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ON THE NUMBER OF REPRESENTATIONS AS SUM OF FOUR SQUARES OF NUMBERS OF THE FORM $4^a(8b+7)$

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1§. It is well known that any integer can be represented as the sum of atmost four squares and that the numbers of the class $4^a(8b+7)$ can not be represented as sum of three squares. Hence all such numbers are representible as sum of four squares none of which is zero.

Among the numbers of this class, different sub-classes can be found out which can be represented as sum of four squares in a fixed number of distinct ways, and no number of the class has the same property.

Throughout this paper the letter n is used for a positive integer of the form $4^a(8b+7)$ $(a \geqslant 0)$, and u for an odd integer of the form $8b+7$.

Now we give the following results which will be used in the detailed discussion to follow.

(a) The representation of a number as the sum of four squares may be either of the following three forms :

$$x^2_1+x^2_2+x^2_3+x^2_4 \qquad (\alpha)$$
$$x^2_1+x^2_2+x^2_3+x^2_3 \qquad (\beta)$$
$$x^2_1+x^2_2+x^2_2+x^2_2 \qquad (\gamma)$$

The case of all the four x's equal is impossible for then the number n itself is a perfect square which is therefore not of the form $4^a(8b+7)$.

(b) One distinct representation of (α) in (a) will give rise to 384 different representations by considering change of sign and change of position.

s(u) means $\sum_{\substack{\delta | n \\ \delta \text{ odd}}} \delta$

Similarly one distinct representation each of form (β) and (γ) will give rise to 192 and 64 representations respectively.

(c) The number of representations as sum of four squares including change of position and change of sign of a number n is given by :

$$Q(n) = 24\, s(u) \text{ when n is even.}$$
$$ = 8\, s(u) \text{ when n is odd.}$$

(d) The number of representations of numbers of the form $4^a(8b+7)$ is the same for all values of $a \geqslant 1$, for the number of representations Q(n) clearly depends upon the odd factor u.

(e) The different representations of 4u belonging to different forms of (a) will give rise to the representations of the same form of the numbers $4^a u$ ($a > 1$) for, all representations of $4^a u$ can be obtained from those of 4u by multiplying each number in the representation by 2^{a-1} and obviously such multiplication will not change the form.

(f) From (c) the number of representations of $4^a u$ is greater than that of u. Hence each representation of u of the forms (α) (β) and (γ) will give rise to at least one representation of $4^a u$ of the same form.

2§. First we shall find out the numbers which have exactly three representations (partitions) as sum of four squares. The possible combinations of three distinct representations as sum of four squares are as follows :—

(2.1) All the three may be of the form (α).
(2.2) All the three may be of the form (β).
(2.3) All the three may be of the form (γ).
(2.4) Two of form (α) and one of form (β).
(2.5) Two of form (α) and one of form (γ).
(2.6) One of form (α) and two of form (β).
(2.7) One of form (α) and two of form (γ).
(2.8) Two of form (β) and one of form (γ).
(2.9) Two of form (γ) and one of form (β).
(2.10) One of form (α) one of form (β) and one of form (γ).

We consider all the above cases one by one, and find out all the numbers under each head.

Case 2.1. In this case $Q(n)=1152$

If an even number belongs to this class :

Then $$Q(n)=24.s(u)=1152$$

$$\text{i.e. } s(u)=48$$

which gives only one value of $u=47$*

But $47=6^2+3^2+1^2+1^2$, which is of the form (β).

Therefore at least one representation of $4^a.47$ will also be of the same form. Hence it does not come under this head.

If an odd number belongs to this class :

$$8s(u)=1152$$

$$s(u)=144$$

which gives $u=119=10^2+3^2+3^2+1^2$ of the form (β). And it also does not belong to the class.

