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Article:

The sums of powers of integers natural even, odd

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Abstract:

In number theory, the sums of natural integers from 1 degree up to p th degree are known:

$$\sum_{i=1}^{n} i = 1+2+3+\dots+n = \frac{1}{2}n\,(n+1)$$

$$\sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+\dots+n^2 = \frac{1}{6}n\,(n+1)\,(2n+1)$$

$$\sum_{i=1}^{n} i^3 = 1^3+2^3+3^3+\dots+n^3 = \frac{1}{4}n^2\,(n+1)^2$$

$$\sum_{i=1}^{n} i^4 = 1^4+2^4+3^4+\dots+n^4 = \frac{1}{30}n\,(6n^4+15n^3+10n^2-1)$$

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$$\sum_{i=1}^{n} i^p = 1^p+2^p+3^p+\dots+n^p = \frac{1}{p+1}\sum_{j=0}^{p}\binom{P+1}{j} B_j\, n^{p+1-j} \quad (1)$$

That B_j, j=1,2,……..,p are the Bernoulli numbers (2)

n	0	1	2	3	4	5	6	7	8	9	10	11
B_n	1	$\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0

The purpose of article is to find a general formula new which permit to calculate the sums of powers of integers natural even and odd, using the formula of the binomial of Newton, the formula of Faulhaber as well as numbers of Bernoulli

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(1) Formula Faulhaber

(2) We take the Bernoulli number$B_1=+\frac{1}{2}$

1. The sum of the p th level of peer integers.

1.1 The sum of the integers natural peers.

For any natural number n, we have:

$$\sum_{i=1}^{n} 2i = 2+4+6+.....+2n = n\,(n+1)$$

Proof:

For any natural number n, we have:

$$\sum_{i=1}^{n} 2i = 2\sum_{i=1}^{n} i = n\,(n+1)$$

1.2 The sum of the integers natural peers of 2 nd degree.

For any natural number n, we have:

$$\sum_{i=1}^{n} (2i)^2 = 2^2+4^2+6^2+......+(2n)^2 = \frac{2}{3}\, n\,(n+1)\,(2n+1)$$

Proof:

For any natural number n, we have:

$$\sum_{i=1}^{n} (2i)^2 = 4\sum_{i=1}^{n} i^2 = 4 \times \frac{1}{6}\, n\,(n+1)\,(2n+1) = \frac{2}{3}\, n\,(n+1)\,(2n+1)$$

1.3 The sum of the integers natural peers of 3 nd degree.

For any natural number n, we have:

$$\sum_{i=1}^{n}(2i)^3 = 2^3+4^3+\ldots + (2i)^3 = 2n^2 (n+1)^2$$

Proof:

For any natural number n, we have:

$$\sum_{i=1}^{n}(2i)^3 = 8\sum_{i=1}^{n} i^3 = 8\times\frac{1}{4} n^2 (n+1)^2 = 2 n^2 (n+1)^2$$

In the same way, we found:

$$\sum_{i=1}^{n}(2i)^4 = \frac{8}{15} n (6n^4+15n^3+10n^2-1)$$

$$\sum_{i=1}^{n}(2i)^5 = \frac{8}{3} n^2 (2n^4+6n^3+5n^2-1)$$

1.4 The sum of the integers natural peers of p th degree.

Formula. for any natural number n, we have:

$$\sum_{i=1}^{n}(2i)^p = \frac{2\text{^}p}{p+1}\sum_{j=0}^{p}\binom{p+1}{j} B_j\, n^{p+1-j}$$

où $p \in N$, $p<n$ et B_j the numbers of Bernoulli.

Proof:

It can be easily seen that for any integer n, it was:

$$\sum_{i=1}^{n}(2i)^p = 2^p \sum_{i=1}^{n} i^p$$

We apply the formula of de Faulhaber, there is:

$$\sum_{i=1}^{n}(2i)^p = \frac{2^p}{p+1}\sum_{j=0}^{p}\binom{p+1}{j} B_j\, n^{p+1-j}$$

Example:

1) **For p=1, it was:** $\sum_{i=1}^{n} 2i = \frac{2}{2}\sum_{j=0}^{1}\binom{2}{j} B_j\, n^{2-j}$

$$= \binom{2}{0} B_0\, n^2 + \binom{2}{1} B_1\, n$$

$$= n^2+n = n\,(n+1)$$

2) **For p=2, it was:** $\sum_{i=1}^{n}(2i)^2 = \frac{4}{3}\sum_{j=0}^{2}\binom{3}{j} B_j\, n^{3-j}$

$$=\frac{4}{3}\left[\binom{3}{0} B_0\, n^3 + \binom{3}{1} B_1\, n^2 + \binom{3}{2} B_2\, n\right]$$

$$=\frac{4}{3}\left[n^3 + \frac{3}{2} n^2 + \frac{1}{2} n\right] = \frac{2}{3}\,(2n^3+3n^2+n)$$

$$=\frac{2}{3}\, n\,(2n^2+3n+1) = \frac{2}{3}\, n\,(n+1)\,(2n+1)$$

2. The sum of natural integers odd of p th degree.

2.1 The sums of natural integers odd.

For any natural number n, we have:

$$\sum_{i=0}^{n}(2i+1) = 1+3+5+.....+(2n+1) = (n+1)^2$$

Proof: for any natural number n, we have:

$$\sum_{i=0}^{n}(2i+1) = 2\sum_{i=1}^{n} i + (n+1) = n(n+1) + (n+1) = (n+1)(n+1) = (n+1)^2$$

2.2 The sum of natural integers odd of 2 nd degree.

For any natural number n, we have:

$$\sum_{i=0}^{n}(2i+1)^2 = 1^2 + 3^2+........+(2n+1)^2 = \frac{1}{3}(n+1)(2n+1)(2n+3)$$

