On a Conformal Mapping of Regular Hexagons and the Spiral of its Centers

Wolfdieter Lang

Abstract

A sequence of regular hexagons used in a geometrical proof of the incommensurability of the shorter diagonal and the side of a hexagon is obtained by iteration of a conformal mapping. The centers form a discrete spiral and are interpolated by two continuous spirals, one with discontinuous curvature the other one a logarithmic spiral.

1 Introduction

A geometrical proof by contradiction of the incommensurability of the shorter diagonal and its side can be given by considering an infinite process of ever smaller hexagons. This is explained in Havil’s book [2] on irrationals. It shows the irrationality of $\sqrt{3}$, the length ratio between the shorter diagonal and the side of a regular hexagon. We use this geometrical construction of a sequence of translated, rotated and down-scaled hexagons (always regular ones) $\{H_k\}_{k=0}^\infty$ inscribed in circles $\{C_k\}_{k=0}^\infty$ of radius $\sigma^k r_0$, with $\sigma = -1 + \sqrt{3}$ and centers $\{O_k\}_{k=0}^\infty$. These centers build a discrete spiral. The interpolation of the centers by a continuous curve is immediately given by patching together circular arcs of radius $\sigma^k$ with one of the $H_k$ vertices as centers. The curvature of this spiral is therefore discontinuous. Due to a conformal mapping of the loxodromic type whose iteration produces the sequence of hexagons an interpolating logarithmic spiral ensues with the finite fixed point $S$ as its center. These two spirals are analogous to the ones in a regular pentagon with a sequence of golden triangles (or rectangles) shown, e.g., in the book of Livio [4], as figures 40 and 41 on p. 119. For these triangles the conformal mapping has been given in [3]. The completion of the hexagon sequence and the spirals using negative $k$ values is also considered.

2 Hexagon Descent

For the following geometrical construction see Figure 1 with $k = 0$. One starts with a circle $C_0$ with center $O_0$ and radius $r_0$ (this will be taken in the sequel as length unit. Hence, lengths will always be lengths ratios w.r.t. $r_0$), and inscribes a regular hexagon (the standard construction with a pair of compasses). The vertices of the hexagon (only regular hexagons will be considered) are denoted by $V_k(j)$, for $j = 0, 1, ..., 5$, taken in the positive (anti-clockwise) sense. The choice of $V_0(0)$ defines the non-negative $x_0$ axis as prolongation of $O_0, V_0(0)$. These Cartesian coordinates are named $(x_0, y_0)$ (or in the complex plane $z = x_0 + y_0 i$).

The next (smaller) hexagon $H_1$ is inscribed in a circle $C_1$ with center $O_1$ and radius $r_1 = \sigma := -1 + \sqrt{3}$. This center is obtained by drawing the smaller diagonal in $H_0$, viz, $D_0 = V_0(0), V_0(2)$, which has length $\sqrt{3}$, intersecting it with a circle of radius 1 around $V_0(2)$. Then on the circle $C_1(O_1, r_1)$, with radius $\sigma^k$ the vertices of the subsequent $H_k$ are found. The completion of the hexagon sequence and the spirals using negative $k$ values is also considered.

1 wolfdieter.lang@partner.kit.edu, http://www/kit.edu/~wl/
$r_1 = \overline{O_1V_0(0)} = \sigma = -1 + \sqrt{3}$, the vertex $V_1(3)$ of $H_1$ is the intersection point with the $x_0$ axis, i.e., the prolongation of $O_1V_0(0)$ or $V_0(3)V_0(0)$. From this vertex $V_1(3)$ one finds the vertex $V_1(0)$ as antipode on $C_1$. $V_1(5)$ coincides with $V_0(0)$.

In the second step the new center $O_2$ of $H_2$ is constructed in the same way by drawing the smaller diagonal $D_1 = \overline{V_1(0)V_1(2)}$ ($V_1(2)$ happens to lie on the diagonal $D_0$, and $D_1$ is parallel to the $x_0$ axis). Then the circle around $V_1(2)$ with radius $r_1$ intersects $D_1$ at $O_2$. The vertex $V_2(3)$ on $C_2(O_2, r_2)$, with $r_2 = \overline{O_2V_1(0)} = \sigma r_1 = \sigma^2$, is the point of intersection of $C_2$ with the $x_1$ axis (prolongation of $O_1$, $V_1(0)$). The antipode of $V_2(3)$ on $C_2$ is $V_2(0)$, etc.

This construction implies the following data (besides some obvious ones for a hexagon).

**Lemma 1**

1) $|V_0(2), V_0(0)| = \sqrt{3}$, $|O_1, V_0(0)| = \sigma := -1 + \sqrt{3}$. $|V_1(3), O_0| = \frac{\sigma^2}{2} = 2 - \sqrt{3}$.

2) The two circles $C_0$ and $C_1$ intersect at $(1, 0)$ and $S = (0, 1)$.

**Proof:** (In Cartesian coordinates $(x_0, y_0)$)

1) $V_0(2) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, hence $\angle(V_0(2), V_0(1), O_0) = \frac{\pi}{6}$. Therefore, $O_1 = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, and $\angle(V_0(0), O_0, O_1) = \frac{\pi}{4}$. $\angle(V_0(2), V_1(3), O_1) = \frac{\pi}{6}$. From $\triangle(V_1(3), O_1, V_0(0))$ one has $|V_1(3), V_0(0)| = 2 \cdot \left(\frac{\sigma \sqrt{3}}{2}\right)$. On the other hand, the $y_0$ component of $V_1(0)$ is $\sin\left(\frac{\pi}{6}\right)2\sigma = \sigma$, hence $|V_0(0)| = V_1(5)$, and $V_1(1), V_0(0)$ is parallel to the $y_0$ axis. Therefore $V_1(0)$, $V_1(2)$ is parallel to the $x_0$ axis, and $V_1(2)$ with $y_0$ component $\sigma$ lies on the diagonal $D_0$. $|V_1(3), O_0| = \sigma \frac{\sqrt{3}}{2} - \frac{\sigma}{2} = \frac{\sigma^2}{2} = 2 - \sqrt{3}$.

2) With $C_0 : x_0^2 + y_0^2 = 1$ and $C_1 : \left(x_0 - \frac{\sigma}{2}\right)^2 + \left(y_0 - \frac{\sigma}{2}\right)^2 = \sigma^2$ one finds the intersections $(1, 0)$ and $S = (0, 1)$.

Thus the new hexagon $H_1$ is obtained from the old one, $H_0$, by a translation with $\vec{v}_0 := \overrightarrow{O_0O_1}$ (a column vector), followed by a rotation about the axis perpendicular to the plane (the $z$ axis) through $O_1$ by the angle $\angle(V_1(0), V_1(2), V_1(5)) = \frac{\pi}{6}$ and scaling down by a factor $\sigma$. This process is iterated to find $H_{k+1}$ from $H_k$, for $k = 0, 1, ...$ (see Figure 1).

Next, the vectors $\vec{v}_k = \overrightarrow{O_{k-1}O_k}$ are given in polar coordinates.

**Lemma 2: Vectors** $\vec{v}_k, k = 1, 2, ...$

$$\vec{v}_k \doteq v_k \left(\frac{\cos \alpha_k}{\sin \alpha_k}\right), \quad \text{with} \quad v_k = \sigma^k \frac{\sqrt{3}}{2}, \quad \text{and} \quad \alpha_k = (2k + 1) \frac{\pi}{12}, \quad \text{for} \quad k \in \mathbb{N}, \quad (1)$$

$v_k = (a_k + b_k \sqrt{3}) \frac{\sqrt{2}}{2}$, where $a_k = (-1)^k A026150(k)$, and $b_k = (-1)^{k+1} A002605(k)$.