Case 2.2. In this case $Q(n)=576$

If an even n exists : $24s(u)=576$

$$s(u)=24$$

$$u=15 \ \& \ 23$$

Both of these have representations of the form (β).

$$15=3^2+2^2+1^2+1^2 \ ; \ 23=3^2+3^2+2^2+1^2$$

Hence nos. of the form 4^a15 and 4^a23 will yield three representations of the desired form, for 60 and 92 give representations under this head.

$$60=6^2+4^2+2^2+2^2 \qquad 92=9^2+3^2+1^2+1^2$$
$$=7^2+3^2+1^2+1^2 \qquad =7^2+5^2+3^2+3^2$$
$$=5^2+5^2+3^2+1^2 \qquad =6^2+6^2+4^2+2^2$$

*The value of u so found only fixes the value of Q(n). But as Q(n) may have numbers of other partitions with numbers 64, 192 and 384, u may also be such other numbers which do not come under the head. In order to exclude such numbers, we note all the representations of u. If even a single representation of u is not of the desired form the number is excluded, for in that case at least one representation of $4^a u$ must also be of the same form. (From 1.f). A second possibility is that even if one representation of u is of different form, 4u may give three representations but that will itself come under some other head.

If an odd n exists : $8s(u)=576$

$$s(u)=72$$
$$u=71 \ \& \ 55$$

But $71=6^2+5^2+3^2+1^2$ of the form (α) and hence ruled out. The only number satisfying the condition of the class is 55.

$$55=7^2+2^2+1^2+1^2$$
$$=6^2+1^2+3^2+3^2$$
$$=5^2+5^2+2^2+1^2$$

Case 2.3. In this case $Q(n)=192$

If n even, $24s(u)=192$

$$s(u)=8$$
$$u=7$$

7 has only one representation $7=2^2+1^2+1^2+1^2$ of the form (γ) so that 28 yields three representations :

$$28=5^2+1^2+1^2+1^2$$
$$=4^2+2^2+2^2+2^2$$
$$=3^2+3^2+3^2+1^2$$

4^a7 will always yield three representations.

if n odd, $8s(u)=192$

$$s(u)=24$$
$$u=15 \ \& \ 23$$

But $15=3^2+2^2+1^2+1^2$; $23=3^2+3^2+2^2+1^2$

Both being of the form (β) do not come in this head.

Case 2.4. $Q(n)=960$

If n even $24s(u)=960$

$$s(u)=40$$

No such u exists.

If n odd $8s(u)=960$

$$s(u)=120$$
$$u=87 \ \& \ 95$$

but $87=9^2+2^2+1^2+1^2=7^2+6^2+1^2+1^2$ i.e. two representations of the form (β) ; hence does not come under this head.

And $95=9^2+3^2+2^2+1^2$

$$=7^2+6^2+3^2+1^2$$
$$=6^2+5^2+5^2+3^2$$

Case 2.5. $Q(n)=832$

If n even $24s(u)=832$

$s(u)=$ A fraction which is impossible.

If n odd, $8s(u)=832$

$s(u)=104$

$u=63$ & 103

$63=6^2+5^2+1^2+1^2$; $103=9^2+3^2+3^2+2^2$ of the form (β) and hence are ruled out.

Case 2.6. $Q(n)=768$

If n even, $24s(u)=768$

$s(u)=32$

$u=31$

$31=3^2+3^2+3^2+2^2$ of the form (γ) so does not come under this head.

If n odd, $8s(u)=768$

$s(u)=96$

No such u exists.

Case 2.7. $Q(n)=512$

If n even, $24s(u)=512$

$s(u)=$ A fraction, hence impossible.

If n odd, $8s(u)=512$

$s(u)=64$

No such u exists.

Case 2.8. $Q(n)=448$

If n even, $24s(u)=448$

$s(u)=$ A fraction, hence impossible.

If n odd, $8s(u)=448$

$s(u)=56$

$u=39$

But $39=5^2+3^2+2^2+1^2$ of the form (α), hence ruled out.

Case 2.9. $Q(n)=320$

If n even, $24s(u)=320$

$s(u)=$ A fraction, impossible.

If n odd, $8s(u)=320$

$s(u)=40$

No such u exists.

s(u) means $\sum \delta$

Case 2.10. $Q(n)=640$

If n even $24s(u)=640$

$s(u)=$ A fraction ; impossible.

If n odd, $8s(u)=640$

$s(u)=80$

$u=79$

And 79 comes in the class, for

$$79=7^2+5^2+2^2+1^2$$
$$=6^2+5^2+3^2+3^2$$
$$=5^2+5^2+5^2+2^2$$

Hence, if n is of the form $4^a(8b+7)$, it can be represented as sum of four squares exactly in three ways, then and only then, when

$$n=55,\ 79,\ 95.$$
$$=4^a.7,\ 4^a.15,\ 4^a,23\ (a\geqslant 1)$$

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3§ A similar treatment as Art. 2 gives the following results :

(a) No even number of the form $4^a(8b+7)$ has got less than three representations.

