CHORDLESS CYCLES IN THE n-ANDRÁSFAI GRAPH

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ABSTRACT. The n-Andrásfai graph is a simple, n-regular graph on $N = 3n-1$ nodes. We put their chordless cycles into equivalence classes, equivalent in the sense of being the same apart from rotations and flips in the planar drawings, and illustrate these chordless cycles for $n \le 8$ by plotting one member of each class.

1. NOTATION

The n-Andrásfai graph $Ant(n)$ with $N = 3n-1$ vertices is constructed by first drawing the 2-regular cyclic graph of the N-gon. A natural labelling of its vertices is $i = 0, 1, \ldots N-1$ counter-clockwise starting at the East as if the vertices were the roots of the cyclotomic polynomial ϕ_N on the unit circle in the complex plane. Then edges are added between all vertex pairs (v_i, v_j) where where $|i-j| \equiv 1 \pmod 3$. The result is a n-regular graph (including the slightly degenerate simple graph on 2 vertices) with $n(3n-1)/2$ edges. The subgraph of the N-gon is a natural Hamiltonion cycle.

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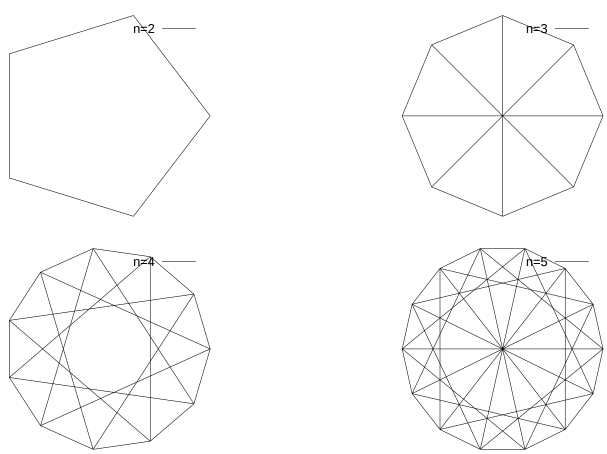


FIGURE 1. The graphs A_n for $n = 2 \ldots 5$. There is no vertex at the crossing lines at the center. All vertices are on the outer rim of the regular N-gon.

A walk on the graph can be encoded in a Lederberg-Coxeter-Frucht (LCF) notation [3]. Once an initial vertex is selected, the notation is a list of integer strides between adjacent vertices of the walk in square brackets determined by the graph-theoretical distance of the two vertices on the subgraph of the N-gon (Hamiltonian cycle). The stride is this positive distance if the next node is reached faster circling the N-gon counter-clock-wise, and the negative of that distance if it is reached faster circling it clock-wise. So the Hamiltonian cycle is endowed with an orientation and the stride's sign indicates moving forward or backward on it. In formula: A closed walk $v_0 \to v_1 \to v_2 \to \ldots \to v_l = v_0$ of length l, $0 \le v_i < N$, is encoded as $[s_1, s_2, \ldots s_l]$ where

$$s_i = \begin{cases} v_i - v_{i-1}, & 0 < v_i - v_{i-1} \le N/2 \\ v_i - v_{i-1} - N, & N/2 < v_i - v_{i-1} \\ v_i - v_{i-1}, & -N/2 < v_i - v_{i-1} < 0 \\ v_i - v_{i-1} + N, & v_i - v_{i-1} \le -N/2 \end{cases} \tag{1}$$

In the n-Andrásfai graphs these strides are $s_i \in \{\pm 1, \pm 4, \pm 7, \pm 10, \pm 13 \ldots\}$ with $|s_i| \le N/2$. (For even N, an ambiguity about the sign of walking the edge through the center of the planar graph is resolved by specifying $+N/2$ in this case.)

Remark 1. *These strides could be mapped to lattice paths of up- and down steps akin to Dyck-, Motzkin and similar paths.*

The graphs have an obvious symmetry illustrated in Figure 1: rotating the graph around the $N-$fold axis of the N-gon leaves it invariant, and flipping the graph along a line that connects a vertex with the opposite vertex (N odd) or the center of the opposite edge (N even) also leaves it invariant. Therefore the Dihedral Group of $2N$ elements is a subgroup of the automorphism group [1, §3].