For the first $a_k$ and $b_k$ entries see Table 6, column $r_k$. For the components of the first twelve vectors $\vec{v}_k$ see Table 1.

**Proof:**

i) The polar angle $\alpha$ is obtained recursively from $\alpha_k = \alpha_{k-1} + \frac{\pi}{4}$, for $k = 2, 3, ..., \text{with input} \alpha_1 = \frac{\pi}{4}$ which follows from the rotation by an angle of $\frac{\pi}{4}$ to obtain $H_k$ from $H_{k-1}$.

ii) The length $v_k$ is obtained recursively from $v_k = v_{k-1} \sigma$ for $k = 2, 3, ... \text{with input} v_1 = \sigma \sqrt{2}$. One may take formally $v_0 = \frac{6}{\sqrt{2}}$ and then $v_k = \sigma^k v_0$ for $k = (0), 1, 2, ...$. For $\{a_k\}_{k=0}^\infty$ and $\{b_k\}_{k=0}^\infty$ one obtains the mixed recurrence $a_k = -a_k_{k-1} + 3b_k_{k-1}$ and $b_k = a_k_{k-1} - b_k_{k-1}$, for $k = 0, 1, ...$, and inputs $a_0 = 1$ and $b_0 = 0$. This decouples, inserting $b_k + b_{k-1} = a_{k-1}$ into $a_k + a_{k-1}$, to the three term recurrences $b_k = 2(-b_{k-1} + b_{k-2})$ with inputs $b_0 = 0$ and $b_1 = 1$, and $a_k = 2(-a_{k-1} + a_{k-2})$ with inputs $b_0 = 0$ and $b_1 = 1$. The vertices of the new hexagon $H_k$ are obtained from the old ones by translation and rotation through $\sigma^k \frac{\sqrt{3}}{2}$, $\sigma^k \frac{\sqrt{2}}{2}$, respectively.
\(a_0 = 1\) and \(a_1 = -1\). The Binet formulae are, with \(\tau := \frac{2}{\sigma} = 1 + \sqrt{3} =: -\sigma\), \(a_k = \frac{1}{2} \left( \sigma^k + (-\tau)^k \right)\) and \(b_k = \frac{1}{2 \sqrt{3}} \left( \sigma^k - (-\tau)^k \right)\). The o.g.f.s (ordinary generating functions) are \(G_a(x) = \frac{1}{1 + x} \) and \(G_b(x) = \frac{x}{1 + 2x - 2x^2}\). This explains the given result involving \(A026150\) and \(A002605\).

In Cartesian coordinates one can write the recurrence as

\[
\vec{v}_k = \sigma \mathbf{R} \vec{v}_{k-1}, \quad k = 2, 3, \ldots \quad \text{with} \quad \vec{v}_1 = \frac{\sigma}{2} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \quad \text{and} \quad \mathbf{R} = \frac{1}{2} \left( \begin{array}{cc} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{array} \right).
\]

\(\mathbf{R}\) is the rotation matrix for angle \(\frac{\pi}{6}\). This leads to

\[
\vec{v}_{k+1} = (\sigma \mathbf{R})^k \vec{v}_1, \quad \text{for} \quad k = (0), 1, 2, \ldots
\]

The powers of \(\sigma\) have been given above as \(\sigma^k = a_k + b_k \sqrt{3}\).

The powers of \(R\) are found as an application of the Cayley – Hamilton theorem, e.g., \([8],[7]):

\[
\mathbf{R}^k = S_{k-1}(\sqrt{3}) \mathbf{R} - S_{k-2}(\sqrt{3}) \mathbf{1}_2, \quad \text{for} \quad k = 1, 2, \ldots
\]

Where \(S_n(x)\) is the Chebyshev polynomial with coefficients given in \(A049310\) with \(S_{-1}(x) = 0\) and \(S_{-2}(x) = -1\). Here \(S_2(\sqrt{3}) = A057079(l)\) and \(S_{2l+1}(\sqrt{3}) = A019892(l) \sqrt{3}\), for \(k = 0, 1, \ldots\) \(A057079\) and \(A019892\) are period length 6 sequences, repeat(1, 2, 1, -1, -2, -1) and repeat(1, 1, 0, -1, -1, 0), respectively. \(I.e.,\) \(S_n(\sqrt{3}) = s_n + t_n \sqrt{3}\), with \(\{s_n\}_{n=0}^\infty = \text{repeat}(1, 0, 2, 0, 1, 0, -1, 0, -2, 0, -1, 0, 0, -1, 0, -1, 0, 0)\).

**Corollary 1:** \(\vec{v}_k\) Periodicity modulo 12 up to scaling

\[
\vec{v}_{k+12l} = \sigma^{12l} \vec{v}_k, \quad \text{for} \quad k \in \mathbb{N}, \ l \in \mathbb{N}_0.
\]

This follows from the periodicity of the angle \(\alpha_k\) in eq. (1).

The calculation of the \(\vec{v}_{2l}\) and \(\vec{v}_{2l+1}\) components w.r.t. the \((x_0,y_0)\) coordinate system leads to

**Proposition 1:** Components of \(\vec{v}_k\), \(k = 1, 2, \ldots\)

\[
\vec{v}_{2l} = \frac{1}{4} \left( \begin{array}{c} ve_{1l}(l) + we_{1l}(l) \sqrt{3} \\ ve_{2l}(l) + we_{2l}(l) \sqrt{3} \end{array} \right), \quad l \geq 1,
\]

\[
\vec{v}_{2l+1} = \frac{1}{4} \left( \begin{array}{c} vo_{1l}(l) + wo_{1l}(l) \sqrt{3} \\ vo_{2l}(l) + wo_{2l}(l) \sqrt{3} \end{array} \right), \quad l \geq 0,
\]

with

\[
ve_{1l}(l) = -a_{2l} A(l-1) + 3 b_{2l}(A(l-1) - 2 B(l-2)),
\]

\[
we_{1l}(l) = +a_{2l} A(l-1) - 2 B(l-2) - b_{2l} A(l-1),
\]

\[
ve_{2l}(l) = +a_{2l} A(l-1) + 3 b_{2l}(A(l-1) - 2 B(l-2)),
\]

\[
we_{2l}(l) = +a_{2l} A(l-1) - 2 B(l-2) + b_{2l} A(l-1),
\]

and

\[
vo_{1l}(l) = a_{2l+1} (3 B(l-1) - 2 A(l-1)) - 3 b_{2l+1} B(l-1),
\]

\[
wo_{1l}(l) = -a_{2l+1} B(l-1) + b_{2l+1} (3 B(l-1) - 2 A(l-1)),
\]

\[
vo_{2l}(l) = +a_{2l+1} (3 B(l-1) - 2 A(l-1)) + 3 b_{2l+1} B(l-1),
\]

\[
wo_{2l}(l) = +a_{2l+1} B(l-1) + b_{2l+1} (3 B(l-1) - 2 A(l-1)),
\]

where \(A(l) = S_{2l}(\sqrt{3})\), \(B(l) = S_{2(l-1)}(\sqrt{3})/\sqrt{3}\), and \(a_k\) and \(b_k\) are given in Lemma 2.

See Table 1 for the coordinates of \(\vec{v}_k\) for \(k = 1, 2, \ldots, 12\).
The vertices \( V \) in the complex plane. It is a periodicity modulo 12 up to a scaling and a translation.

**Proposition 2:** Vertices of hexagons

The relation between \( \vec{O} \), \( \vec{O}_0 \), \( \vec{O}_k \)

**Corollary 2:** Components of \( y \) lie on the \( 1 \) axis. This will be proved in the next section in \( \text{Proposition 4} \).