(b) The numbers of the form $4^a(8b+7)$ which yield only one representation as sum of four squares are exclusively 7, 15 and 23.

(c) The numbers of the form $4^a(8b+7)$ which yield two representations as sum of four squares are exclusively 31, 39, 47, 71.

(d) The numbers of the form $4^a(8b+7)$ which yield four representations as sum of four squares are exclusively 63, 87 and 119.

(e) The same procedure may be adopted to frame other sub-classes of the above mentioned type.

4§ In this section we shall use the above properties in generating multigrade equations of order m and degree three having eight elements in each set.

If a number n has m distinct representations as sum of four squares then:

$$n = a_1^2 + b_1^2 + c_1^2 + d_1^2 = a_2^2 + b_2^2 + c_2^2 + d_2^2 = \ldots$$
$$= a_m^2 + b_m^2 + c_m^2 + d_m^2$$

It immediately follows that,

$$(a_1, b_1, c_1, d_1, -a_1, -b_1, -c_1, -d_1)_3 = (a_2, b_2, c_2, d_2, -a_2, -b_2, -c_2, -d_2)_3$$
$$= \ldots\ldots = (a_m, b_m, c_m, d_m, -a_m, -b_m, -c_m, -d_m)_3$$

Adding 'h' to all sets where $h > \max(|a_r|\ |b_r|\ |c_r|\ |d_r|)$ for all r we shall get all elements positive.

The results of Art. 2 & 3 give that

(a) The number 55, 79, 95 and others of the type $4^a.7$, $4^a.15$, $4^a.23$, will generate multigrade equations of order 3 and degree 3 with eight numbers in each set.

(b) The numbers 31, 39, 47 and 71 will generate multigrade equations of order 2 and degree 3 with eight numbers in each set.

(c) The numbers 63, 87, 119 will generate multigrade equations of order 4. and degree 3. with eight numbers in each set. For illustration:

$$55 = 7^2 + 2^2 + 1^2 + 1^2$$
$$= 6^2 + 1^2 + 3^2 + 3^2$$
$$= 5^2 + 5^2 + 2^2 + 1^2$$

so, $(7, 2, 1, 1, -7, -2, -1, -1)_3 = (6, 1, 3, 3, -6, -1, -3, -3)_3$
$= (5, 5, 2, 1, -5, -5, -2, -1)_3$

$(15, 10, 9, 9, 7, 7, 6, 1)_3 = (14, 11, 11, 9, 7, 5, 5, 2)_3$
$= (13, 13, 10, 9, 7, 6, 3, 3)_3$

i.e. 55 generates multigrade equations of order 3 and degree 3.

Also $31 = 5^2 + 2^2 + 1^2 + 1^2$
$= 3^2 + 3^2 + 3^2 + 2^2$

so $(5, 2, 1, 1, -5, -2, -1, -1)_3 = (3, 3, 3, 2, -3, -3, -3, -2)_3$

$(11, 8, 7, 7, 5, 5, 4, 1)_3 = (9, 9, 9, 8, 4, 3, 3, 3,)_3$

Again $63 = 7^2 + 3^2 + 2^2 + 1^2$
$= 6^2 + 3^2 + 3^2 + 3^2$
$= 6^2 + 5^2 + 1^2 + 1^2$
$= 5^2 + 5^2 + 3^2 + 2^2$

so $(7, 3, 2, 1, -7, -3, -2, -1)_3 = (6, 3, 3, 3, -6, -3, -3, -3)_3$
$= (6, 5, 1, 1, -6, -5, -1, -1)_3 = (5, 5, 3, 2, -5, -5, -3, -2)_3$
Therefore $(15, 11, 10, 9, 7, 6, 5, 1)_3 = (14, 11, 11, 11, 5, 5, 5, 2)_3$
$(14, 13, 9, 9, 7, 7, 3, 2)_3 = (13, 13, 11, 10, 6, 5, 3, 3)_3$

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References

1. E. Landau ... Vorlesungen über Zahlen theorie Band 1, Satz 172, p. 113.
2. E. Landau ... Vorlesungen über Zahlen theorie Band 1, Satz 186, p. 122.