Remark 2. *For $n = 1$, the connected simple graph on 2 vertices, the group is only the cyclic group of 2 elements, given by swapping the two vertices.*

2. Chordless Cycles

Cycles are defined as usual: closed trails where each vertex is visited at most once and each edge used at most once. (The starting vertex is, philosophically speaking, visited twice, but on the unlabelled graph the vertex is not marked, and that is a moot point.)

For closed walks the arithmetic sum of these LCF strides is $\equiv 0 \pmod N$.

Definition 1.

$$w = \frac{\sum_{i=1}^{l} s_i}{N} \tag{2}$$

could be called the winding number of the cycle.

Theorem 1. *Cycles of length $l = 3$ don't exist in the Andrásfai graphs* [1].

Chordless cycles are defined as usual: cycles where each edge of the graph that is not in the cycle does not connect two vertices of the cycle. (In short: there is no shortcut that allows to reach one vertex of the cycle from another along an edge of the graph besides walking along the cycle.)

The following illustrations plot the cycle edges in red and the edges of the graph that are not in cycles as dashed lines. The LCF notation and a multiplicity of the graphs (by rotations and/or flips) are noticed. The multiplicity is the full $2N$ if the cycle is asymmetric, N if it contains a mirror line, or even smaller if it contains a rotation axis of higher order.

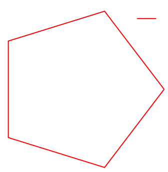
Cycles that can be mapped on each other by rotations around the central N-fold axes and/or flips are plotted only with one representative of the class.

Remark 3. *The convention here is that the representative contains the vertex $i = 0$ "most eastwards" on the N-gon, and that the cycle starts with small positive strides.*

Remark 4. *(i) Cyclic rotation of the LCF strides s_i—meaning selecting a different start vertex, (ii) reading the s_i in backward order—meaning flipping it around the diagonal through the start vertex, or (iii) flipping all their signs at the same time—meaning walking it in the opposite direction—do not change the class of the cycle.*

If multiplicities of cycles of odd length are summed up, sequence [2, A301771] results.

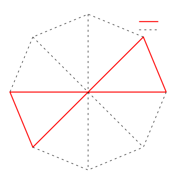
2.1. $n = 2$, $N = 5$.



[1, 1, 1, 1, 1] $\times$ 1

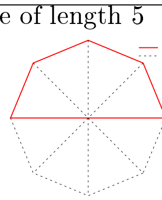
3. $n = 3$, $N = 8$

1 cycle of length 4



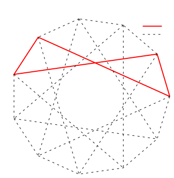
[1, 4, -1, 4], $\times N/2$

1 cycle of length 5

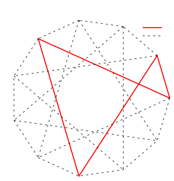


[1, 1, 1, 1, 4] $\times N$

3.1. $n = 4$, $N = 11$. 2 cycles of length 4

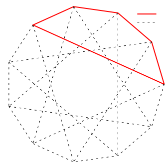


[1, 4, -1, -4] $\times N$

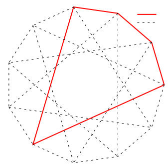


[1, -4, -4, -4] $\times N$

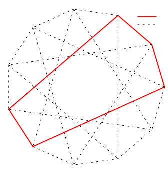
3 cycles of length 5



[1, 1, 1, 1, -4] $\times N$

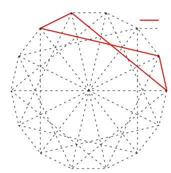


[1, 1, 1, 4, 4] $\times N$

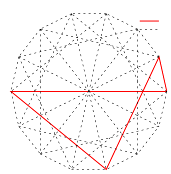


[1, 1, 4, 1, 4] $\times N$

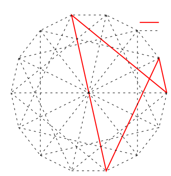
3.2. $n = 5$, $N = 14$. 5 cycles of length 4