**Lemma 3:** Triangles \( H \)

For the hexagon \( H \), the endpoint of the vector \( \vec{O}_k := \overrightarrow{O_0, O_k} \), is obtained from (undefined sums are set to 0)

\[
\vec{O}_k = \sum_{j=1}^{k} \vec{v}_j, \quad k = 1, 2, \ldots \quad \text{and} \quad \vec{O}_0 = \vec{0},
\]

\[
\vec{O}_k = \left( 1_2 + \sum_{j=1}^{k-1} (\sigma \mathbf{R})^j \right) \vec{v}_1.
\]

In the coordinate system \((x_0, y_0)\) the components of center \( O_k \) follow from \textit{Proposition 1}.

**Corollary 2:** Components of \( O_k, k = 1, 2, \ldots \)

\[
(\vec{O}_k)_{x_0} = \frac{1}{4} \left( \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} (ve1(j) + ve1(j) \sqrt{3}) + \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (vo1(j) + wo1(j) \sqrt{3}) \right),
\]

\[
(\vec{O}_k)_{y_0} = \frac{1}{4} \left( \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} (ve2(j) + ve2(j) \sqrt{3}) + \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (vo2(j) + wo2(j) \sqrt{3}) \right).
\]

See Table 1 for the components of \( O_k \) for \( k = 1, 2, \ldots, 12 \). It seems that the centers \( O_{l1} \), for \( l = 0, 1, \ldots \) lie on the \( y_0 \) axis. This will be proved in the next section in \textit{Proposition 4}.

The relation between \( \vec{O}_{k+12l} \) and \( Q_k \) will also be considered in the next section in \textit{Proposition 6}, part 7), in the complex plane. It is a periodicity modulo 12 up to a scaling and a translation.

The vertices \( V_k(j) \), for \( j = 0, 1, \ldots, 5 \), of the hexagon \( H \) follow from \( \vec{V}_k(j) := \overrightarrow{O_0, V_k(j)} \).

**Proposition 2:** Vertices of hexagons \( H \)

\[
\vec{V}_k(j) = \vec{O}_k + \sigma^k \mathbf{R}^{k+2j} \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad \text{for} \quad k = 0, 1, \ldots, \quad \text{and} \quad j = 0, 1, \ldots, 5.
\]

**Proof:**

For the hexagon \( H \) the vector \( \overrightarrow{O_k, V_k(0)} \) is obtained from the unit vector in \( x_0 \) direction of the original coordinate system \((x_0, y_0)\) for the first hexagon \( H_0 \) by \( k \)-fold rotation with \( \mathbf{R} = \mathbf{R} R \left( \frac{\pi}{6} \right) \) and down-scaling by \( \sigma \) as

\[
\overrightarrow{O_k, V_k(0)} = (\sigma \mathbf{R})^k \left( \begin{array}{c} 1 \\ 0 \end{array} \right).
\]

Then the vectors for the other vertices are obtained by repeated rotation of \( 60^\circ \), i.e., by application of \( \mathbf{R}^2 \) leading to the assertion.

For the \((x_0, y_0)\) components of \( \vec{V}_k(0) \), for \( 0, 1, \ldots, 12 \), see Table 2, and for the other vertices, for \( j = 1, 2, \ldots, 5 \), see Tables 3, 4 and 5.

**Lemma 3:** Triangles \( T_k \)

The triangle \( T_k = \triangle(O_k, V_k(2), O_{k+1}) \), for \( k = 0, 1, \ldots \), is isosceles with basis \( v_{k+1} = \frac{1}{\sqrt{2}} \sigma^{k+1} \) and two sides of length \( r_k = \sigma^k \). The angles are \( \angle(O_{k+1}, V_k(2), O_k) = \frac{\pi}{6} = 30^\circ \) and twice \( \frac{5\pi}{12} = 75^\circ \).

**Proof:** This is clear from the construction and the values for \( v_k \) given above in Lemma 2 and \( r_k \). See Figure 1.
The polar coordinates of $O_k$, the center of hexagon $H_k$ are given as follows. Note that $\varphi \in [0, 2\pi)$. The number of revolutions, using also $\varphi \geq 2\pi$ (sheets in the complex plane), will be considered in the next section.

**Corollary 3: Polar coordinates of $O_k$**

In the complex plane $O_k \cong z_k = \rho_k \exp(i \varphi_k)$ with $\rho_k = |\overrightarrow{O_0, O_k}|$, one has

$$\rho_k = \sqrt{((O_k)_{x0})^2 + ((O_k)_{y0})^2}, \quad \text{with eq.}(17)$$

$$\varphi_k = \hat{\varphi}_k \text{ in quadrant I, } = \hat{\varphi}_k + \pi \text{ in quadrants II and III, } = \hat{\varphi}_k + 2\pi \text{ in quadrant IV, with}$$

$$\hat{\varphi}_k = \arctan \left( \frac{(O_k)_{y0}}{(O_k)_{x0}} \right). \quad (21)$$

$\rho_k^2$ is integer in the real quadratic number field $\mathbb{Q}(\sqrt{3})$. For the values for $k = 0, 1, ..., 12$, see Table 2. The corresponding angles are $(\varphi_k 180/\pi)$. The values for $\tan \hat{\varphi}_k$ are elements of $\mathbb{Q}(\sqrt{3})$. For their components see also Table 2, for $k = 1, 2, ..., 12$ (for $k = 0$, with $z_0 = 0$, the value of $\varphi_0$ is arbitrary; in Table 2 we have set it to 0).

### 3 Conformal mapping and the Hexagon Spiral

The discrete spiral formed by the hexagon centers $O_0$ and $O_k$ given in eq. (17) for $k = 0, 1, ..., n$, are shown as dots in Figure 2 for $k = 0, 1, ..., 11$. In the complex plane $\mathbb{C} = \mathbb{C} \cup \infty$ these centers will be called $z_k = (O_k)_{x0} + (O_k)_{y0} i$. The construction of these hexagon described in sect. 1 is obtained by repeated application of a conformal M"obius transformation. It is determined by mapping the triangle $T_0$ of $H_0$ with vertices $z(1) = V_0(2) = -\frac{1}{2}(-1 + \sqrt{3})$, $z(2) = z_0 = 0 + 0 i$ and $z(3) = z_1 = \frac{1}{2}(1 + 1 i)$ to the translated, rotated and scaled triangle $T_1$ of $H_1$ with vertices $w(1) = V_1(2) = (-2 + \sqrt{3}) + (-1 + \sqrt{3}) i$, $w(2) = z_1 = \frac{1}{2}\left(-1 + \sqrt{3} + (-1 + \sqrt{3}) i\right)$ and $w(3) = z_2 = (-3 + 2 \sqrt{3}) + (-1 + \sqrt{3}) i$. See Figure 1 for these two triangles, setting $k = 0$. In general triangle $T_k$ is mapped to $T_{k+1}$ by this conformal transformation, especially $w(z_k) = z_{k+1}$, for $k = 0, 1, ...$. The unique M"obius transformation which maps the vertices of $T_0$ to those of $T_1$ is given by solving the double quotient equation for $w = w(z)$ (see. e.g., [6], [9])

$$DQ(w(1), w(2), w(3), w) = DQ(z(1), z(2), z(3), z), \quad \text{with } DQ(z1, z2, z3, z4) := \frac{z4 - z3}{z4 - z1} / \frac{z2 - z3}{z2 - z1}. \quad (22)$$

The solution is a M"obius transformation of the loxodromic type, having besides one fixed point at $\infty$ another finite one $S$ with $(w - S) = a(z - S)$, where $a$ is not real non-negative, and $|a| \neq 1$.

$$w(z) = \frac{A}{D} z + \frac{B}{D}, \quad \text{with}$$

$$A = 2 \left((-2 + \sqrt{3}) + (-7 + 4 \sqrt{3}) i\right),$$

$$B = (-9 + 5 \sqrt{3}) + (5 - 3 \sqrt{3}) i,$$

$$D = (1 - \sqrt{3}) + (-5 + 3 \sqrt{3}) i. \quad (23)$$

The determinant of this transformation is $AD = 8(-19 + 11 \sqrt{3})$. $A$, $B$ and $D$ are integers in $\mathbb{Q}(\sqrt{3})$. This is rewritten in the following Proposition.
Proposition 3: Loxodromic map \( w \)