Proof:

For any natural number n, we have:

$$\sum_{i=0}^{n}(2i+1)^2 = 4\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + (n+1)$$

$$= \frac{2}{3} n(n+1)(2n+1) + 2n(n+1) + (n+1)$$

$$= (n+1)\left[\frac{2}{3} n(2n+1) + 2n+1\right]$$

$$= (n+1)(2n+1)\left(\frac{2}{3} n+1\right)$$

$$= \frac{1}{3}(n+1)(2n+1)(2n+3)$$

2.3 The sum of natural integers odd of 3 th degree.

For any natural number n, we have:

$$\sum_{i=0}^{n}(2i+1)^3 = 1^3+3^3+.....+(2n+1)^3 = (n+1)^2\,(2n^2+4n+1)$$

Proof:

For any natural number n, we have:

$$\sum_{i=0}^{n}(2i+1)^3 = (n+1) + \sum_{i=1}^{n}(2i)^3 + 3\sum_{i=1}^{n}(2i)^2 + 3\sum_{i=1}^{n}2i$$

$$= (n+1) + 8\sum_{i=1}^{n} i^3 + 12\sum_{i=1}^{n} i^2 + 6\sum_{i=1}^{n} i$$

$$= (n+1) + 2\,n^2\,(n+1)^2 + 2\,n\,(n+1)\,(2n+1) + 3\,n\,(n+1)$$

$$= (n+1)\,[1+2n^2\,(n+1) + 2n\,(2n+1) + 3n]$$

$$= (n+1)\,(2n^3+6n^2+5n+1)$$

$$= (n+1)^2\,(2n^2+4n+1)$$

In the same way, we found:

$$\sum_{i=0}^{n}(2i+1)^4 = \frac{1}{15}\,(n+1)\,(48n^4+192n^3+248n^2+112n+15)$$

$$\sum_{i=0}^{n}(2i+1)^5 = \frac{1}{3}\,(n+1)\,(16n^5+80n^4+140n^3+100n^2+27n+3)$$

2.4 The sum of natural integers odd of p th degree.

Formula. for any natural number n, we have:

$$\sum_{i=0}^{n}(2i+1)^p = n+1 + \sum_{k=1}^{p} \left[\binom{p}{k}\sum_{i=1}^{n}(2i)^k\right]$$

Proof:

$$\sum_{i=0}^{n}(2i+1)^p = 1^p + 3^p + 5^p + \ldots\ldots + (2n+1)^p$$

$$= 1 + (2+1)^p + (4+1)^p + \ldots\ldots + (2n+1)^p$$

$$= 1+\sum_{i=0}^{p}\binom{p}{i}2^i + \sum_{i=0}^{p}\binom{p}{i}4^i + \ldots\ldots + \sum_{i=0}^{p}\binom{p}{i}(2n)^i$$

$$= n+1 + \sum_{i=1}^{p}\binom{p}{i}2^i + \sum_{i=1}^{p}\binom{p}{i}4^i + \ldots\ldots + \sum_{i=1}^{p}\binom{p}{i}(2n)^i$$

$$= n+1 + \binom{p}{1}2 + \binom{p}{2}2^2 + \ldots\ldots + \binom{p}{p}2^p$$

$$+\binom{p}{1}4 + \binom{p}{2}4^2 + \ldots\ldots + \binom{p}{p}4^p$$

$$+\ldots\ldots$$

$$+\binom{p}{1}(2n) + \binom{p}{2}(2n)^2 + \ldots\ldots + \binom{p}{p}(2n)^p$$

$$= n+1 + \binom{p}{1}[2+4+6+\ldots\ldots+2n]$$

$$+ \binom{p}{2}[2^2 + 4^2 + \ldots\ldots + (2n)^2]$$

$$+ \ldots\ldots$$

$$+ \binom{p}{p}[2^p + 4^p + \ldots\ldots + (2n)^p]$$

$$= n+1 + \binom{p}{1}\sum_{i=1}^{n}2i + \binom{p}{2}\sum_{i=1}^{n}(2i)^2 + \ldots\ldots + \binom{p}{p}\sum_{i=1}^{n}(2i)^p$$

$$= n+1 + \sum_{k=1}^{p}\left[\binom{p}{k}\sum_{i=1}^{n}(2i)^k\right]$$

Example: if we apply this formula for p=0, p=1, p=2, we find:

P=0:

$\sum_{i=0}^{n}(2i+1)^0 = 1^0 + 3^0 + \ldots\ldots + (2n+1)^0 = n+1$

P=1:

$\sum_{i=0}^{n}(2i+1) = 1 + 3 + \ldots\ldots + (2n+1)$

$= n+1+ \sum_{i=1}^{n} 2i$

$= n+1 + n(n+1)$

$= (n+1)^2$

P=2:

$\sum_{i=0}^{n}(2i+1)^2$

$= 1^2 + 3^2 + \ldots\ldots + (2n+1)^2$

$= n+1 + \sum_{k=1}^{2} [\binom{2}{k} \sum_{i=1}^{n}(2i)^k]$

$= n+1 + \binom{2}{1} \sum_{i=1}^{n}(2i) + \binom{2}{2} \sum_{i=1}^{n}(2i)^2$

$= n+1 + 2n(n+1) + \frac{2}{3} n(n+1)(2n+1)$

$= (n+1)[1+2n + \frac{2}{3} n(2n+1)]$

$= (n+1)(2n+1)(1+\frac{2}{3} n)$

$= \frac{1}{3} (n+1)(2n+1)(2n+3)$

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5. **(en)** Cet article est partiellement issu de l'article de Wikipédia en anglais intitulé « Bernoulli number » (voir la liste des auteurs).

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