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FORESTS OF LABEL-INCREASING TREES

by

John Riordan
The Rockefeller University
New York, N. Y. 10021

Abstract

Label-increasing trees are fully labeled rooted trees with the restriction that the labels are in increasing order on every path from the root; the best known example is the binary case - no tree with more than two branches at the root, or internal vertices of degree greater than three - extensively examined in [5]. The forests without branching restrictions are enumerated by number of trees by $F_n(x) = x(x+1)...(x+n-1)$, $n>1$ ($F_o(x) = 1$), whose equivalent:

$$F_n(x) = Y_n(xT_1,...,xT_n),\ F_n(1) = T_{n+1} = n!$$

is readily adapted to branching restriction.

1. Introduction

What H. W. Becker, in the dim past of 1949, (private communication) called triangular words, (Δ-words for short) are families of sequences $(s_1, s_2, \ldots, s_n)$, generated by the rule : $s_i = 1, 2, \ldots i$. For $n = 3$ the sequences are: 111, 112, 113, 121, 122, 123. In [4] their complements: $s_i = i, i+1, \ldots, n$, are called "exeedent maps" .

It is immediate that the number of words for any n is n!. The enumerator for words by number of fixed points: $s_i = i$ (which is also the enumerator of the associated forests by number of trees, $F_n(x)$ has the immediate recurrence

$$F_n(x) = (x+n-1)\ F_{n-1}(x)$$

since the only fixed point for n is $s_n = n$. Hence, since $F_1(x) = x$,

$$F_n(x) = x(x+1)\ldots(x+n-1),\ n = 1,2,\ldots \tag{1}$$

and, by convention, $F_o(x) = 1$.

The right hand side of (1) is the enumerator of permutations by number of cycles, designated by $c_n(x)$ in [7, p.71], (which incidentally implies a mapping of trees to cyclic graphs). Also [7, p.70]

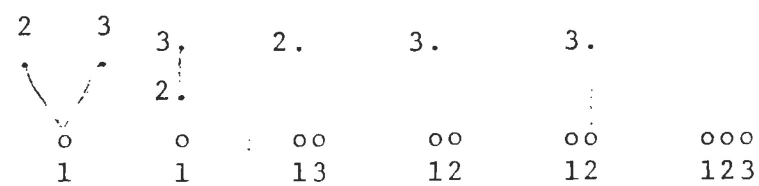
$$c_n(x) = C_n(x, \ldots, x) = Y_n(x, x, 2x, \ldots, (n-1)!x)$$

with C_n the cycle indicator of the symmetric group, a multi-variable function, and Y_n its relative, the Bell multivariable polynomial.

Thus a companion form of (1) is

$$F_n(x) = Y_n(xT_1,\ldots,xT_n),\ F_n(1) = n! = T_{n+1} \qquad (1a)$$

The mapping of Δ-words to forests follows the procedure described in [8]: a fixed point, $s_i = i$, is a tree root, and $s_i = j$, $j \neq i$ is a line from i to j. Thus $F_n(x)$ is an enumerator of forests by number of trees, with n point labels clearly in increasing order on every branch from a root, and at most n trees. The forests for n = 3 are



Also, the identity $F_n(1) = T_{n+1}$ in equation (1a), reflects the fact that the forests for n become trees for n+1 by adding a new root with single lines to each of the forest roots (which are then ignored).

The virtue of (1a) is the ease with which it is adapted to branching restrictions. If the restriction is to at most k branchings (at most k branches at a root, and internal virtices of degree at most k+1), the terms in Y_n are those of degree of most k, and the forest enumerator $F_n(x,k)$ is given by

$$F_n(x;k) = Y_n(fxT_1(k),\ldots,fxT_n(k)),\ F_n(1,k) = T_{n+1}(k) \qquad (2)$$

with f an umbral variable, $f^j \equiv f_j$ and $f_1 = f_2 = \ldots = f_k = 1$, $f_{k+n} = 0$, $n = 1,2,\ldots$. Of course $F_n(x;k) = F_n(x)$, $n = 0(1)k$.

Then

$$F(x,y;k) = \sum_{0} F_n(x;k)y^n/n!$$

$$= \exp\ (yY(fxT_1(k),\ldots,fxT_j(k),\ldots),\quad Y^n \equiv Y_n$$

$$= \exp\ (fx \sum_{1} T_n(k)\ y^n/n!)$$

$$= \exp\ (fx\ (T(y;k)-1) \qquad (3)$$

with

$$T(y;k) = \sum_{0} T_n(k)\ y^n/n!$$

Hence

$$F(x,y;k) = \sum_{0}^{k} x^j\ (T(y;k)-1)^j/j! \qquad (4)$$

and, with a prime denoting a derivative, by (2)

$$T'(y,k) = F(1,y;k) = \sum_{0}^{k} (T(y,k)-1)^j/j! \qquad (5)$$

The right hand side of (5) is the "displacement" polynomial $D_k(T(x))/k!$ defined in [7, p.59]; hence

$$k!\ T'(y,k) = D_k(T(x)) \qquad (5a)$$

For k = 2, (5a) becomes

$$2T'(y;2) = 1+T^2(y;2) \qquad (6)$$

whose solution, going back to D. André [1], is

$$T(y;2) = \tan y + \sec y \qquad (7)$$

The first few values of $T_n(2)$ are as follows

n	0	1	2	3	4	5	6	7	8
$T_n(2)$	1	1	1	2	5	16	61	272	1385

Further values appear in [9, sequence 587], but the most extensive table: n = 0(1)120 seems to be in [6]. Their relation to "binary increasing trees" has already appeared in [3] and [5].

