Continued fractions with period 1 or 2

If k is an integer between m^2 and $(m+1)^2, m > 0$, it can be written as

$$k = m^2 + r, \qquad 1 \le r \le 2m$$

$\sqrt{k}$ has a continued fraction $k = [a_0;\ a_1, a_2, a_1, a_2, \ldots]$ (period 1 or 2), if, and only if, r is a divisor of 2 m. Then $a_0 = m, a_1 = \frac{2m}{r}, a_2 = 2m$.

<u>Proof:</u>

$a_0 = m$, remainder: $\varepsilon_0 = \sqrt{m^2 + r} - m, 0 \le \varepsilon_0 < 1$

Following remainders: $\varepsilon_j = \frac{1}{\varepsilon_{j-1}} - a_j$

Period 1: $\varepsilon_1 = \varepsilon_0$, period 2: $\varepsilon_2 = \varepsilon_0$

$\frac{1}{\varepsilon_0} = \frac{1}{\sqrt{m^2+r}-m} = \frac{\sqrt{m^2+r}+m}{r} = a_1 + \varepsilon_1$ with $a_1 = \left\lfloor \frac{2m}{r} \right\rfloor$

or (1) $a_1 = \frac{2m-s}{r}, s = (2m)\ mod\ r.$

<u>$r = 1$:</u> $a_1 = 2m$ and $\varepsilon_1 = \sqrt{m^2 + 1} - m = \varepsilon_0$ (period 1)

<u>$r > 1$:</u> $\varepsilon_1 = \frac{1}{\varepsilon_0} - a_1 = \frac{\sqrt{m^2+r}+m}{r} - \frac{2m-s}{r} = \frac{\sqrt{m^2+r}-(m-s)}{r}$

$$\frac{1}{\varepsilon_1} = \frac{r}{\sqrt{m^2 + r} - (m - s)} = \frac{r(\sqrt{m^2 + r} + m - s)}{m^2 + r - (m - s)^2}$$

Denominator: $r + s(2m - s) = r + sra_1$, see (1).

Reduced fraction: $\frac{1}{\varepsilon_1} = \frac{\sqrt{m^2+r}+m-s}{1+sa_1}$.

The period 2 requires $\varepsilon_2 = \varepsilon_0$ after the next step and therefore

$$\frac{1}{\varepsilon_1} = \sqrt{m^2 + r} + c$$

For that, $1 + sa_1 = 1$ or $s = 0$.

Then $c = m\ and\ a_2 = 2m\ and\ \varepsilon_2 = \frac{1}{\varepsilon_1} - a_2 = \sqrt{m^2 + r} - m = \varepsilon_0$ (period 2)

With $s = 0$, $a_1 = \frac{2m}{r}$ is an integer and r a divisor of $2m$. (q.e.d.)

As a trivial consequence, the number of terms $k = m^2 + r$, which are between m^2 and $(m+1)^2$, is equal to the number of divisors of $2m$.

Continued fraction: $k = \left[m;\ \frac{2m}{r}, 2m, \frac{2m}{r}, 2m, \ldots\right]$