Maple-assisted proof of empirical formula for A267244

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Sep 8 2019

Consider a "state" of the system to be a 1×13 binary array x, where $x_1 .. x_7$ represent a row, and for j from 8 to 13, $x_j = 1$ if this row and the previous ones have already determined that column $j-6$ is lexicographically greater than column $j-7$. In particular if $x_1 \neq x_2$ we must have $x_8 = 1$ and similarly for the others.

We enumerate the 1458 possible states:

```
> S[2]:= [[0,0,0],[0,0,1],[0,1,1],[1,0,1],[1,1,0],[1,1,1]]:
  for i from 3 to 7 do
    S[i]:= map(proc(t) [op(t[1..i-1]),t[i-1],op(t[i..-1]),0], [op(t
  [1..i-1]),t[i-1],op(t[i..-1]),1],
       [op(t[1..i-1]),1-t[i-1],op(t[i..-1]),1] end proc, S[i-1])
  od:
  states:= S[7]:
  nops(states);
```

$$1458 \quad (1)$$

Although it is not part of the $n \times 7$ array, we may imagine that we start in state [0,0,0,0,0,0,0,0,0,0,0,0,0]. Let T be the 1458×1458 transition matrix where $T_{ij} = 1$ if state j can be followed by state i.

```
> T:= Matrix(1458,1458,proc(i,j) local k;
    if add(states[j,k]-states[i,k],k=1..7) > 0 then return 0 fi;
    for k from 8 to 13 do if states[j,k]>states[i,k] then return 0
  fi od;
    for k from 1 to 6 do if states[i,k]>=states[i,k+1] and states
  [j,k+7]<>states[i,k+7] then return 0 fi od;
  1
  end proc):
```

Then we should have $a_n = u^T T^n e$ where $u = (1, ..., 1)^T$ and $e = (1, 0, ..., 0)^T$. To check, we compute the first few terms of the sequence. .

```
> E:= Vector(1458): E[1]:= 1:
  U[0]:= Vector[row](1458,1):
  for k from 1 to 32 do U[k]:= U[k-1].T od:
  seq(U[j] . E, j=1..25);
```

8, 70, 904, 18205, 516084, 17892539, 683027146, 27044976947, 1079112886476, 42860145907558, 1687239907979286, 65777529883058423, 2540922972496976428, 97351678797063744735, 3703224984260808730288, 139993814565092144904305, 5263581017833119467436816, 196967402441342127901105822, 7340014308853073595836121020, 272519639757003747065668917769, 10084933379767056583424487670132, 372110360485418692723491624924283, 13693715083431328559152054166409534, 502725568194081684245914670566316615, 18416003801147421401186018197525511572 **(2)**

Now the empirical formula is

```
> Emp:=  a(n) = 236*a(n-1) -25680*a(n-2) +1717504*a(n-3) -79417394*
  a(n-4) +2707798440*a(n-5) -70899406188*a(n-6) +1465896913824*a
  (n-7) -24421757248431*a(n-8) +332861244138564*a(n-9)
  -3755300016546300*a(n-10) +35390628699049728*a(n-11)
  -280610566308801516*a(n-12) +1882413463252467120*a(n-13)
  -10729331312513919192*a(n-14) +52123544280277991616*a(n-15)
  -216277785263000273775*a(n-16) +767370439659990868020*a(n-17)
  -2328674591889971376488*a(n-18) +6039623808173911907968*a(n-19)
  -13364805995823788545362*a(n-20) +25161918805489259088488*a(n-21)
  -40140151907739595227388*a(n-22) +53955000634729356546720*a(n-23)
  -60650423670523051920321*a(n-24) +56445409553303282568732*a(n-25)
  -42910761548685014780364*a(n-26) +26160176586524646234240*a(n-27)
  -12460398044348337274800*a(n-28) +4460624272170497592000*a(n-29)
  -1127355192728480520000*a(n-30) +179146175950526400000*a(n-31)
  -13449500030736000000*a(n-32):
```

This corresponds to $u^T P(T) T^n e = 0$ where $P(x)$ is the following polynomial of degree 32:

```
> P:= x^32 - add(coeff(rhs(Emp),a(n-i))*x^(32-i),i=1..32);
```

$$P := x^{32} - 236\,x^{31} + 25680\,x^{30} - 1717504\,x^{29} + 79417394\,x^{28} - 2707798440\,x^{27} + 70899406188\,x^{26} - 1465896913824\,x^{25} + 24421757248431\,x^{24} - 332861244138564\,x^{23} + 3755300016546300\,x^{22} - 35390628699049728\,x^{21} + 280610566308801516\,x^{20} - 1882413463252467120\,x^{19} + 10729331312513919192\,x^{18} - 52123544280277991616\,x^{17} + 216277785263000273775\,x^{16} - 767370439659990868020\,x^{15} + 2328674591889971376488\,x^{14} - 6039623808173911907968\,x^{13} + 13364805995823788545362\,x^{12} - 25161918805489259088488\,x^{11} + 40140151907739595227388\,x^{10} - 53955000634729356546720\,x^{9} + 60650423670523051920321\,x^{8} - 56445409553303282568732\,x^{7} + 42910761548685014780364\,x^{6} - 26160176586524646234240\,x^{5} + 12460398044348337274800\,x^{4} - 4460624272170497592000\,x^{3} + 1127355192728480520000\,x^{2} - 179146175950526400000\,x + 13449500030736000000 \quad (3)$$

It turns out that $u^T P(T) = 0$. The verification of this completes the proof.

```
> UP:= add(coeff(P,x,j)*U[j],j=0..32):
  UP . UP^%T;
```

$$0 \quad (4)$$