A 3 parameter family of Riordan subarrays of a Riordan array

Peter Bala, Nov 30 2015

Let $R = (R(n,k))_{n,k\geq 0}$ be a proper Riordan array. Let m be a nonnegative integer and let $a > b$ be integers. We show the array $(R((m+1)n - ak, mn - bk))\, n, k \geq 0$ is also a Riordan array (not necessarily proper).

Let $f(x) = 1 + f_1x + f_2x^2 + f_3x^3 + \cdots$ and $g(x) = 1 + g_1x + g_2x^2 + g_3x^3 + \cdots$ be a pair of formal power series with, say, integer coefficients. The proper Riordan array $R = (f(x), xg(x))$ is defined as the lower unitriangular array whose k-th column has the ordinary generating function $f(x)(xg(x))^k$ [6, Section 5], [3, Section 1]. The elements of the array R are thus given by

$$R(n,k) \quad = \quad [x^n]f(x)(xg(x))^k \;\; k = 0,1,2...,\tag{1}$$

where $[x^n]$ is the coefficient extraction operator. The most well-known example of a proper Riordan array is Pascal's triangle $\left(\binom{n}{k}\right)$, which is the Riordan array $\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$.

More generally, a vertically stretched Riordan array $S = (f(x), x^cg(x))$, where c is a positive integer greater than 1, is defined as the lower triangular array whose k-th column has the ordinary generating function $f(x)(x^cg(x))^k$ [3, Section 2]. The elements of the array S are given by

$$S(n,k) \quad = \quad [x^n]f(x)(x^cg(x))^k.\tag{2}$$

We are interested in finding Riordan arrays which are, so to speak, contained in R. We will refer to these as Riordan subarrays of R. One simple method to construct a Riordan subarray is by selecting columns from R. For example, if $R = (f(x), xg(x))$ then the Riordan array $(f(x), xg^2(x))$ is constructed by taking the even indexed columns of R and arranging then in a lower unitriangular array. Similarly, the Riordan array $(f(x)g(x), xg^2(x))$ is obtained by taking the odd indexed columns of R. More generally, let $p \geq 0, q > 1$ be integers then the Riordan array $(f(x)g^p(x), xg^q(x))$ is made up of columns $p, p+q, p+2q, ..$ of R.

A less obvious example was given by Sprugnoli [7, Section 5]. The array $\left(\binom{2n-k}{n}\right)$, which is A092392 in the database, is a subarray of Pascal's triangle constructed by taking the first $k+1$ elements from column k and arranging them in a lower unit triangular array:

$$\binom{n}{k} \qquad\qquad\qquad\qquad\qquad \binom{2n-k}{n}$$

$$\begin{pmatrix} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ 1 & 5 & 10 & 10 & 5 & 1 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 2 & 1 \\ 6 & 3 & 1 \\ 20 & 10 & 4 & 1 \\ 70 & 35 & 15 & 5 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Using the Lagrange inversion formula, Sprugnoli shows that $\left(\binom{2n-k}{n}\right)$ is the Riordan array $(r'(x), r(x))$, where $r(x) = \text{Revert}(x(1-x)) = \frac{1-\sqrt{1-4x}}{2}$. By the above remarks we can then construct further Riordan arrays from the columns of A092392. For example, the array $\left(\binom{2n}{n-k}\right)$ formed from the even numbered columns of A092392 will be the Riordan array $(r'(x), \frac{r^2(x)}{x})$. This is A094527. The array $\left(\binom{2n+1}{n-k}\right)$ formed from the odd numbered columns of A092392 will be the Riordan array $(r'(x)\frac{r(x)}{x}, \frac{r^2(x)}{x})$. This is A111418 in the database.

Sprugnoli says his approach extends to deal with binomial coefficients of the form $\binom{pn+ak}{n-ck}$. Indeed, as we shall see in a moment, it can be extended further to find Riordan subarrays of an arbitrary proper Riordan array. We shall need the following version of the Lagrange-Bürmann formula for formal power series [4, Theorem 1.2.4] , [9]: if $f(x) = 1 + f_1x + f_2x^2 + f_3x^3 + \cdots$ and $H(x) = h_0 + h_1x + h_2x^2 + h_3x^3 + \cdots$ are formal power series and $G(x) = \text{Revert}\left(\frac{x}{f(x)}\right)$ then

$$[x^n]H(G(x)) \quad = \quad \frac{1}{n}\left[x^{n-1}\right] H^{'}(x)f(x)^n, \text{ for } n \geq 1. \tag{3}$$

The following result extends the calculations in [2, Theorem 3.1].

Theorem 1. Let m be a nonnegative integer. Let $a > b$ be integers. Let $R = (R(n,k))_{n,k\geq 0} = (f(x), xg(x))$ be a proper Riordan array. Define the power series $r(x)$ by

$$r(x) \quad = \quad \text{Revert}\left(\frac{x}{g^m(x)}\right). \tag{4}$$

Then the array $(R((m+1)n - ak, mn - bk))_{n,k\geq 0}$ is a Riordan array given by

$$(R((m+1)n - ak, mn - bk))_{n,k\geq 0} \quad = \quad \left(xf(r(x))\frac{r'(x)}{r(x)}, x^{a-b}\left(\frac{r(x)}{x}\right)^{a-b-\frac{b}{m}}\right).$$

If $a = b + 1$ the array is a proper Riordan array, while if $a > b + 1$ the array is a vertically stretched Riordan array.

