Goursat & Elliptic Curves

periods converter

See also: https://community.wolfram.com/groups/-/m/t/2926119

In[1]:=
```
$WPFCs = {…}
```

Out[1]=
$$\{-21\,G2[0]\,G2[1]^3\,G3[0]+18\,G2[0]^2\,G2[1]^2\,G3[1]+8\,G2[0]^3\,G2[2]\times G3[1]-216\,G2[2]\,G3[0]^2\,G3[1]+108\,G2[1]\times G3[0]\,G3[1]^2-120\,G2[0]\,G3[1]^3-8\,G2[0]^3\,G2[1]\times G3[2]+216\,G2[1]\,G3[0]^2\,G3[2],\\ -16\,(9\,G2[0]^2\,G2[1]^2\,G3[0]-3\,G2[0]^3\,G2[2]\times G3[0]+81\,G2[2]\,G3[0]^3-7\,G2[0]^3\,G2[1]\times G3[1]-135\,G2[1]\,G3[0]^2\,G3[1]+108\,G2[0]\times G3[0]\,G3[1]^2+2\,G2[0]^4\,G3[2]-54\,G2[0]\,G3[0]^2\,G3[2]),\\ 16\,(G2[0]^3-27\,G3[0]^2)\,(-3\,G2[1]\times G3[0]+2\,G2[0]\times G3[1])\}$$

In[2]:=
```
EllipticPeriodsAnnihilator[
  g2x_, g3x_] := With[{res0 = ReplaceAll[
     $WPFCs, {
      G2[n_] :↦ D[g2x, {x, n}],
      G3[n_] :↦ D[g3x, {x, n}]
     }
    ]},
  Factor[Cancel[Divide[#, PolynomialGCD @@ #
      ] &[Together[res0]]
   ]]
]
```

After Herfurtner 1991

https://link.springer.com/article/10.1007/BF01445211

2,3,5,II ~ A318495

$$I_2 I_3 I_5 II \quad (-\tfrac{5}{9}, 0, \infty, 3)$$

$$G_2 = 3(X-3Y)(81X^3-9X^2Y-53XY^2-27Y^3)$$
$$G_3 = (X-3Y)(3^6X^5-3^5\cdot 5X^4Y-2\cdot 3^3\cdot 5^2X^3Y^2-350X^2Y^3 - 3^3\cdot 5^2XY^4-243Y^5)$$
$$\Delta = -2^{14}\cdot 3^4X^3Y^5(X-3Y)^2(9X+5Y)^2$$
$$\mathcal{J} = -\frac{1}{2^{14}\cdot 3}\frac{(X-3Y)(81X^3-9X^2Y-53XY^2-27Y^3)^3}{X^3Y^5(9X+5Y)^2}$$
$$CR = \tfrac{27}{32}$$

In[4]:=
```
data = Together[{
      3 (x - 3 y) (81 x^3 - 9 x^2 y - 53 x y^2 - 27 y^3),
      (x - 3 y) * (3^6 x^5 - 3^5 * 5 * x^4 y - 2 * 3^3 × 5^2 x^3 y^2 - 350 * x^2 * y^3
         - 3^3 * 5^2 x y^4 - 243 * y^5)
    } /. y → 1 / 3 (1 + x) /. x → 27 x / 32 - 1]
```

Out[4]=
$$\left\{-\frac{243}{256}\left(-256+640\,x-520\,x^2+135\,x^3\right),\ -\frac{729\left(-8192+30\,720\,x-44\,160\,x^2+29\,680\,x^3-8775\,x^4+729\,x^5\right)}{8192}\right\}$$

In[5]:=
```
odeA = Times[-1, EllipticPeriodsAnnihilator @@ data]
```

Out[5]=
$$\{5\,(-16+21\,x),\ 4\left(32-118\,x+81\,x^2\right),\ 4\,(-1+x)\,x\,(-32+27\,x)\}$$

In[6]:=
```
previous = Factor[{5 * (21 * x - 16), D[#, x], #} &[4 * x * (x - 1) * (27 * x - 32)]]
```

Out[6]=
$$\{5\,(-16+21\,x),\ 4\left(32-118\,x+81\,x^2\right),\ 4\,(-1+x)\,x\,(-32+27\,x)\}$$

In[7]:=
```
odeA - previous
```

Out[7]=
$$\{0, 0, 0\}$$

Our previous result!

2,3,4, III ~ A318245

Table 3 (continued)

In[8]:=

$$I_2 I_3 I_4 III \quad (-\tfrac{1}{3}, 0, \infty, 1)$$

$$G_2 = 3(X-Y)(16X^3-3XY^2-Y^3)$$
$$G_3 = (X-Y)^2(64X^4+32X^3Y+6X^2Y^2+5XY^3+Y^4)$$
$$\Delta = 2^2\cdot 3^6X^3Y^4(X-Y)^3(3X+Y)^2$$
$$\mathcal{J} = \frac{1}{108}\frac{(16X^3-3XY^2-Y^3)^3}{X^3Y^4(3X+Y)^2}$$
$$CR = \tfrac{3}{4}$$

In[9]:=
```
data = Together[{
       3 (x - y) (16 x^3 - 3 x y^2 - y^3),
       (x - y)^2 (64 x^4 + 32 x^3 y + 6 x^2 y^2 + 5 x y^3 + y^4)
      } /. y → (1 + x) /. x → 3 / 4 x - 1]
```

Out[9]=
$$\left\{-\frac{3}{16}\left(-256+576\,x-405\,x^2+81\,x^3\right),\ \frac{1}{64}\left(4096-13\,824\,x+17\,496\,x^2-9963\,x^3+2187\,x^4\right)\right\}$$

In[10]:=
```
odeA = Times[-1, EllipticPeriodsAnnihilator @@ data]
```

Out[10]=
$$\left\{9\,(-4+5\,x),\ 16\left(4-14\,x+9\,x^2\right),\ 16\,(-1+x)\,x\,(-4+3\,x)\right\}$$

In[11]:=
```
previous = Factor[{9 * (5 * x - 4), D[#, x], #} &[16 * x * (x - 1) * (3 * x - 4)]]
```

Out[11]=
$$\left\{9\,(-4+5\,x),\ 16\left(4-14\,x+9\,x^2\right),\ 16\,(-1+x)\,x\,(-4+3\,x)\right\}$$

In[12]:=
```
odeA - previous
```

Out[12]=
$$\{0, 0, 0\}$$

Our previous result!

Acknowledgements

Many thanks to Duco Van Straten for bringing this reference to our attention.

Author

Bradley Klee

May 29, 2023