1) The unique conformal \( \text{M"obius} \) transformation \( w \) which maps the corners of triangle \( T_0 \) to those of \( T_1 \) (keeping the orientation), and hence \( T_k = \Delta(V_k(2), O_k, O_{k+1}) \) to \( T_{k+1} \), is given by the loxodromic map

\[
  w(z) = az + b, \quad \text{with}
  \begin{align*}
    a &= \frac{1}{2} \left( (3 - \sqrt{3}) + (-1 + \sqrt{3})i \right), \\
    b &= \frac{1}{2} (-1 + \sqrt{3}) (1 + i) = (1 - a)i.
  \end{align*}
\]

(24)

2) \( a = \sigma e^{i \frac{\pi}{6}} \), and \( |a| = \sigma = -1 + \sqrt{3} \neq 1 \). The finite fixed point of this map is \( S = i \). \( S \) is the common intersection point of all circles \( C_k \).

Proof:

1) This is clear from the construction and the previous form of \( w \) from eq. (23), and the computation has been checked with the help of Maple [5].

2) The values of \( a \) and \( |a| \) show that this \( \text{M"obius} \) transformation is loxodromic with finite fixed point \( S = i \). \( S \) has to lie on each circle \( C_k \), for \( k = 0, 1, \ldots \), because \( w \) maps \( C_k \) to \( C_{k+1} \).

Corollary 4: Inverse map \( w^{-1} \)

The inverse of map \( w^{-1} \) of \( w \) is given by

\[
  w^{-1}(z) = a^{-1} z + (1 - a^{-1})i
  = \frac{1}{4} \left[ \left( (3 + \sqrt{3}) - (1 + \sqrt{3})i \right) z + \left( - (1 + \sqrt{3}) + (1 - \sqrt{3})i \right) \right], \quad \text{for} \ z \in \mathbb{C}. \quad (25)
\]

Check: \( w^{-1}(w(z)) \equiv z \).

With the help of the conformal map \( w \) it is now easy to prove that points \( z_{6,j} \) (corresponding to the centers \( O_{6,j} \)) lie on the imaginary axis (the \( y_0 \)-axis).

Proposition 4: Centers \( z_{6,j} \) lie on the imaginary axis

\[ \Re(z_{6,j}) = 0, \quad \text{for} \ j \in \mathbb{N}_0. \]

Proof:

Compute \( w^6(z) \) for \( z \) on the imaginary axis, \( z = y i \), with real \( y \): \( w^6(y i) = (y + (209 - 120 \sqrt{3}) (1 - y)) i = (y + (O_{6,y_0}(1 - y)) i \). See the last column of Table 1 for \( (O_{6,y_0}) \). Therefore, points on the non-negative imaginary axis are mapped by \( w^6 \) again on this axis. Because \( z_0 = 0 \) lies on the imaginary axis also \( z_{6,j} \), for \( j = 1, 2, \ldots \), have to lie on the imaginary axis.

Corollary 5: Number of centers for each revolution of the spiral

The number of centers \( 0_k \) for each revolution is 12.

See Figure 4 for the first revolution, except for \( 0_{12} \) on the imaginary axis where the second revolution starts.

The discrete hexagon spiral can be interpolated between \( O_k \) and \( O_{k+1} \) by circular arcs \( A_k \) of the circles \( \hat{C}_k(V_k(2), r_k) \). See Figure 4. These arcs \( A_k \) belong to a sector of \( \hat{C}_k \) of angle \( \frac{5\pi}{12} \) (see Lemma 3). The precise form is given by
Proposition 5: Interpolating circular arcs $A_k$

The circular arc with center $V_k(2)$ and radius $r_k = \sigma^k$ which interpolates between the centers $O_k$ and $O_{k+1}$ of the hexagon $H_k$ is given by

$$A_k = \text{arc} \left( V_k(2), r_k, \frac{(k-2)\pi}{6}, \frac{(k-1)\pi}{6} \right).$$ \hspace{1cm} (26)

Proof:

From Lemma 3 the range of the angle $\varphi$ is $\frac{\pi}{6}$. The angles are counted in the positive sense with respect to the horizontal line, defined by the $x_0$-axis. It is therefore sufficient to know the angle for one of the lines $V_k(2), O_{k+1}$ which corresponds to the larger of the angles for arc $A_k$. For $k = 1$ this angle vanishes because the $y_0$ components of $V_1(2)$ and $O_2$ coincide, they are $\sigma r_0$. Hence the angle for arc $A_2$ starts with $0$ ($V_2(2)$ is on the line segment $V_1(2), O_2$) and ends with $\frac{\pi}{6}$. This proves the given range for each $A_k$.

This interpolation by circular arcs is continuous but has discontinuous curvature with increases at each center $O_k$ by a factor of $1/\sigma = \frac{\tau}{2} = \frac{1}{2}(1 + \sqrt{3}) \approx 1.366025403$.

An interpolation with continuous curvature is given by the equal angle spiral (the logarithmic) spiral (Jacob I Bernoulli: spira mirabilis), defined in the complex plane by $LS(\phi) = r(\phi) \exp(i\phi)$, with $r(\phi) = r(0) \exp(-\kappa \phi)$ where the constant $\kappa$ defines the constant angle $\alpha$ between the radial ray and the tangent (taken in the direction of increasing angle $\phi$) at any point of the spiral by $\alpha = \arccot(-k)$. Here the center of the logarithmic spiral is at the finite fixed point $S$ and we choose a coordinate system $(X, Y)$ with the positive $X$ direction along the vertical line (the $y_0$-axis in the negative sense) and the positive $Y$ axis in the horizontal direction to the right, parallel to the positive $x_0$ axis. I.e., $X = -y_0 + 1$ and $Y = x_0$. In this system $0_0 = (1, 0)$ and $r(0) = r_0 = 1$. The angle $\phi_1$ for $0_1 = \left( \frac{2 - \sigma}{2}, \frac{\sigma}{2} \right)$ (in the $(x_0, y_0)$ system) becomes in the $(X, Y)$ system $\frac{\pi}{6}$ because $\tan(\phi_1) = \frac{\sigma}{2 - \sigma} = \frac{\sqrt{3}}{3}$. $r(\frac{\pi}{6}) = r_1 = \sigma$.

Therefore the constant of the logarithmic spiral is $\kappa = -\frac{6}{\pi} \log(\sigma) \approx -0.5956953531$. This corresponds to $\arccot(-\kappa) \approx 1.033548020$, corresponding to about 59.216°. To summarize:

Proposition 6: Logarithmic Spiral for non-negative $k$

1) In the coordinate system $(X, Y)$ of the logarithmic spiral with origin $S$ and $X = -y_0 + 1$, $Y = x_0$ the spokes $S\phi_k = S, O_k$ have lengths $r_k = \sigma^k$. The angles $\phi_k$ are obtained by $\sin(\phi_k) = (O_k)_{x_0} \sigma^{-k}$ where $\sigma^{-k} = \left( \frac{\tau}{2} \right)^k = a_{-k} + b_{-k} \sqrt{3}$, where $\tau = 1 + \sqrt{3} = -\sigma$ and $a_{-k} = A_002531(k)/2^{[\frac{k+1}{2}]}$, $b_{-k} = A_002530(k)/2^{[\frac{k+1}{2}]}$ for $k = 0, 1, \ldots$. I.e., $\{\sin(\phi_k)\}_{k=0}^\infty = \text{repeat} \left( 0, \frac{1}{2}, \frac{1}{2} \sqrt{3}, 1, \frac{1}{2} \sqrt{3}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2} \sqrt{3}, -1, -\frac{1}{2} \sqrt{3}, -\frac{1}{2} \right)$. The first period applies to the first revolution of the spiral (sheet $S_1$ in the complex plane). The corresponding angles are for the $N$-th revolution (sheet $S_N$ in the complex plane) $\phi_k = 2\pi (N - 1) + \frac{\pi}{3} k$ (mod 12), I.e., an addition of $\frac{\pi}{6}$ or $30^\circ$ from spoke $S\phi_k$ to $S\phi_{k+1}$ for each $k = 0, 1, \ldots$. The periodicity modulo 12 is proved in part 6).

