A319929(n, k) can be discovered and motivated as follows.

Ask: When is f(a, b) = xa + yb + z associative?

f(f(a, b), c) = f(a, f(b, c))
f(xa+yb+z, c) = f(a, xb+yc+z)
x(xa+yb+z) + yc + z = xa + y(xb+yc+z) + z
$x^2a + xyb + xz + yc + z = xa + xyb + y^2c + yz + z$
$(x^2 – x)a – (y^2 – y)c + (x – y)z = 0$

Since the a term, the c term and the constant term are independent, all three components must equal 0.

(1) $x^2 – x = 0$ and
(2) $y^2 – y = 0$ and
(3) (x – y)z = 0

(1) x = 1 or x = 0
(2) y = 1 or y = 0
(3) x = y or z = 0

Case 1: choose x=1 and y=1, then z is unrestricted
Case 2: choose x=1 and y=0, then z is 0
Case 3: choose x=0 and y=1, then z is 0
Case 4: choose x=0 and y=0, then z is unrestricted

We've learned that f(a, b) is associative if and only if

	f(a, b) = a + b + z	for any z
or	f(a, b) = a	
or	f(a, b) = b	
or	f(a, b) = z	for any z

There is no reason a, b and z cannot be complex numbers.

The last three functions are trivial examples of associativity. f(a, b) = a + b + z is addition with an offset. Its identity element is -z.

Now, there is another way to interpret the result of this investigation into simple associative arithmetic. We can come away with a single function that is split into four cases.

$$f(a, b) = \begin{cases} a + b + z1 & \text{if a and b are in class z1} \\ a & \text{if a is in class z2 and b is in class z1} \\ b & \text{if a is in class z1 and b is in class z2} \\ z2 & \text{if a and b are in class z2} \end{cases}$$

We assume the following. The numbers z1 and z2 are not equal. The two classes are mutually exclusive and together they include all a and b. There is a suitable rule for adding elements of the two classes and placing the result in one of the classes.

A function of this form is globally associative. It has -z1 for its unique two-sided identity element. z2 is its unique two-sided zero element, meaning f(a, z2) = z2 for all a. By itself, f(a, b) = a + b + z1 has an identity element, -z1; f(a, b) = z2 has a zero element, z2. For f(a, b) = a by itself, any number can be a left zero element or a right identity element. The situation is reversed for f(a, b) = b.

If we restrict a and b to positive integers, use the positive odd numbers and the positive even numbers as the two classes, and choose z1 = -1 and z2 = 0, we get A319929(n, k).