Description of dense centipedes

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A *centipede* is a tree T which consists of a diametrical path P whose deletion leaves only isolated vertices [1]. A non-terminal vertex of the diametrical path P which has at least one pendant vertex attached to it is called a *joint*. A centipede in which every non-terminal vertex is a joint is defined to be a *dense centipede*. The subgraph of a dense centipede formed with a joint and all the pendants attached to it is defined to be a *cluster*. (Figure 1)

It is to be noted that there are exactly $r = d(T) - 1$ non-terminal vertices in any diametrical path of a centipede. We denote these vertices as $v_1, v_2, \ldots, v_r$. These are the same for any diametrical path of a centipede. We define the joints v_1 and v_r as *extreme joints*. The joints other than v_1 and v_r are the *internal joints*. It is to be noted that the degree of an extreme joint is at least 2 and the degree of an internal joint is at least 3. Thus the maximum degree of a dense centipede is at least 3. We introduce a new scheme of labelling the vertices of a dense centipede. For this, we consider any diametrical path and group together the pendant vertices attached to each joint of this diametrical path. It can be seen that this grouping is independent of the chosen diametrical path. This is illustrated in Figure 1. We denote the cluster corresponding to the non-terminal vertex v_i by C_i. So there are exactly $d(T) - 1$ clusters $C_1, C_2, \ldots, C_r$ in a dense centipede T. Let each cluster has N_i vertices. By definition of a dense centipede, $N_i \geq 2$ for all $i = 1, 2, \ldots, r$. If N is the total number of vertices in T, then we have $N = \sum_{i=1}^{r} N_i$.

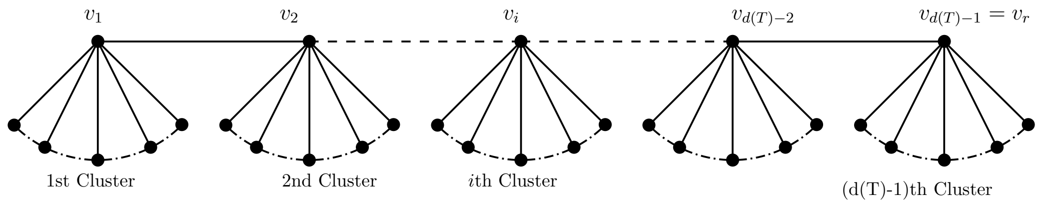

Figure 1: A dense Centipede

We also introduce a new nomenclature for dense centipedes. It is easily seen that a centipede is totally described if we know its diameter, the number of vertices in each of its clusters and the order in which the clusters appear. Thus, a dense centipede of diameter $d(T)$ can be represented by an ordered tuple with $r = d(T) - 1$ coordinates, where the ith coordinate gives the number of vertices in the the ith cluster. Thus, the dense centipede described in Figure 1 will be named as $C(N_1, N_2, \ldots, N_r)$. But we need to add something more to this scheme of nomenclature. It is to be noted that the dense centipede in Figure 2 will be named as $C(3, 5, 4)$. But a dense centipede named as $C(4, 5, 3)$ will actually represent the same graph as represented by $C(3, 5, 4)$ in Figure 2. Similarly, $C(12, 6, 4, 3)$ and $C(3, 4, 6, 12)$ will represent the same dense centipede. To avoid this redundancy in nomenclature, we further impose the condition that the centipede will be named as $C(N_1, N_2, \ldots, N_r)$ if the number $10^{r-1}N_1 + 10^{r-2}N_2 + \cdots + N_r$ is smaller than the number with its 'digits' reversed i.e. $10^{r-1}N_r + 10^{r-2}N_{r-1} + \cdots + N_1$, otherwise it will be named as $C(N_r, N_{r-1}, \ldots, N_1)$. This resembles the usual ordering in the place value notation. Thus, as per our nomenclature scheme, $C(3, 5, 4)$, $C(3, 4, 6, 12)$ and $C(2, 3, 6, 2)$ are correct names for a dense centipede but $C(4, 5, 3)$, $C(12, 6, 4, 3)$ and $C(2, 6, 3, 2)$ are not.