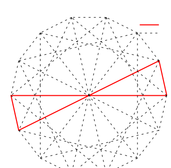
[1, 4, -1, -4] $\times N$



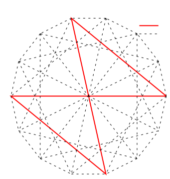
[1, -4, -4, 7] $\times 2N$



[1, -4, 7, -4] $\times N$

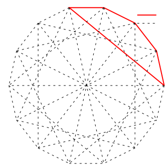


[1, 7, -1, 7] $\times N$

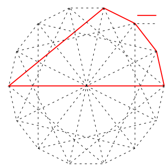


[4, 7, -4, 7] $\times N$

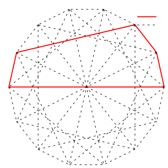
5 cycles of length 5



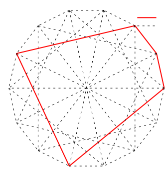
[1, 1, 1, 1, -4] $\times N$



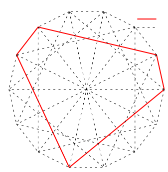
[1, 1, 1, 4, 7] $\times 2N$



[1, 1, 4, 1, 7] $\times 2N$



[1, 1, 4, 4, 4] $\times N$



[1, 4, 1, 4, 4] $\times N$

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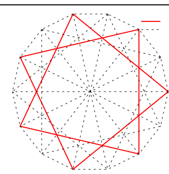
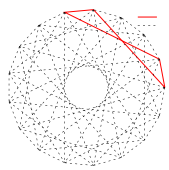
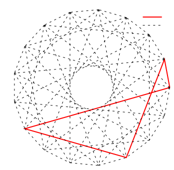
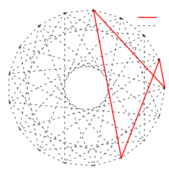
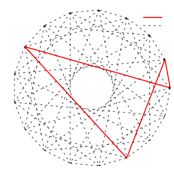
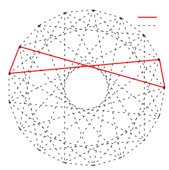
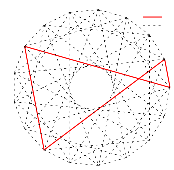
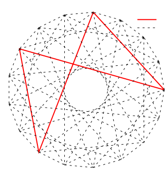
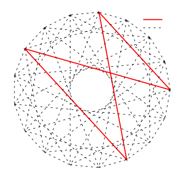
1 cycle of length 7

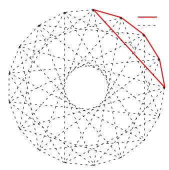


[4, 4, 4, 4, 4, 4, 4] $\times N/7$

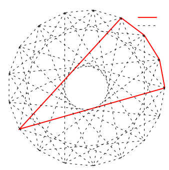
3.3. $n = 6,\ N = 17.$ 8 cycles of length 4

	[1, 4, -1, -4] $\times N$		[1, -4, -4, 7] $\times 2N$
	[1, -4, 7, -4] $\times N$		[1, -4, -7, -7] $\times 2N$
	[1, 7, -1, 7] $\times N$		[1, -7, -4, -7] $\times N$
	[4, 7, -4, -7] $\times N$		[4, -7, -7, -7] $\times N$

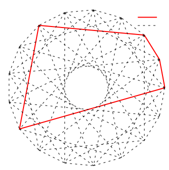
10 cycles of length 5



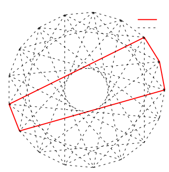
$[1, 1, 1, 1, -4]$ $\times N$



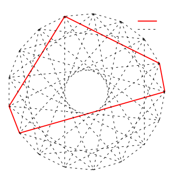
$[1, 1, 1, 7, 7]$ $\times N$



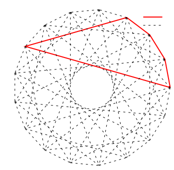
$[1, 1, 4, 4, 7]$ $\times 2N$



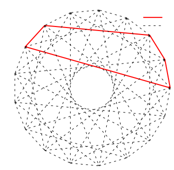
$[1, 1, 7, 1, 7]$ $\times N$



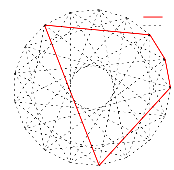
$[1, 4, 4, 1, 7]$ $\times N$



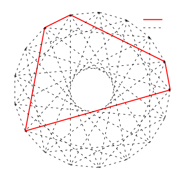
$[1, 1, 1, 4, -7]$ $\times 2N$



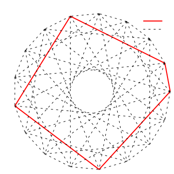
$[1, 1, 4, 1, -7]$ $\times 2N$



$[1, 1, 4, 7, 4]$ $\times N$

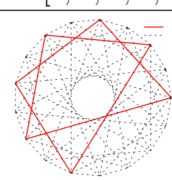


$[1, 4, 1, 4, 7]$ $\times 2N$



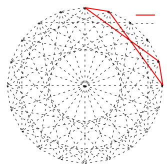
$[1, 4, 4, 4, 4]$ $\times N$

1 cycle of length 7

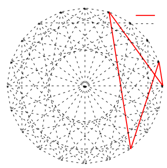


$[4, 4, 4, 7, 4, 4, 7]$ $\times N$

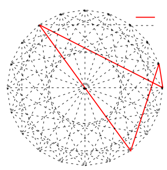
3.4. $n = 7$, $N = 20$. 14 cycles of length 4