2) In the coordinate system $(X, Y)$ with origin $S$ the hexagon centers are $LS(\phi_k) = Z_k = \sigma^k \exp(i \phi_k) = (a_k + b_k \sqrt{3}) \exp(i \frac{\pi}{3} k)$, for $k = 0, 1, \ldots$. This becomes with the help of the $de Moivre$ formula, expressed in terms of Chebyshev's $S$ polynomials evaluated at $\sqrt{3}$:

$$Z_k = \frac{1}{2} \left( (3b_k S_{k-1}(\sqrt{3}) - 2a_k S_{k-2}(\sqrt{3})) + (a_k S_{k-1}(\sqrt{3}) - 2b_k S_{k-2}(\sqrt{3})) \sqrt{3} + (a_k + b_k \sqrt{3}) S_{k-1}(\sqrt{3}) i \right) = (O_k)_X + (O_k)_Y i,$$ \hspace{1cm} (27)
where \(a_k\) and \(b_k\) have been given in Lemma 2, and Chebyshev’s \(S_n(\sqrt{3})\) polynomials entered in connection with eq. (4). See Table 6 for the Cartesian coordinates \(((O_k)\chi, (O_k)\gamma)\) for \(k = 0, 1, ..., 12\).

3) The curvature \(K(\phi)\) of the logarithmic spiral \(r(\phi) = \exp(-\kappa \phi)\) is itself a logarithmic spiral

\[
K(\phi) = \frac{1}{\sqrt{1 + \kappa^2}} \exp(+\kappa \phi) \quad \text{with} \quad \kappa = -\frac{6}{\pi} \log(\sigma).
\]

\(\kappa \approx -0.5956953531\) and \(K(0) = \frac{1}{\sqrt{1+\kappa^2}} \approx 0.8591201770\).

4) The conformal map \(W(Z)\) and its inverse \(W^{-1}\) in the \(S\)-system are for \(Z \in \mathbb{C}\) given by

\[
W(Z) = \frac{1}{2} \left( (3 - \sqrt{3}) + (-1 + \sqrt{3})i \right) Z = aZ,
\]

\[
W^{-1}(Z) = \frac{1}{4} \left( (3 + \sqrt{3}) - (1 + \sqrt{3})i \right) Z = a^{-1}Z.
\]

5) The relation between the conformal maps \(w\) and \(W\) is

\[
W(Z) = iw(z(Z)) + 1, \quad \text{or} \quad w(z) = i(1 - W(Z(z))),
\]

with \(z(Z, \bar{Z}) = z(Z) = i(1 - Z), \quad \text{or} \quad Z(z) = 1 + iz\).

6) Periodicity modulo 12 up to scaling for \(Z_k\):

\[
Z_{k+12l} = \sigma^{12l} Z_k, \quad \text{for} \quad k \in \mathbb{N}_0, \quad l \in \mathbb{N}_0.
\]

7) Periodicity modulo 12 up to scaling and translation for \(z_k\):

\[
z_{k+12l} = \sigma^{12l} z_k + i(1 - \sigma^{12l}), \quad \text{for} \quad k \in \mathbb{N}_0, \quad l \in \mathbb{N}_0.
\]

Proof:

1) The length ratio of the spokes is clear: \(S\) is the intersection of all circles \(C_k\), for \(k = 0, 1, ..., \), and \(O_k\) is the center of \(C_k\). The periodicity modulo 12 of the angles \(\phi_k\) follows conjecturally from the \(\sin(\phi_k)\) formula if the \(x_0\) component of \(O_k\) from eq. (17) is inserted. Later, under part 6), this is proved. The values for the first revolution then show that in general \(\phi_{k+1} = \phi_k + \frac{\pi}{6}\). One has to take into account the quadrants when interpreting the angles from the \(\sin(\phi_k)\) result.

2) This uses a standard reformulation of the trigonometric quantities obtained from the de Moivre formula in terms of Chebyshev’s polynomials (they are the circular harmonics). The powers of \(\sigma\) have already been treated in Lemma 2.

3) The formula for the curvature \(K\) of a curve in two-dimensional polar coordinates \(r = r(\phi)\) is \(K(\phi) = \frac{r^2 + 2r' r'' - r'''}{r^2 + r'^2} \), e.g., [1]. As explained in the preamble to this Proposition the logarithmic spiral is \(r(\phi) = \exp(-\kappa \phi)\), and with \(r_1 = r\left(\frac{\pi}{6}\right) = \sigma\) one determines the constant \(-\kappa\). The curvature \(K\) becomes itself a logarithmic spiral with \(K(0) = \frac{1}{\sqrt{1 + \kappa^2}}\) and the constant \(+\kappa\).

4) Like for the conformal map \(w\), the unique Möbius transformation \(W\) which maps the points \((S = 0, Z_0, Z_1)\) to \((S, Z_1, Z_2)\) is obtained by solving the double quotient equation \(DQ(0, Z_0, Z_1, Z) = DQ(0, Z_1, Z_2, W)\) for \(W = W(Z)\). The real and imaginary parts of \(Z_k\), for \(k = 0, 1, ..., 12\) are shown in Table 6 as \((O_k)\chi\) and \((O_k)\gamma\). In general \(W(Z_k) = Z_{k+1}\), for \(k = 0, 1, ...\). The same \(a\) as in eq. (24) appears. The inverse map \(W^{-1}\) satisfies \(W^{-1}(W(Z)) = Z\), identically. Note that, in contrast to \(w\), the map \(W\), hence \(W^{-1}\), is linear.
5) The coordinate transformation \( X = 1 - y_0 \) and \( Y = x_0 \) leads for \( z = x_0 + y_0 \) and \( Z = X + Y \) to \( z(Z, \overline{Z}) = \frac{Z - \overline{Z}}{2i} + \left(1 - \frac{Z + \overline{Z}}{2}\right) i = i(1 - Z) + 0\overline{Z} = i(1 - Z) = z(Z) \). With \( w(z) = az + (1 - a)i \) from eq. (24), one obtains \( w(z(Z)) = a(1 - Z)i + (1 - a)i = i(1 - aZ) = i(1 - W(Z)) \). I.e., \( W(Z) = i w(z(Z)) + 1 \). Or, with \( Z(z) = 1 + z_i \), \( W(z(Z)) = i(az + b) + (-ib + a) = iw(z) + 1 \), because \( a - ib = 1 \). Therefore, \( w(z) = i(1 - W(Z(z))) \).

6) The linearity of \( W \) means that \( W^{[p]}(Z) = a^p Z \) for the \( p \)-fold iterated map \( W \) for \( Z \in \mathbb{C} \). Now, with \( Z_0 = 1 \), one has \( Z_{k+12l} = W^{[k+12l]}(1) = W^{[k+12l]}(W^{[k]}(1)) = W^{[k+12l]}(Z_k) \). By linearity this is \( a^{12l} Z_k = (\sigma^{12})^l Z_k \). Here \( a^{12} = \sigma^{12} \) even though \( a \neq \sigma \). This follows from \( Z_{12} = W^{[12]}(1) = a^{12}1 = a^{12} \), and by computation (see the last two columns of Table 6) \( Z_{12} = 86464 - 49920\sqrt{3} + 0i = \sigma^{12} \) by the first column of this Table.

