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# Numbers n such that n * Rev(n) is square.
# The different families

Definition: The reversal of a number n is the number which is obtained when n is written backwards (see A004086).

Convention: “Leading zeros (after the reversal has taken place) are omitted” (NJAS).

Notation in OEIS: Rev(n) or rev(n) or R(n)

There are in OEIS eight different sequences with an infinite number of terms n such that n * Rev(n) is square; here is a brief methodical presentation of these sequences with some properties and examples. These numbers belong also to seven families of integers (see p. 5).

There is now in OEIS a new sequence containing all these numbers: A306273.

## I) Numbers n such that n = Rev(n)

These numbers are exactly palindromic numbers in A002113 (family 1).

Formula: $n * Rev(n) = n^2$

Example: $131 * 131 = 131^2$

## II) Numbers such that n != Rev(n) with n * Rev(n) is square

All these numbers can be found in A070760, where there are six hidden different families in this sequence.

### 2.1) n and Rev(n) have the same number of digits

These numbers are in A062917, but there are two different families of such integers here.

#### 2.1.1) n and Rev(n) are both square numbers.

These numbers seem happy in A035090 (family 2).

This sequence is infinite, since it includes e.g.

$10^{2k} + 4 * 10^k + 4 = (10^k + 2)^2$ (from Robert Israel), and also,

$10^{2k} + 6 * 10^k + 9 = (10^k + 3)^2$ for $k >= 1$.

Examples:

$144 * 411 = 12^2 * 21^2 = 252^2$

$10404 * 40401 = 102^2 * 201^2 = 20502^2$

$10609 * 90601 = 103^2 * 301^2 = 31003^2$

$1089 * 9801 = (3^2 * 11^2) * (3^4 * 11^2) = (3^3 * 11^2)^2 = 3267^2$

### 2.1.2) n and Rev(n) are not both square

These numbers sleep quietly in A082994 (family 3).

This sequence has also an infinite number of terms, two infinite such subsequences are described below.

With 1089 = 3^2 * 11^2 which belongs so to A035090, we obtain 10x89 which is a term of A082994 where x is a string of k repeated digits 9 with k >= 1, i.e. 10989, 109989,...

With 2178 = 2 * 3^2 * 11^2 which belongs to A082994 this time, we obtain 21x78 which is a term of A082994 where x is a string of k repeated digits 9 with k >= 1, i.e. 21978, 219978, 2199978, 21999978, 219999978,...

Remark: The numbers 1089, 2178, 10989, 21978,... are the numbers n such that the reversal of these integers n is a nontrivial multiple of n. These integers are in A008919 which is so a subsequence of A062917, with 1089 which is square in A035090, and all the other terms are in A082944. We have more with these integers n:

n * m^2 = Rev(n) with m = 2 (for 2178 and its family) or 3 (for 1089 and its family)
1089 * 9 = 9801 = Rev(1089)
2178 * 4 = 8712 = Rev(2178)
10989 * 9 = 98901 = Rev(10989)

Examples:
n * Rev(n) = n * (n * m^2) = (n * m)^2
10989 * 98901 = (3^3 * 11 * 37) * (3^5 * 11 * 37) = (3^4 * 11 * 37)^2 = 32967^2
2178 * 7812 = (2 * 3^2 * 11^2) * (2^3 * 3^2 * 11^2) = 4356^2
21978 * 87912 = (2 * 3^3 * 11 * 37) * (2^3 * 3^3 * 11 * 37) = 43956^2, also,
528 * 825 = 660^2

### 2.1.3) Relation between these last sequences.

A062917 = A035090 Union A082994 with empty intersection.

## <u>2.2) n and Rev(n) have not the same number of digits.</u>

These numbers are all present in A322835.

If n and Rev(n) have a different number of digits, so, n ends necessary with one or more zeros.

### 2.2.1) n ends with an even number of zeros.

There is not OEIS specific sequence for these numbers with an even number of zeros at the end. The three present families below come from the three previous families met above (from & I, 2.1.1 and 2.1.2).

A002113(j) * 10^(2k) for k >= 1 (family 4)
Example: 13100 * 131 = 131^2 * 10^2 = 1310^2

A035090(j) * 10^(2k) for k >= 1 (family 5)
Example: 1440000 * 441 = 1200^2 * 21^2 = 25200^2

A082994(j) * 10^(2k) for k >= 1 (family 6)
Example: 288000000 * 882 = (2 * 144) * 10^6 * (2 * 441) = 504000^2

When I created the sequence A322835 in OEIS, it was in order to bring together all the numbers n which end by some zeros, when n * Rev(n) is square and which belong to the three previous families above.