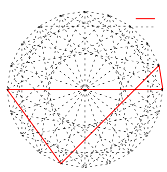
[1, 4, -1, -4] $\times N$



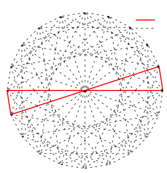
[1, -4, 7, -4] $\times N$



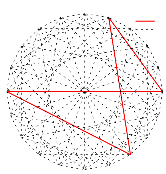
[1, -4, 10, -7] $\times 2N$



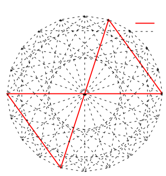
[1, -7, -4, 10] $\times 2N$



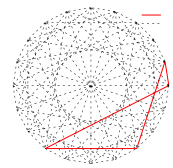
[1, 10, -1, 10] $\times N/2$



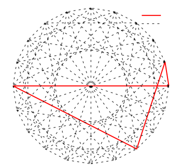
[4, -7, -7, 10] $\times 2N$



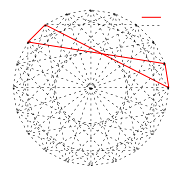
[4, 10, -4, 10] $\times N/2$



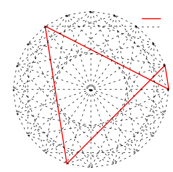
[1, -4, -4, 7] $\times N$



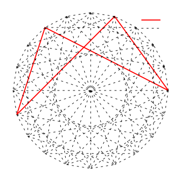
[1, -4, -7, 10] $\times 2N$



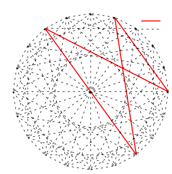
[1, 7, -1, -7] $\times N$



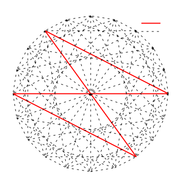
[1, -7, -7, -7] $\times N$



[4, 7, -4, -7] $\times N$

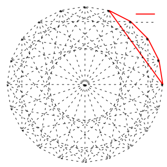


[4, -7, 10, -7] $\times N$

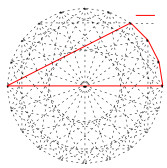


7, 10, -7, 10] $\times N/2$

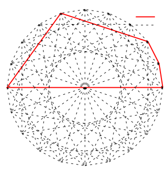
16 cycles of length 5



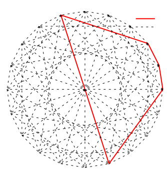
$[1, 1, 1, 1, -4]$ $\times N$



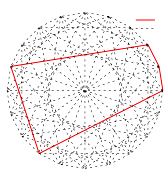
$[1, 1, 1, 7, 10]$ $\times 2N$



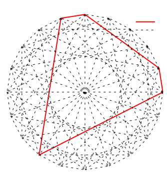
$[1, 1, 4, 4, 10]$ $\times 2N$



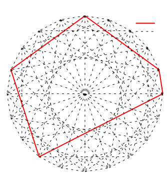
$[1, 1, 4, 10, 4]$ $\times N$



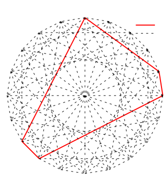
$[1, 1, 7, 4, 7]$ $\times N$



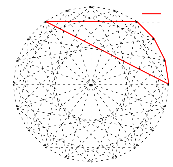
$[1, 4, 1, 7, 7]$ $\times N$



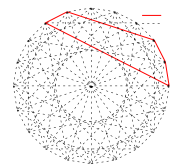
$[1, 4, 4, 4, 7]$ $\times 2N$



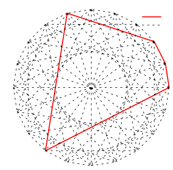
$[1, 4, 7, 1, 7]$ $\times 2N$



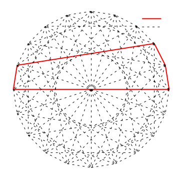
$[1, 1, 1, 4, -7]$ $\times 2N$



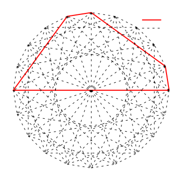
$[1, 1, 4, 1, -7]$ $\times 2N$



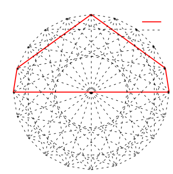
$[1, 1, 4, 7, 7]$ $\times 2N$



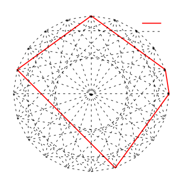
$[1, 1, 7, 1, 10]$ $\times 2N$



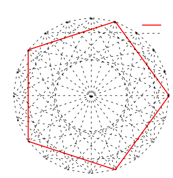
$[1, 4, 1, 4, 10]$ $\times 2N$



$[1, 4, 4, 1, 10]$ $\times N$

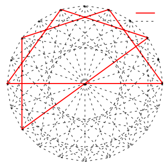


$[1, 4, 4, 7, 4]$ $\times 2N$

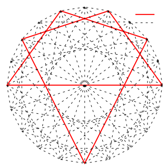


$[4, 4, 4, 4, 4]$ $\times N/5$

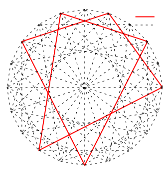
4 cycles of length 7



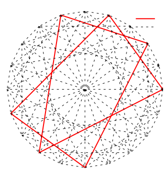
[4, 4, 4, 10, 4, 4, 10] $\times N$



[4, 4, 7, 7, 4, 4, 10] $\times N$

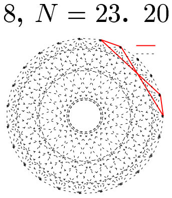


[4, 4, 7, 7, 4, 7, 7] $\times N$

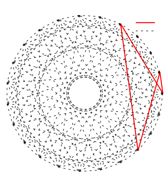


[4, 7, 4, 7, 4, 7, 7] $\times N$

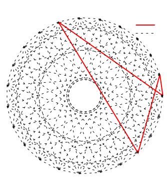
3.5. $n = 8$, $N = 23$. 20 cycles of length 4