7) This periodicity modulo 12 up to scaling translates into a periodicity modulo 12 up to translation and scaling for the centers \( z_k \) of the circles \( C_k \) in the coordinate system \((x_0, y_0)\) due to the transformation given in part 5) applied to these centers, viz, \( z_k(Z_k) = i(1 - Z_k) \) for \( k \in \mathbb{N}_0 \). Therefore, \( z_{k+12l} = i(1 - \sigma^{12l}Z_k) \) from part 4). With \( Z = Z_k(z_k) = 1 + z_k \) this becomes \( z_{k+12l} = \sigma^{12l}z_k + i(1 - \sigma^{12l}) \).

4 Hexagon Ascent

It is straightforward to continue the discrete spiral and its interpolations to \( n \) \( k \) values. In the coordinate system \((x_0, y_0)\) with origin \( O_0 = 0 \) the vectors \( \vec{v}_{-k} = O_{-(k+1)}, O_{-k} \) have polar coordinates following from extending eq. (1).

\[
\vec{v}_{-k} = \begin{pmatrix} \cos \alpha_k \\ \sin \alpha_k \end{pmatrix}, \quad \text{with} \quad v_{-k} = \sigma^{-k} \frac{\sqrt{2}}{2}, \quad \text{with} \quad \alpha_k = (1 - 2k) \frac{\pi}{12} \quad \text{for} \quad k \in \mathbb{N}_0,
\]

\[
v_{-k} = (a_{-k} + b_{-k} \sqrt{3}) \frac{\sqrt{2}}{2}, \quad \text{where} \quad a_{-k} = \frac{A002531(k)}{2^{\left\lfloor \frac{k+1}{3} \right\rfloor}}, \quad \text{and} \quad b_{-k} = \frac{A002530(k)}{2^{\left\lfloor \frac{k+1}{3} \right\rfloor}}.
\]

\( \sigma^{-k} \) appeared already in Proposition 5, part 1). See also the second column of Table 3 for \( \{a_{-k}, b_{-k}\} \) for \( k = 0, 1, ..., 12 \).

This can be written as

\[
\vec{v}_{-k} = (\sigma R^{-1})^k \vec{v}_1, \quad \text{with} \quad R^{-1} = \frac{1}{2} \begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix}, \quad \text{for} \quad k \in \mathbb{N}_0.
\]

For \( \vec{v}_1 \) and \( R \) see eq. (2). E.g., \( \vec{v}_0 = \frac{1}{2} \begin{pmatrix} \tau \\ \sigma \end{pmatrix} \).

The formula eq. (4) can be used to obtain \( R^{-k} \) with the Chebyshev polynomials \( S_{-n}(x) = -S_{n-2}(x) \), for \( n \in \mathbb{N}_0 \), with \( S_{-1}(x) = 0 \).

\[
R^{-k} = -S_{k-1}(\sqrt{3}) R + S_k(\sqrt{3}) 1_2, \quad \text{for} \quad k = 0, 1, 2, ....
\]

The components of \( \vec{v}_{-k} \) can be computed from this. Similarly to Corollary 1 these vectors are periodic modulo 12 up to scaling:

Corollary 6 = 1': \( \vec{v}_{-k} \) periodicity up to scaling

\[
\vec{v}_{-(k+12l)} = (\sigma)^{-12l} \vec{v}_{-k} = \left(\frac{\tau}{2}\right)^{12l} \vec{v}_{-k}, \quad \text{for} \quad k \in \mathbb{N}_0, \ l \in \mathbb{N}_0.
\]

In order to obtain components of \( \vec{v}_{-k} \) which are integers in the real quadratic number field \( \mathbb{Q}(\sqrt{3}) \) the largest denominator \( q^{(k)} \) with \( s(k) = A300068(k) \) has been multiplied. This sequence \( \{s(k)\}_{k \geq 6} \) is obtained from the periodic sequence \( A300067 \), repeat(0, 0, 0, 1, 2, 2, ...).
Lemma 4: Sequence $s$

The formula for the members of sequence $s$ and its o.g.f. is

$$s(k) = 2 + \left\lfloor \frac{k \pmod{6}}{3} \right\rfloor + \left\lfloor \frac{k \pmod{4}}{4} \right\rfloor + 3 \left\lfloor \frac{k}{6} \right\rfloor,$$

for $k \in \mathbb{N}_0$.

O.g.f.: $G(x) = \frac{2 + x^3 + x^4 - x^6}{(1 - x^3)(1 - x)}$.

(39)

Proof:

Due to the periodicity up to scaling (Corollary 6) it is sufficient to consider $s(k)$, for $k = 0, 1, ..., 11$. These values are given from the first twelve vectors $\vec{v}_{-k}$ by the second column of Table 7 with the first six members 2, 2, 2, 3, 4, 4, and the other six ones are obtained by adding 3 to each member. The scaling factor $\sigma^{-12l}$ (see Proposition 6, part 1) and $r_{-k}$ in Table 6) has the denominator $2\left\lfloor \frac{6}{12} \right\rfloor = 2^6$ because $\gcd(A002531(k), A002530(k)) = 1$ due to the fact that they are denominators and numerators in lowest terms of fractions (they give the continued fraction convergents of $\sqrt{3}$). Therefore, for each period of length 12 a new factor $2^6$ has to be multiplied, which means for the exponents that $s(k + 12l) = 6l s(k)$. Because in the first period 3 is added to the first six entries of $s$ this results in a period of length 6 and the periodicity up to scaling formula for $s$ becomes $s(k + 6l) = 3l s(k)$. This explains the last term in the explicit formula for $s$. The second and third terms result from $A300067$, repeat(0, 0, 0, 1, 2, 2, ), and the 2 has then to be added to produce the first six entries of the sequence $s$. The o.g.f. of $\{s(k) - 2\}_{k \geq 0}$ is found from the obvious ones of $A300067$ and $3 \left\lfloor \frac{k}{6} \right\rfloor$.

For the scaled vectors components $\vec{v}_{-k}$, for $k = 0, 1, ..., 12$, see Table 7. The centers $O_{-k}$ are then given by

$$\vec{O}_{-k} = \vec{O}_0, O_{-k} = -\sum_{j=0}^{k-1} \vec{v}_{-j}, \quad \text{for} \quad k \in \mathbb{N}, \quad \text{and} \quad \vec{O}_{-0} = \vec{0}. \quad \tag{40}$$

Again, some scaling $2^l(k)$ is applied to obtain integers in $\mathbb{Q}(\sqrt{3})$ for the components of $\vec{O}_{-k}$. For $k = 0$, the zero-vector $\vec{0}$, no scaling is needed and $t(0) = 0$. The above reasoning for sequence $s$ does not apply immediately because $O_{-k}$, like $O_k$, is not periodic up to scaling, but in the $y_0$ component also a translation appears (for $O_k$, in the complex plane called $z_k$, see the Proposition 6, part 7)). Later, in Proposition 9, part 5), it will be seen that for $Z_{-k}$, in the coordinate system $(X, Y)$ with origin $S$, the same sequence $t$ is used to obtain integers in $\mathbb{Q}(\sqrt{3})$ for the real and imaginary parts of $2^l(k)Z_{-k}$. Then by the coordinate transformation $x_0 = Y = 3(Z)$ and $y_0 = 1 - X = 1 - \Re(Z)$ this will imply integer coordinates in $\mathbb{Q}(\sqrt{3})$ also for $O_{-k}$. It is therefore again sufficient to consider $t(k)$ for the first period $k = 1, 2, ..., 12$. These values are given in the fifth column of Table 7 as 2, 2, 2, 3, 4, 3, and the next six numbers are obtained by adding 3 to these members. This results in the following formula based on the period length 6 sequence $A300069$, repeat(0, 0, 0, 1, 2, 1, ) (but there the offset is 0, not 1).