Happily, Chai Wah Wu came to visit this new sequence on Jan 07 2019...and...

### 2.2.2) n ends with an odd number of zeros

Chai Wah Wu has dicovered a new family of integers where n and Rev(n) have not the same number of digits and n * Rev(n) is square.

This new family is A323061(j) * 10^(2k+1) for k >= 0 (family 7).

As these numbers end with an odd numbers of zeros, so, in order to get a new zero which is necessary for that n * Rev(n) is square, “the product of the first and last digit of a term (of A323061) is a multiple of 10, i.e. the first and last digit are the digit 5 and an even non zero digit”. (from Chai Wah Wu)

Chai Wah Wu finds that there are only 30 such terms among the first 1000 terms of A322835. The first one is 5449680, it’s the 755$^{th}$ term.

Chai Wah Wu proves also that the sequence A323061 has an infinite number of terms:

“For instance, 601x065 is a term where x is a string of k repeated digits 6 where k >= 0, i.e., 601065, 6016065, 60166065, etc. Similarly 560x106 are also terms”.

Examples:
5449680 * 869445 = 2176740^2
5601060 * 601065 = 1834830^2
5606106000 * 6016065 = 183648300^2

Why not call these numbers: Chai Wah Wu numbers? It looks like these numbers were not known before.

### 2.2.3) Relation between all these sequences

A070760 = A062917 Union A322835 with empty intersection.
A306273 = A002113 Union A070760 with empty intersection

The sets {A002113}, {A062917} and {A322835} realize such a partition of {A306273}. In page 5, the links between these different sequences and families appear.

## III) Squares N equal to the product of an integer and its reversal in (at least) two ways

The numbers which are equal to the product of an integer and its reversal in (at least) two ways are called EPRN = Equal Product of Reversible Numbers in OEIS, with reference to Shyam Sunder Gupta website.

### 3.1) Squares N don't end with 0

These square numbers are square of palindromes.

The smallest square is 63504 = 252 * 252 = 144 * 441, with 144 in A035090 (family 2).

The second one is 7683984 = 2772 * 2772 = 1584 * 4851 with 1584 in A082994 (family 3).

The palindromes whose squares can be expressed as the product of an other integer and its reversal are in A117281. There is an infinite number of terms in this sequence.
For instance, 27x72 is a term where x is a string of k repeated digits 9 where k >= 0, with:

(27x72)^2 = 15x84 * 48x51
2772^2 = 1584 * 4851
27972 ^2 = 15984 * 48951
279972^2 = 159984 * 489951

### 3.2) Squares N with trailing zeros

The smallest square is here 435600 = 660^2 = 6600 * 66 = 528 * 825.

The next one is 6350400 =  2520^2 = 25200 * 252 =  14400 * 441 = 44100 * 144.
So, 6350400 is also the smallest square which is the product of a number and its reversal in three different ways.

### 3.3) The sequences in OEIS

The squares equal to the product of an integer and its reversal in (at least) two ways are in:

A083406 when these squares are even,
A083407 when these squares are odd,
A083408 for all these square numbers; we have:
A083408 = A083406 Union A083407 with empty intersection.

The first such odd square is 1239016098321 = 1113111 * 1113111 = 1022121 * 1212201.
This number is only the $50^{th}$ term of A083408.

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n * Rev(n) is square
Families and subsequences of A306273

n = Rev(n)
Palindromes
A002113
Family 1

n <> Rev(n)
A070760

n and Rev(n) have
the same number of digits
A062917

n and Rev(n)
are both square
A035090
Family 2

n and Rev(n)
are not square
A082994
Family 3

n and Rev(n) have not
the same number of digits
<==> n ends with 0
A322835

n ends with an even
number of zeros
(no OEIS sequence, but
three families)

Family 4: A002113(j) * 10^(2k) for k >= 1
Family 5: A035090(j) * 10^(2k) for k >= 1
Family 6: A082994(j) * 10^(2k) for k >= 1

n ends with an odd
number of zeros
A323061(j) * 10^(2k+1) for k >= 0
Family 7

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