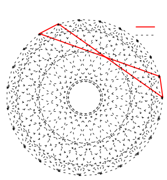
[1, 4, -1, -4] $\times N$



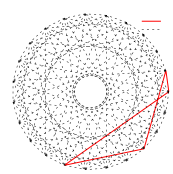
[1, -4, 7, -4] $\times N$



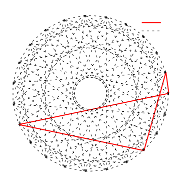
[1, -4, 10, -7] $\times 2N$



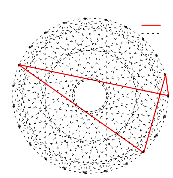
[1, 7, -1, -7] $\times N$



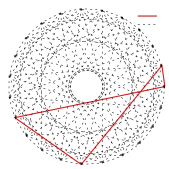
[1, -4, -4, 7] $\times 2N$



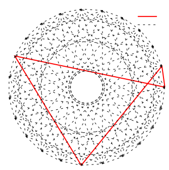
[1, -4, -7, 10] $\times 2N$



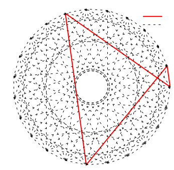
[1, -4, -10, -10] $\times 2N$



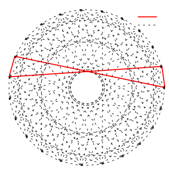
[1, -7, -4, 10] $\times 2N$



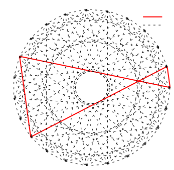
[1, -7, -7, -10] $\times 2N$



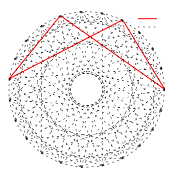
[1, -7, -10, -7] $\times N$



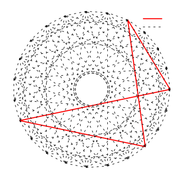
[1, 10, -1, -10] $\times N$



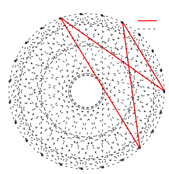
[1, -10, -4, -10] $\times N$



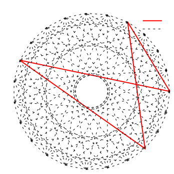
[4, 7, -4, -7] $\times N$



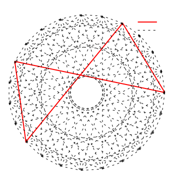
[4, -7, -7, 10] $\times 2N$



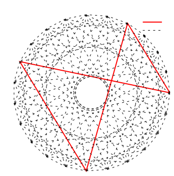
[4, -7, 10, -7] $\times N$



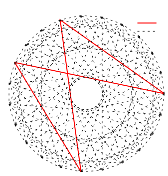
[4, -7, -10, -10] $\times 2N$



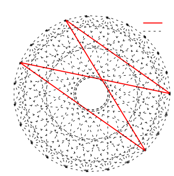
[4, 10, -4, -10] $\times N$



[4, -10, -7, -10] $\times N$

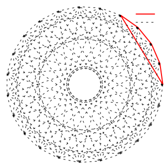


[7, 10, -7, -10] $\times N$

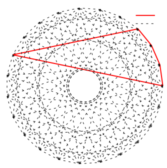


[7, -10, -10, -10] $\times N$

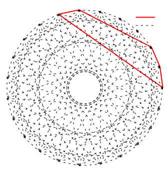
26 cycles of length 5



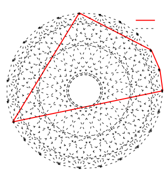
[1, 1, 1, 1, -4] $\times N$



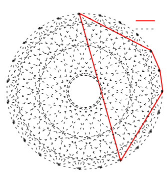
[1, 1, 1, 7, -10] $\times 2N$



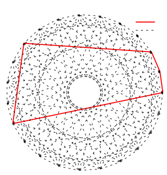
[1, 1, 4, 1, -7] $\times 2N$



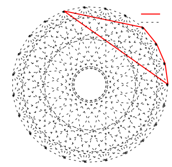
[1, 1, 4, 7, 10] $\times 2N$



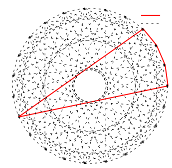
[1, 1, 4, -10, 4] $\times N$



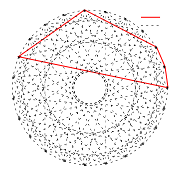