Lemma 5: Sequence $t$

The formula for the members of sequence $t$ and its o.g.f. is

$$t(0) = 0, \quad \text{and}$$

$$t(k) = 2 + \left\lfloor \frac{k - 1 \pmod{6}}{3} \right\rfloor + \left\lfloor \frac{k \pmod{6}}{5} \right\rfloor + 3 \left\lfloor \frac{k - 1}{6} \right\rfloor = 2 + A174257(k), \quad \text{for} \quad k \in \mathbb{N}.$$

O.g.f.: $G(x) = \frac{x(2 + 2x - x^3)}{(1 + x - x^3 - x^4)(1 - x)}$.

(41)

The proof is analogous to the one of the preceding Lemma 4 but the different offset has to be taken into account.
The vertices of the hexagons $H_{-k}$, for $k \in \mathbb{N}_0$, are given in the obvious extension of Proposition 2 with 
\[\sigma^{-1} = \frac{\tau}{2}\] as follows.

**Proposition 7: Vertices of hexagons** $H_{-k}$, $k \in \mathbb{N}_0$,

\[\vec{V}_{-k}(j) = \vec{O}_{-k} + \left(\frac{\tau}{2}\right)^k R^{-k+2j} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{for } k = 0, 1, \ldots, \text{ and } j = 0, 1, \ldots, 5. \tag{42}\]

In order to obtain integers in $\mathbb{Q}(\sqrt{3})$ after some scaling of the components of $\vec{V}_{-k}(j)$ it turns out that one needs only the three scaling sequences $2^{v0(k)}$, $2^{v1(k)}$, $2^{v2(k)}$ for $\vec{V}_{-k}(0)$, $\vec{V}_{-k}(1)$, $\vec{V}_{-k}(2)$, which also work for $\vec{V}_{-k}(3)$, $\vec{V}_{-k}(4)$, $\vec{V}_{-k}(5)$, respectively. Again it is sufficient for the sequences $v0$, $v1$ and $v2$ to concentrate on the first six entries besides the values for $k = 0$ (the original hexagon $H_0$) which are 0, 1 and 1, respectively (for $\vec{V}_{0}(0)$ see Table 2 for $k = 0$ which does not need a scaling). The other six values are obtained by adding 3, and for each new period of length 12 (starting with $k = 1$) another 3 is added. We skip the proof (see the one for the sequence $t$ which is similar), and give the results for these three sequences.

**Lemma 6: Sequences** $v0$, $v1$, $v2$

$v0(0) = 0$, and

\[v0(k) = 1 + \left[\frac{k \pmod{6}}{2}\right] + 2 \left[\frac{(k-1) \pmod{6}}{5}\right] + 3 \left[\frac{k-1}{6}\right]\hspace{1cm} (k \in \mathbb{N}). \tag{43}\]

$v0(k) = A300068(k+2)$, for $k \in \mathbb{N}_0$.

O.g.f.: $G0(x) = \frac{x(1 + x + x^3)}{(1 - x^6)(1 - x)}$.

$v1(0) = 1$, and

\[v1(k) = 1 + (k-1) \pmod{6} - \left[\frac{(k-1) \pmod{6}}{3}\right] - \left[\frac{(k-1) \pmod{6}}{5}\right] + 3 \left[\frac{k-1}{6}\right]\hspace{1cm} (k \in \mathbb{N}). \tag{45}\]

O.g.f.: $G1(x) = \frac{1 + x^2 + x^3 + x^5 - x^6}{(1 - x^6)(1 - x)}$.

$v2(0) = 1$, and

\[v2(k) = 2 + 2 \left[\frac{(k-1) \pmod{6}}{5}\right] + \left[\frac{k \pmod{6}}{3}\right] + 3 \left[\frac{k-1}{6}\right]\hspace{1cm} (k \in \mathbb{N}). \tag{47}\]

O.g.f.: $G2(x) = \frac{1 + x + x^3}{(1 - x^6)(1 - x)}$.

The o.g.f.s show that $v2(k) = v0(k+1)$, for $k \in \mathbb{N}_0$.

The discrete hexagon spiral with points $O_{-k}$ can again be interpolated by circular arcs $A_{-k}$ between $O_{-k}$ and $O_{-k+1}$. The centers of the circles are $\hat{C}_{-k} = V_{-k}(2)$ and the radius is $r_{-k} = \sigma^{-k} = \left(\frac{\tau}{2}\right)^k$ (see Table 6 for $2^{k+1}r_{-k}$). The precise statement is given in
**Proposition 8: Interpolating circular arcs \(A_{-k}, k \in \mathbb{N}_0\)**

The circular arcs \(A_{-k}\) interpolation between the centers \(O_{-k}\) and \(O_{-k+1}\) of the discrete hexagon spiral are, for \(k \in \mathbb{N}\) given by

\[
A_{-k} = \arccos \left( V_{-k}(2), r_{-k}, \frac{-(k+2)\pi}{6}, \frac{-(k+1)\pi}{6} \right). 
\]  

In *Figure 6* this interpolation by arcs is shown in dashed blue (almost coinciding with the later discussed logarithmic spiral shown there in solid red).

**Proof:**

This is simply the generalization of eq. (26) for negative \(k\). The angle \(-\frac{2\pi}{6}\), the first angle for \(A_0\) becomes the second angle for \(A_{-1}\) and then \(-\frac{\pi}{6}\) has to be added in order to obtain the first angle. This continues for each step \(A_{-k} \to A_{-(k+1)}\).

**Proposition 9: Logarithmic Spiral for non-positive \(k\)**

1) The centers of the circles \(C_{-k}\) are

\[
Z_{-k} = (W^{-1}[k](1)) = (a^{-1})^k, \quad \text{for} \quad k \in \mathbb{N}_0, \quad \text{and} \quad a_{-1} = \frac{\tau}{2} e^{-i k \frac{\pi}{6}} .
\]  

2) The spokes \(S_{pk} = \overline{Z_{-k}}\) have lengths \((\frac{\tau}{2})^k\) and the angles \(\phi_{-k} = -k \frac{\pi}{6}\), for \(k \in \mathbb{N}_0\). For \(\sigma^{-k} = (\frac{\tau}{2})^k\) see *Proposition 6*, part 1.

3) The explicit form, using *de Moivre’s* formula expressed in terms of the *Chebyshev’s* \(S\) polynomials with negative index \(S_{-n}(x) = -S_{n-2}(x)\) is like eq. (27) with \(k \to -k\):

\[
Z_{-k} = \frac{1}{2} \left( (-3 b_{-k} S_{k-1}(\sqrt{3}) + 2 a_{-k} S_{k}(\sqrt{3})) + (-a_{-k} S_{k-1}(\sqrt{3}) + 2 b_{-k} S_{k}(\sqrt{3})) \sqrt{3} - (a_{-k} + b_{-k} \sqrt{3}) S_{k-1}(\sqrt{3}) \right) .
\]  

4) The logarithmic spiral in the complex plane

\[
LS(\phi) = e^{(-\kappa+i)\phi}, \quad \text{with} \quad \kappa = -\frac{\pi}{6} \log(\sigma). 
\]  

interpolates between all points \(Z_k\) for \(k \in \mathbb{Z}\).

5) Periodicity modulo 12 up to scaling for \(Z_{-k}\):

\[
Z_{-(k+12l)} = (\frac{\tau}{2})^{12l} Z_{-k}, \quad \text{for} \quad k \in \mathbb{N}_0, \quad l \in \mathbb{N}_0.
\]

Therefore one has eq. (33) with \(k \in \mathbb{Z}\) and \(l \in \mathbb{Z}\).