Proof. By (1)

$$
\begin{aligned}
R((m+1)n-ak, mn-bk) &= [x^{(m+1)n-ak}]f(x)(xg(x))^{mn-bk} \\
&= [x^n]x^{(a-b)k}f(x)g(x)^{mn-bk} \\
&= (n+1)\left\{\frac{1}{n+1}[x^n]\frac{f(x)}{g^m(x)}\left(\frac{x^{a-b}}{g^b(x)}\right)^k (g^m(x))^{n+1}.\right\}
\end{aligned} \tag{5}
$$

Define a power series $H(x)$ by

$$
H^{'}(x) = \frac{f(x)}{g^m(x)}\left(\frac{x^{a-b}}{g^b(x)}\right)^k, \text{ with } H(0)=0. \tag{6}
$$

Then (5) becomes

$$
\begin{aligned}
R((m+1)n-ak, mn-bk) &= (n+1)\left\{\frac{1}{n+1}[x^n]H^{'}(x)\,(g^m(x))^{n+1}\right\} \\
&= (n+1)[x^{n+1}]H(r(x)) \quad \text{by (3) and (4) since } n+1\geq 1 \\
&= [x^n]\frac{d(H(r(x)))}{dx} \\
&= [x^n]H^{'}(r(x))r'(x) \\
&= [x^n]\frac{f(r(x))}{g^m(r(x))}r'(x)\left(\frac{r(x)^{a-b}}{g^b(r(x))}\right)^k \quad \text{by (6).}
\end{aligned}
$$

Thus $(R((m+1)n-ak, mn-bk))_{n,k\geq 0}$ is the Riordan array

$$
(F(x), x^{a-b}G(x)),
$$

where

$$
F(x) = \frac{f(r(x))}{g^m(r(x))}r'(x), \;\; G(x) = \frac{\left(\frac{r(x)}{x}\right)^{a-b}}{g^b(r(x))}. \tag{7}
$$

Now it follows from (4) that

$$
\frac{r(x)}{g^m(r(x))} = x. \tag{8}
$$

We can use this to rewrite (7) in terms of $r(x)$

$$
F(x) = f(r(x))\frac{xr'(x)}{r(x)}, \quad G(x) = \left(\frac{r(x)}{x}\right)^{a-b-\frac{b}{m}}.
$$

Thus

$$
(R((m+1)n-ak, mn-bk))_{n,k\geq 0} = \left(xf(r(x))\frac{r'(x)}{r(x)}, x^{a-b}\left(\frac{r(x)}{x}\right)^{a-b-\frac{b}{m}}\right).
$$

If $a = b+1$ then the array is a proper Riordan array; if $a > b+1$ the array is a vertically stretched Riordan array. $\square$

Example. In the theorem take R to be Pascal's triangle $\left(\binom{n}{k}\right)$ and choose $a = m, b = m-1$ so $a - b = 1$. Taking $m = 0, 1, 2, \ldots$ we get the family of proper Riordan arrays $\left(\binom{n}{k}\right)$, $\left(\binom{2n-k}{n}\right)$, $\left(\binom{3n-2k}{2n-k}\right)$, $\left(\binom{4n-3k}{3n-2k}\right)$, ... given by

$$\left(\frac{1}{1-r(x)}\frac{xr'(x)}{r(x)}, x\left(\frac{r(x)}{x}\right)^{\frac{1}{m}}\right),$$

where

$$r(x) \quad = \quad \text{Revert}\left(x(1-x)^m\right).$$

This result can be expressed in terms of the generalized binomial series of Lambert $\mathcal{B}_t(x)$ [5, Sections 5.4 and 7.5] defined as

$$\mathcal{B}_t(x) \quad = \quad \sum_{n\geq 0}\frac{1}{nt+1}\binom{nt+1}{n}x^n.$$

The Riordan property of our family of arrays is equivalent to a classical identity for Lambert series [5, equation 5.61] , [6, Section 2]:

$$\sum_{n\geq 0}\binom{mn+k}{n}x^n \quad = \quad \frac{\mathcal{B}_m(x)^{k+1}}{m+(1-m)\mathcal{B}_m(x)}.$$

Hence the array $\left(\binom{(m+1)n-mk}{mn-(m-1)k}\right)_{n,k\geq 0}$ equals the proper Riordan array $\left(\frac{\mathcal{B}_{m+1}(x)}{m+1-m\mathcal{B}_m(x)}, x\mathcal{B}_{m+1}(x)\right)$.

REFERENCES

[1] P. Bala, **Fractional iteration of a series inversion operator** - uploaded to A251592.

[2] Paul Barry, On the Central Coefficients of Riordan Matrices, Journal of Integer Sequences, 16 (2013), #13.5.1.

[3] C. Corsani, D. Merlini, R. Sprugnoli, Left-inversion of combinatorial sums, Discrete Mathematics, 180 (1998) 107-122.

[4] I. P. Goulden and D. M. Jackson, Combinatorial Enumeration, Dover Publications Inc., Mineola, NY, 2004. Reprint of the 1983 original.

[5] R. L. Graham, D. E. Knuth and O. Patashnik, **Concrete Mathematics**, Addison-Wesley, Reading, MA. Second ed. 1994.

[6] Elcio Lebensztayn, On the asymptotic enumeration of accessible automata, Discrete Mathematics and Theoretical Computer Science, Vol. 12, No.3, 2010, 75–80.

[7] Renzo Sprugnoli, **An Introduction to Mathematical Methods in Combinatorics**

[8] Sai-nan Zheng and Sheng-liang Yang, On the r-Shifted Central Coefficients of Riordan Matrices, Hindawi Publishing Corporation Journal of Applied Mathematics Volume 2014, Article ID 848374.

[9] Wikipedia, **Lagrange inversion theorem**