[1, 1, 7, 4, 10] $\times 2N$



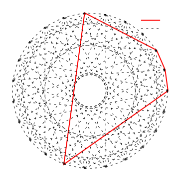
[1, 1, 1, 4, -7] $\times 2N$



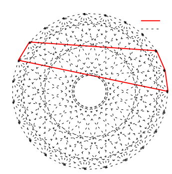
[1, 1, 1, 10, 10] $\times N$



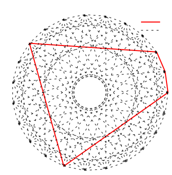
[1, 1, 4, 4, -10] $\times 2N$



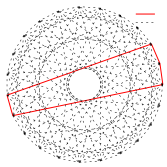
[1, 1, 4, 10, 7] $\times 2N$



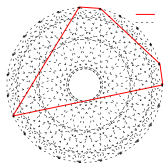
[1, 1, 7, 1, -10] $\times 2N$



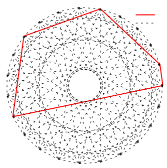
[1, 1, 7, 7, 7] $\times N$



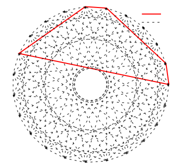
[1, 1, 10, 1, 10] $\times N$



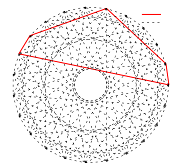
[1, 4, 1, 7, 10] $\times 2N$



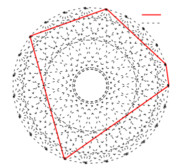
[1, 4, 4, 4, 10] $\times 2N$



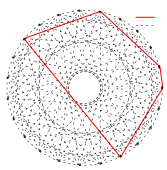
[1, 4, 1, 4, -10] $\times 2N$



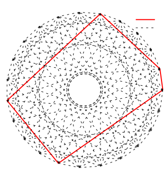
[1, 4, 4, 1, -10] $\times N$



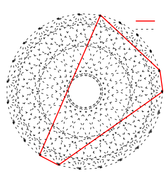
[1, 4, 4, 7, 7] $\times 2N$



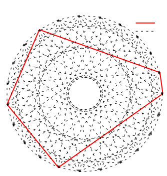
[1, 4, 4, 10, 4] $\times 2N$



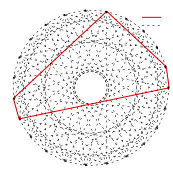
[1, 4, 7, 4, 7] $\times 2N$



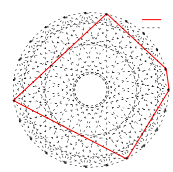
[1, 4, 10, 1, 7] $\times 2N$



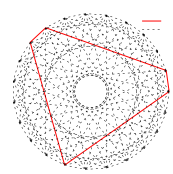
[1, 7, 4, 4, 7] $\times N$



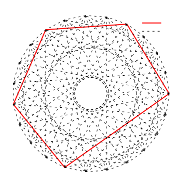
[1, 4, 7, 1, 10] $\times 2N$



[1, 4, 7, 7, 4] $\times N$

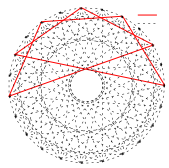


[1, 7, 1, 7, 7] $\times N$

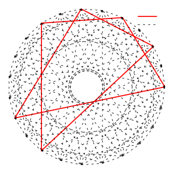


[4, 4, 4, 4, 7] $\times N$

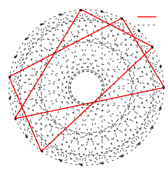
8 cycles of length 7



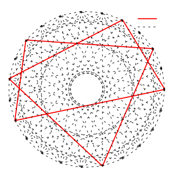
[4, 4, 4, -10, 4, 4, -10] $\times N$



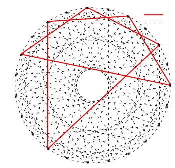
[4, 4, 7, 10, 4, 7, 10] $\times 2N$



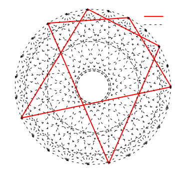
[4, 7, 4, 10, 4, 7, 10] $\times 2N$



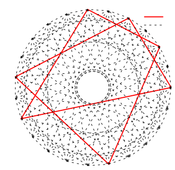
[4, 7, 7, 7, 7, 4, 10] $\times N$



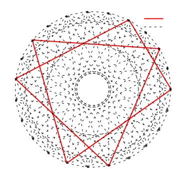