From (4) and (6), it follows that

$$F(x,y;2) = 1-x + (x-x^2)\ T(y;2) + x^2 T'(y;2)$$

so that the forest enumerator $F_n(x;2)$ is given by

$$F_n(x;2) = (1-x)\delta_{no} + (x-x^2)\ T_n(2) + x^2 T_{n+1}(2) \qquad (8)$$

which satifies the initial conditions $F_0 = 1$, $F_1 = x$, $F_2 = x+x^2$.

It is interesting to notice that use of two of the congruences for Bell polynomials in [2], namely

$$Y_p(y_1,\ldots,y_p) \equiv y_1^p + y_p \pmod p$$

$$Y_{n+p} \equiv Y_n Y_p + \sum_{r=1}^{n} \binom{n}{r} y_{r+p} Y_{n-p} \pmod p$$

with p a prime, lead to, with $A_n = T_n(2)$ (A for André)

$$A_{p+n-1} \equiv A_p A_n \pmod p \qquad (9)$$

and by iteration

$$A_{j(p-1)+n} \equiv A_p^j\ A_n \pmod p \qquad (10)$$

Combined with the results of periodicity of residues in [6], (9) and (10) lead to

$$A_p \equiv 1 \pmod p,\ p \equiv 1 \pmod 4$$

$$A_p \equiv -1 \pmod p,\ p \equiv 3 \pmod 4$$

For larger values of k, the familiarity and elegance of k = 2 vanish; the brief description of results for k = 3,4 in the next section, merely underlines this fact.

2. Ternary and Quaternary Trees and Forests

For k = 3, the ternary case, it is convenient to replace $T_n(3)$ by B_n and $T(y;3)$ by $B(y) = \Sigma B_n y^n/n!$. Then by (5a)

$$6B'(y) = 2 + 3B(y) + B^3(y) \qquad (11)$$

or, if $B^j(y) = \Sigma B_n(j)\, y^n/n!$,

$$6B_{n+1} = 2\delta_{n0} + 3B_n + B_n(3) \qquad (12)$$

with δ_{nm} the Kronecker delta; equation (12) of course satisfies the initial conditions: $B_0 = 1$, $B_{n+1} = n!$, $n = 0(1)3$.

By (4), the forest generating function may be written

$$6F(x,y:3) = 6+6(B(y)-1)x + 3(B(y)-1)^2x^2 + (B(y)-1)^3x^3 \qquad (13)$$

Using (11), $(B(y)-1)^3 = 6B'(y) - 3B^2(y)-3$; hence (13) implies

$$6F_n(x:3) = 6\delta_{n0} + 6(B_n-\delta_{n0})x + 3(B_n(2) - 2B_n + \delta_{n0})x^2 \qquad (14)$$
$$+ (6B_{n+1} - 3B_n(2) - 3\delta_{n0})x^3$$

Thus $F_0(x) = 1$, $F_1(x) = x$, $F_2(x) = x+x^2$, and

$$6F_n(x;3) = 6B_nx + 3(B_n(2) - 2B_n)x^2 + (6B_{n+1} - 3B_n(2)), n = 3,4,..$$

Numerical results (obtained by hand) are as follows:

n	0	1	2	3	4	5	6	7	8	9	10
B_n	1	1	1	2	6	23	108	601	3863	28159	229524
$B_n(2)$	1	2	4	10	34	146	752	4520	31062	240188	2063846
$B_n(3)$	1	3	9	30	120	579	3282	21375	157365	1292667	11722416

and $B_{11} = 2068498$

Some general information on the sequence $\{B_n\}$ is in the congruences: $B_{p+n-1} \equiv B_p B_n \pmod p$ for p a prime greater than 3, and its consequence $B_{j(p-1)+n} \equiv B_p^j B_n \pmod p$.

The corresponding results for k = 4 (with $C_n = T_n(4)$) for n>0, are

$$2 + C_{n+1} = 8\, C_n + 6\, C_n(2) + C_n(4)$$

$$24\, F_n(x;4) = 24\, C_n x + 12(C_n(2) - 2\, C_n)x^2 + 4(C_n(3) - 3C_n(2) + 3C_n)x^3 + (24C_{n+1} - 4(C_n(3) + 3C_n))x^4$$

and the table:

n	0	1	2	3	4	5	6	7	8	9	10
C_n	1	1	1	2	6	24	119	703	4819	37596	328871
$C_n(2)$	1	2	4	10	34	148	786	4920	35446	288822	2625844
$C_n(3)$	1	3	9	30	120	582	3351	22395	170505	1457238	13816869
$C_n(4)$	1	4	16	68	324	1776	11204	80512	651076	5859204	58117066

Also $C_{11} = 3187627$

The congruences for C_n are the same as those for B_n, with p a prime greater than 3.

None of the sequences for these two tables appear in [9].

A more extensive table for $T_n(k)$, ignoring the initial values: $n = 1(1)k$, and the trivial $T_n(1) = 1$, is the following

k/n	3	4	5	6	7	8	9	10
2	2	5	16	61	272	1385	7936	50521
3		6	23	108	601	3863	28159	229524
4			24	119	703	4819	37596	328871
5				120	719	5017	39938	357100
6					720	5039	40290	362258
7						5040	40319	362842
8							40320	362879
9								362880

The differences, $U_n(k) = T_n(k) - T_n(k-1)$, show the following regularities:

$U_n(n-1) = 1, \; n>2$

$U_n(n-2) = 1 + \binom{n-1}{2}, \; n>3$

$U_n(n-3) = 2 + 2\binom{n-1}{4} + \binom{n+1}{4}, \; n>5$

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