}), \quad u_2 = (u_{21}, \ldots, u_{2n}), \ldots, \quad u_k = (u_{k1}, \ldots, u_{kn})$$

such that

$$u_{in_i} = 1, \quad u_{ij} = 0 \text{ for } j > n_i, \quad u_{il} = 0 \text{ for } l \neq i,$$

for $1 \leq i \leq k$, where $n \geq n_1 > n_2 > \cdots > n_k \geq 1$. For example, if $n = 8$, $k = 4$, $n_1 = 8$, $n_2 = 5$, $n_3 = 3$, and $n_4 = 2$, the basis has the schematic pattern

$$u_1 = (u_{11}, 0, 0, u_{14}, u_{16}, u_{17}, 1),$$
$$u_2 = (u_{21}, 0, 0, u_{24}, 1, 0, 0, 0),$$
$$u_3 = (u_{31}, 0, 1, 0, 0, 0, 0, 0),$$
$$u_4 = (u_{41}, 1, 0, 0, 0, 0, 0, 0).$$

This basis, obtained by a familiar triangularization procedure, is uniquely defined by the subspace. [See D. E. Knuth, “Subspaces, subsets, and partitions,” J. Combinatorial Theory 10 (1971), 178–180, for further elementary properties of the canonical basis.]

A subspace is full if and only if no component is zero in all of its canonical basis vectors. For example, the basis above defines a full subspace if and only if $u_{11} \lor u_{21} \lor u_{31} \lor u_{41} = u_{14} \lor u_{24} = u_{16} = u_{17} = 1$. We define a Stanley graph corresponding to each canonical basis by making vertex $j$ adjacent to vertex $n$, when $u_{ij} = 1$, for $1 \leq i \leq k$. For example, the subspace spanned by the vectors

$$u_1 = (1, 0, 0, 0, 0, 1, 1, 1),$$
$$u_2 = (0, 0, 0, 1, 1, 0, 0, 0),$$
$$u_3 = (0, 0, 1, 0, 0, 0, 0, 0),$$
$$u_4 = (1, 1, 0, 0, 0, 0, 0, 0)$$

has edges $\{1, 9\}$, $\{7, 9\}$, $\{8, 9\}$, $\{4, 5\}$, and $\{1, 2\}$. It is clear that this procedure defines a Stanley graph, since no vertex has neighbors both to the left and to the right. Furthermore,
the construction is reversible: A Stanley graph with exactly \( n - k \) vertices that have neighbors to the right corresponds to a full subspace of dimension \( k \). Q.E.D.

**Appendix.** (The following remarks are mostly for my own records, but I include them here because I'm sending a copy of this letter to Ira Gessel, Richard Stanley, Herb Wilf, and Neil Sloane.) Stanley graphs are interesting combinatorial objects that do not seem to have been investigated before; at least, the number of such graphs

\[
\begin{align*}
f(0), f(1), f(2), \ldots &= 1, 1, 2, 6, 158, 1330, 15414, 245578, 5382862, \ldots
\end{align*}
\]

is a sequence not yet in Sloane and Plouffe's *Encyclopedia of Integer Sequences*. Nor is the number of such graphs without isolated vertices,

\[
1, 0, 1, 2, 11, 72, 677, 8686, 152191, 3632916, \ldots
\]

obtained by multiplying the exponential generating function by \( e^{-x} \); this corresponds to full subspaces in which the canonical basis vectors all have at least two nonzero components. Nor is the number of connected Stanley graphs,

\[
0, 1, 1, 2, 8, 52, 502, 6824, 127166, 3205924, \ldots
\]

obtained from \( \ln \sum f(n)x^n/n! \). Nor is the number of Stanley trees,

\[
0, 1, 1, 2, 7, 36, 246, 2140, 21652, 260720, 3598120, \ldots
\]

it is interesting to enumerate the latter (see below).

A graph on the vertices \( \{1, 2, \ldots, n\} \) is a Stanley graph if and only if no vertex has both left and right neighbors. Thus every Stanley graph is bipartite: Every edge connects a vertex with right neighbors to a vertex with left neighbors. Conversely, the vertices of every bipartite graph can be labeled \( \{1, 2, \ldots, n\} \) in such a way that we obtain a Stanley graph, simply by assigning the smallest labels to the vertices of one part and the largest labels to the vertices of the other.

Let \( |S| \) be the number of Stanley trees on \( n \) vertices such that \( k \) of the vertices have right neighbors and \( n - k \) have left neighbors. Let

\[
s(x, y) = \sum_{k,n} \frac{n!}{k!} \frac{x^k y^{n-k}}{n!}
\]

\[
= \frac{xy}{2} + \frac{x^2 y + xy^2}{6} + \frac{x^3 y + 5x^2 y^2 + xy^3}{24} + \ldots
\]
Then by standard methods of combinatorial enumeration we find

\[ y + \vartheta s(x, y) = y \exp(x + \vartheta x \cdot s(x, y)), \]

where \( \vartheta_x \) is the differential operator \( x \frac{\partial}{\partial x} \) and \( \vartheta = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \). It follows that we have the recurrence

\[
\begin{align*}
\begin{vmatrix} n+1 \\ k \end{vmatrix} &= \delta_{n1} \delta_{k1} + k \begin{vmatrix} n \\ k \end{vmatrix} + \begin{vmatrix} n \\ k-1 \end{vmatrix} + \sum_{m,j} \begin{vmatrix} n-1 \\ m-1 \end{vmatrix} \begin{vmatrix} m \\ j \end{vmatrix} \begin{vmatrix} n+1-m \\ k-j \end{vmatrix}.
\end{align*}
\]

these numbers \( \begin{vmatrix} n \\ k \end{vmatrix} \) form the “Gessel-Stanley triangle”

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By the correspondence above, \( \begin{vmatrix} n \\ k \end{vmatrix} \) is the number of ways to put 0s and 1s into a tableau shape (Ferrers diagram) with \( k \) rows and \( n-k \) columns in such a way that exactly \( n-1 \) entries are equal to 1, and there is no way to partition the rows and columns into two parts \( R_1 \cup R_2 \) and \( C_1 \cup C_2 \) such that the entries of \( R_1 \cap C_1 \) and \( R_2 \cap C_2 \) are entirely zero. For example, the 17 such arrays when \( n = 5 \) and \( k = 2 \) are

\[
\begin{align*}
111 & 111 111 111 111 111 111 110 110 \\
1 & 10 100 01 010 001 101 011 \\
101 & 101 101 011 011 011 001 010 100 \\
11 & 110 011 11 110 101 111 111 111
\end{align*}
\]

We have \( \sum \begin{vmatrix} n \\ 2 \end{vmatrix} |x^n| = x^3/(1-x)(1-2x)^2 \), according to Sloane and Plouffe.

After writing the above, I happened to see Ira Gessel's paper in *Electronic J. Combinatorics* 3 (2) (1996), #R8 [The Foata Festschrift, 197–201], where the quantity \( \begin{vmatrix} n \\ k \end{vmatrix} \) is called \( u_{n-1,k-1,n-k-1} \); Gessel’s \( u_{n,i,j} \) is the number of forests on vertices \( \{1,2,\ldots,n\} \) having \( i \) descents and \( j+1 \) leaves. So Stanley trees have implicitly been considered before!