[4, 4, 7, 10, 4, 4, -10]



[4, 4, 10, 7, 4, 7, 10]



[4, 7, 7, 7, 4, 7, 10]



[4, 7, 7, 7, 7, 7, 7]

n	$N\backslash l$	4	5	7
3	8	1	1	
4	11	2	3	
5	14	5	5	1
6	17	8	10	1
7	20	14	16	4
8	23	20	26	8
9	26	30	38	20
10	29	40	57	38
11	32	55	79	76
12	35	70	111	133

TABLE 1. The number of classes of the chordless cycles of length l in the n-Andrásfai graph. Only graphs for $n \le 8$ have been plotted here. For $n \le 12$, chordless cycles of length > 7 don't exist.

4. Overview

The number of classes of chordless cycles is summarized in Table 1, complete to $n \le 12$.

The fact that the number of cycles is strictly increasing as a function of n is obvious: consider a chordless cycle in the $Ant(n)$, cut one edge of the Hamiltonian and insert a chain of 3 new vertices (and their edges) at this place to get an $Ant(n+1)$ graph. That construction obviously preserves the cycles, because none of the additional edges and vertices violate the rules. Moreover, these new vertices increase the number of ways to rotate these cycles. (Perhaps the fact that these 3 new vertices could be covered in $2^3 = 8$ ways by the existing cycles is also a strategy to prove the conjectural C-finite linear recurrence in the OEIS.)

Conjecture 1. *Chordless cycles of length 6 don't exist.*

Remark 5. *7 seems to be an upper limit of the lengths of the chordless cycles. A heuristic cause for this is that each of the 7 vertices has 2 of its edges along the cycle, and n-2 of its edges leading to avoided vertices (chords). So the number of avoided vertices grows roughly like $7(n-2)$—certainly an overcount because some of the avoided vertices of distinct vertices are the same—whereas the total number of vertices grows only as $N = 3n - 1$: a starving of accessible vertices for longer cycles.*

References

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