**Proof:**

1) With the map \(W^{-1}\) from *Proposition 6*, eq. (30), with \(a^{-1}\) from eq. (25), the hexagon centers \(O_{-k}\), in the complex plane denoted by \(Z_{-k}\), satisfy

\[
Z_{-k} = W^{-1}[k](Z_{k-1}) = W^{-k}(Z_0) = W^{-k}[1](1) = (a^{-1})^k = a^{-k}, \quad \text{for} \quad k \in \mathbb{N}_0.
\]  

2) This is clear from part 1).

3) This is also clear, repeating the steps which led to *Proposition 6*, part 2), and the rewriting of \(S\) polynomials with negative index, as given.

4) The logarithmic spiral, by construction of the maps \(W\) and \(W^{-1}\), interpolates between all hexagon centers \(Z_k\), for \(k \in \mathbb{Z}\).

5) The periodicity up to scaling is obvious from part 1).
References


Keywords: Conformal Mapping, Hexagon, Spiral

Figure 1: Construction $H_k \rightarrow H_{k+1}: C_k(O_k, r_k), V_k(0), (x_k, y_k), D_k = V_k(0), V_k(2), V_k(2), O_{k+1} = r_k, C_{k+1}(O_{k+1}, r_{k+1} = \sigma^k, V_{k+1}(3), V_{k+1}(0), (x_{k+1}, y_{k+1}), ...$

Figure 2: The first four circles and the first eleven centers.

Figure 3: The first three hexagons and the first four circles.

Figure 4: The discreet hexagon spiral of the first 13 centers. The interpolating circular arcs (dashed blue) and the logarithmic spiral (solid red) are almost indistinguishable.)
Figure 5: The circle $C_0(0, 1)$ and the first six circles $C_{-k}(O_{-k}, \sigma^{-k})$, for $k = 1, 2, ..., 6$.

Figure 6: The fixed point $S$, the center $O_0 = 0$, the first 12 centers $O_{-k}$ with $k = 1, 2, ..., 12$. The interpolating circular arcs (dashed blue) and the logarithmic spiral (solid red) are almost indistinguishable.
In the following tables all length have been divided by $r_0$.

**Table 1**

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<th>$k$</th>
<th>$(\vec{v}<em>k)</em>{x_0}$ / $\sqrt{3}$</th>
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**Table 2**

| $k$ | $(\vec{V}_k(0))_{x_0}$ / $\sqrt{3}$ | $(\vec{V}_k(0))_{y_0}$ / $\sqrt{3}$ | $\rho_k^2 = \frac{|\vec{O}_0 \cdot \vec{O}_k|^2}{1}$ | $\tan \phi_k$ / $\sqrt{3}$ |
|-----|----------------------------------|----------------------------------|------------------------------------------|----------------------------|
| 0   | $1$, $0$                         | $0$, $0$                         | $0$, $0$                                  | $0$, $0$                   |
| 1   | $1$, $0$                         | $-1$, $1$                        | $2$, $-1$                                 | $1$, $0$                   |
| 2   | $-1$, $1$                        | $-4$, $3$                        | $25$, $-14$                               | $1$, $1/3$                 |
| 3   | $-10$, $6$                       | $-9$, $6$                        | $209$, $-120$                            | $5/4$, $3/4$               |
| 4   | $-38$, $22$                      | $-9$, $6$                        | $1581$, $-912$                           | $2$, $3/2$                 |
| 5   | $-104$, $60$                     | $29$, $-16$                      | $11717$, $-6764$                        | $19/4$, $15/4$             |
| 6   | $-208$, $120$                    | $209$, $-120$                    | $87881$, $-50160$                       | $\infty$                  |
| 7   | $-208$, $1204$                   | $777$, $-448$                    | $646361$, $-373176$                     | $-71/8$, $-49/8$           |
| 8   | $568$, $-328$                    | $2121$, $-1224$                  | $4818705$, $-2782080$                  | $-7$, $-35/8$              |
| 9   | $4240$, $-2448$                  | $4241$, $-24488$                 | $35955713$, $-20759040$                | $-265/32$, $-153/32$       |
| 10  | $15824$, $-9136$                 | $4241$, $-2448$                  | $268365505$, $-15490896$               | $-209/16$, $-173/24$      |
| 11  | $43232$, $-24960$                | $-11583$, $6688$                 | $2003139041$, $-1156512864$            | $-989/32$, $-539/32$      |
| 12  | $86464$, $-49920$                | $-86463$, $49920$                | $14951869569$, $-8632465920$           | $\infty$                  |
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### Table 3

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Table 5

| $k$ | $(\vec{V}_k(5))_{x_0}$ | $(\vec{V}_k(5))_{y_0}$ | $2^k \rho_{-k}^2 = 2^k |\vec{O}_0, O_{-k}|^2$ | $\cdot 1, \cdot \sqrt{3}$ |
|-----|------------------------|------------------------|--------------------------------|-------------------------|
| 0   | 1/2, 0                 | 0, −1/2                | 0, 0                           | 0, −1/2                 |
| 1   | 1, 0                   | 0, 0                   | 1, 0                           | 1/2, −1/2               |
| 2   | 1, 0                   | −1, 1                  | 7, 2                           | 2, −1                   |
| 3   | −1, 1                  | −4, 3                  | 34, 15                         | 6, −3                   |
| 4   | −10, 6                 | −9, 6                  | 141, 72                        | 15, −8                  |
| 5   | −38, 22                | −9, 6                  | 526, 285                       | 29, −16                 |
| 6   | −104, 60               | 29, −16                | 1831, 1020                     | 29, −16                 |
| 7   | −208, 120              | 209, −120              | 6154, 3479                     | −75, 44                 |
| 8   | −208, 120              | 777, −448              | 20625, 11760                   | −567, 328               |
| 9   | 568, −328              | 2121, 1224             | 70738, 40545                   | −2119, 1224             |
| 10  | 4240, −2448            | 4241, −2448            | 251527, 144628                 | −5791, 3344             |
| 11  | 15824, −9136           | 4241, −2448            | 925354, 533071                 | −11583, 6688            |
| 12  | 43232, −24960          | −11583, 6688           | 3481569, 2007720               | −11583, 6688            |

Table 6

| $k$ | $r_k = \frac{|S, \vec{O}_k|}{\cdot 1, \cdot \sqrt{3}}$ | $2^k \rho_{k+1}^2 = \vec{r}_k \cdot 1, \cdot \sqrt{3}$ | $(O_k)_X$ | $(O_k)_Y$ |
|-----|--------------------------------------------------------|-----------------------------------------------------|------------|------------|
| 0   | 1, 0                                                   | 1, 0                                                | 1, 0       | 0, 0       |
| 1   | −1, 1                                                  | 1, 1                                                | 3/2, −1/2  | −1/2, 1/2  |
| 2   | 4, −2                                                  | 2, 1                                                | 2, −1      | 2, 2       |
| 3   | −10, 6                                                 | 5, 3                                                | 0, 0       | −10, 6     |
| 4   | 28, −16                                                | 7, 4                                                | −14, 8     | −24, 14    |
| 5   | −76, 44                                                | 19, 11                                              | −66, 38    | −38, 22    |
| 6   | 208, −120                                              | 26, 15                                              | −208, 120  | 0, 0       |
| 7   | −568, 328                                              | 71, 41                                              | −492, 284  | 284, −164  |
| 8   | 1552, −896                                             | 97, 56                                              | −776, 448  | 1344, −776 |
| 9   | −4240, 2448                                            | 265, 153                                            | 0, 0       | 4240, −2448|
| 10  | 11584, −6688                                           | 362, 209                                            | 5792, −3344| 10032, −5792|
| 11  | −31648, 18271                                          | 989, 571                                            | 27408, −15824| 15824, −9136|
| 12  | 86464, −49920                                          | 1351, 780                                           | 86464, −49920| 0, 0       |

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Table 10

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