Example for $n = 7$ ($U'(7)$), $m = 2$ ($\mathbb{Z}/3\mathbb{Z}$)

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As an example, consider $n = 7$. We have $U(7) = \{1, 2, 3, 4, 5, 6\}$. Meanwhile, $U'(7) = \{1, 2, 3\}$. The following table presents the Cayley table of $U'(7) \cong U(7)/\{\pm 1\}$, and shows that it is a cyclic group isomorphic to $\mathbb{Z}/3\mathbb{Z}$:

$\cdot$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{2}$	$\overline{2}$	$\overline{3}$	$\overline{1}$
$\overline{3}$	$\overline{3}$	$\overline{1}$	$\overline{2}$

TABLE 1. Cayley table for $U'(n)$, $n = 7$

On the other hand, we have the Cayley table for $\mathbb{Z}/3\mathbb{Z}$:

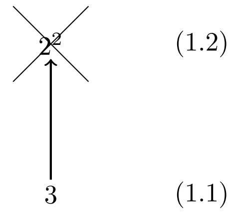
$+$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{0}$
$\overline{2}$	$\overline{2}$	$\overline{0}$	$\overline{1}$

TABLE 2. Cayley table for $\mathbb{Z}/3\mathbb{Z}$

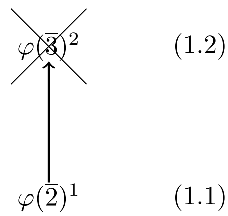
We shall give an example illustrating this algorithm for $n = 7$. In this case, $U'(n)$ has 3 elements, $\overline{1}, \overline{2}, \overline{3}$. Observe the Cayley table for $U'(n)$ in Table 1. Since all the elements in any element form are all irreducible or are all not irreducible, it suffices to work with one representative of each element, so that in this specific case, we can consider $13 \in \overline{1}, 2 \in \overline{2}$, $3 \in \overline{3}$. We can represent the algorithm as the following graph, for a_2:

$$\begin{tikzcd} \cancel{3^2} & (1.2) \\ 2 \arrow[u] & (1.1) \end{tikzcd}$$

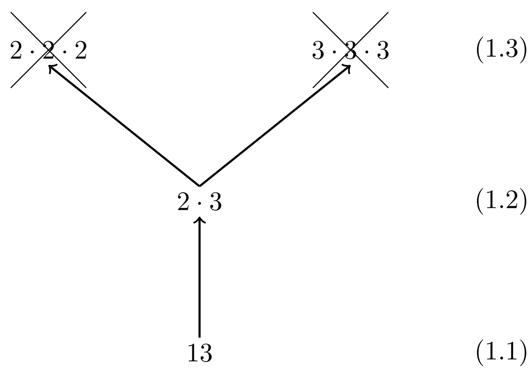
Note that at each level, every product has the same number of factors as the number of the level it is in. So we start with a single node, as in (1.1). We then do all simple substitutions, which can be seen in (1.2), in the second step. Now, with this level, (1.2), we see that the number turns out to factorize. Thus, we cross it out as it does not produce any irreducible elements. This concludes the process for a_2. Similarly for a_3:



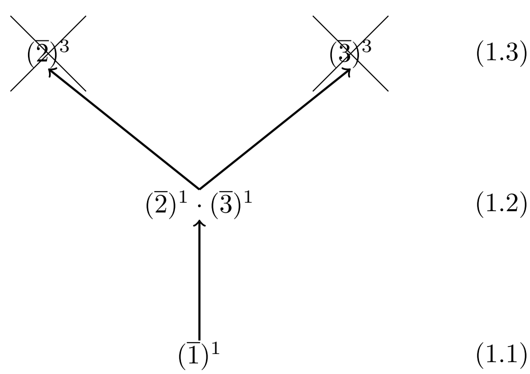
Since the order of $\overline{2}$ is the same as the order of $\overline{3}$, these graphs can be summarized as the following graph:



where φ is the automorphism that is either the identity or the isomorphism that takes $\overline{2}$ to $\overline{3}$. Repeating this process for $\overline{1}$, we obtain:



Noting that all automorphisms leave the graph identical, we obtain that the graph of irreducible element forms for $\overline{1}$ is:



And so, the four irreducible element forms for $n = 7$ are $(\overline{1})^k(\overline{2})^1$, $(\overline{1})^k(\overline{3})^1$, $(\overline{1})^1$, $(\overline{2})^1(\overline{3})^1$.

Repeating the exact same process described above, but for the Cayley table of $\mathbb{Z}/3\mathbb{Z}$ we obtain the following graphs:

$$\begin{tikzcd}
\cancel{2+2} & (1.2) \\
1 \arrow[u] & (1.1)
\end{tikzcd}$$

$$\begin{tikzcd}
\cancel{1+1} & (1.2) \\
2 \arrow[u] & (1.1)
\end{tikzcd}$$

$$\begin{tikzcd}
\cancel{1+1+1} & & \cancel{2+2+2} & (1.3) \\
& 1+2 \arrow[ul] \arrow[ur] & & (1.2) \\
& 0 \arrow[u] & & (1.1)
\end{tikzcd}$$

The same analogy can be made about the automorphisms of $\mathbb{Z}/3\mathbb{Z}$. In the end, the irreducible element forms for $\mathbb{Z}/3\mathbb{Z}$ are $1, 2, 0, 1+2$, which are in one to one correspondance (and are obtained in the same way) with the ones above. The last three graphs contain all of the necessary information to create any of the graphs for $U'(7)$.

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