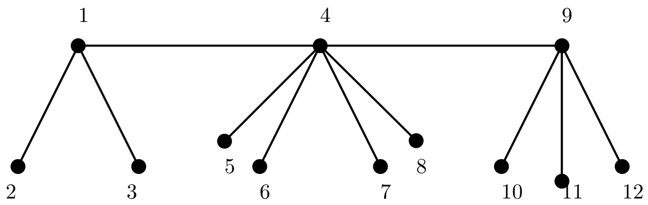

Figure 2: A dense centipede of diameter 4

Since each $N_i \geq 2$ and $N = \sum_{i=1}^{r} N_i$, so a dense centipede with N vertices can have at most $\lfloor N/2 \rfloor$ joints (here $\lfloor \cdot \rfloor$ is the greatest integer function). Hence the maximum possible diameter of a dense centipede with N vertices is $\lfloor N/2 \rfloor + 1$. The minimum number of vertices required to construct a dense centipede is 4 and there is only one such dense centipede having 4 vertices, namely $C(2, 2)$. The dense centipede on 5 vertices is $C(2, 3)$. The dense centipedes on 6 vertices are $C(2, 4)$, $C(3, 3)$ and $C(2, 2, 2)$, in *increasing order of nomenclature* i.e. the order of the names in the list obeys the order $24 < 33 < 222$. We list all possible dense centipedes of at most 10 vertices in the following table :

N	Names of dense centipedes in increasing order of nomenclature
4	$C(2,2)$
5	$C(2,3)$
6	$C(2,4)$, $C(3,3)$, $C(2,2,2)$
7	$C(2,5)$, $C(3,4)$, $C(2,2,3)$, $C(2,3,2)$
8	$C(2,6)$, $C(3,5)$, $C(4,4)$, $C(2,2,4)$, $C(2,3,3)$, $C(3,2,3)$, $C(2,2,2,2)$
9	$C(2,7)$, $C(3,6)$, $C(4,5)$, $C(2,2,5)$, $C(2,3,4)$, $C(2,4,3)$, $C(2,5,2)$, $C(3,2,4)$, $C(2,2,2,3)$, $C(2,2,3,2)$,
10	$C(2,8)$, $C(3,7)$, $C(4,6)$, $C(5,5)$, $C(2,2,6)$, $C(2,3,5)$, $C(2,4,4)$, $C(2,5,3)$, $C(2,6,2)$, $C(3,2,5)$, $C(3,3,4)$, $C(3,4,3)$, $C(4,2,4)$, $C(2,2,2,4)$, $C(2,2,3,3)$, $C(2,2,4,2)$, $C(2,3,2,3)$, $C(2,3,3,2)$, $C(3,2,2,3)$, $C(2,2,2,2,2)$
11	$C(2,9)$, $C(3,8)$, $C(4,7)$, $C(5,6)$, $C(2,2,7)$, $C(2,3,6)$, $C(2,4,5)$, $C(2,5,4)$, $C(2,6,3)$, $C(2,7,2)$, $C(3,2,6)$, $C(3,3,5)$, $C(3,4,4)$, $C(3,5,3)$, $C(4,2,5)$, $C(4,3,4)$, $C(2,2,2,5)$, $C(2,2,3,4)$, $C(2,2,4,3)$, $C(2,2,5,2)$, $C(2,3,2,4)$, $C(2,3,3,3)$, $C(2,3,4,2)$, $C(2,4,2,3)$, $C(3,2,2,4)$, $C(3,2,3,3)$, $C(2,2,2,2,3)$, $C(2,2,2,3,2)$, $C(2,2,3,2,2)$

Matrices described by such dense centipedes have associated with them some interesting inverse eigenvalue problems. Some special cases of centipedes are brooms. Some related inverse eigenvalue problems for matrices described by brooms were studied in [2]. It would thus be good to fix a standard scheme of nomeclature of dense centipedes by including the scheme in OEIS.

References

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